"Continuous Functions"

الدوال المتصلة

<u>Definition:</u> A function f is said to be continuous at a if the following conditions are satisfied.

1. $f(a)$ defined
2. $\lim\limits_{x \to a} f(x)$ exists
3. $f(a) = \lim\limits_{x \to a} f(x)$.

<u>Examples:</u>

① Is the function $f(x) = \begin{cases} 2x+3 & , x \leq 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$ continuous? Explain.

<u>Sol.</u> at $(a=4)$

1. $f(4) = 2(4)+3 = 8+3 = 11$

2. $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \left(7+\frac{16}{x}\right) = 7 + \frac{16}{4} = 7+4 = 11$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (2x+3) = 2(4)+3 = 8+3 = 11$

$\therefore \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^-} f(x) \Rightarrow \lim\limits_{x \to 4} f(x) = 11$ exists

3. $f(4) = \lim\limits_{x \to 4} f(x)$

$\therefore f(x)$ is continuous at $a=4$.

② Is the function $f(x) = \begin{cases} 4x-2 & , x > 2 \\ 2 & , x = 2 \\ 3x & , x < 2 \end{cases}$

continuous? Explain.

<u>Sol.</u> at $(a=2)$

1. $f(2) = 2$

2. $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x-2) = 4(2)-2 = 6$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3x) = 3(2) = 6$

$\therefore \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) \Rightarrow \lim\limits_{x \to 2} f(x) = 6$ exists

3. $f(2) \neq \lim\limits_{x \to 2} f(x)$

$2 \neq 6$

$\therefore f(x)$ is <u>not continuous</u> at $a=2$.
(discontinuous)

(3) Is the function $f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 4 & , x = 2 \end{cases}$

continuous? Explain.

Sol: at $(a=2)$

1- $f(2) = 4$

2- $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2}$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= \lim_{x \to 2} (x+2) = 2+2 = 4$

$\therefore \lim_{x \to 2} f(x) = 4$

3- $f(2) = \lim_{x \to 2} f(x)$

$\therefore f(x)$ is continuous at $a=2$.

H.w: Is the following functions continuous or discontinuous? Explain your answer.

(1) $f(x) = \begin{cases} x^2+5, & x \geq -1 \\ 6x, & x < -1 \end{cases}$

(2) $f(x) = \begin{cases} 2x+5 & , x > 3 \\ x^2+2 & , x \leq 3 \end{cases}$

"Slope of Line" (4) ?

ميل الخط

Def: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are the end points of nonvertical line segment, then the slope m of the line segment defined by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Examples: Find the slope of the line passing through the points:

(1) $P_1(11, 3)$ and $P_2(14, 7)$
(2) $P_1(3, 6)$ and $P_2(-5, -9)$
(3) $P_1(4, 5)$ and $P_2(8, 13)$

Sol:

1- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-3}{14-11} = \frac{4}{3} = 1.3$

2- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-9-6}{-5-3} = \frac{-15}{-8} = \frac{15}{8} = 1.875$

3- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13-5}{8-4} = \frac{8}{4} = 2$

"Slope of Line"

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Equation of Line: معادلة الخط

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$\boxed{m = \frac{y - y_1}{x - x_1}} \longrightarrow (1)$$

$y - y_1 = m(x - x_1)$
$y - y_1 = mx - mx_1$
$y - mx + (mx_1 - y_1) = 0$

$$\boxed{y = mx + b} \longrightarrow (2)$$

Examples: ① Find the equation of the line passing through the point $P_1(2, 5)$ with slope $\frac{3}{4}$.

Sol.

$m = \frac{y - y_1}{x - x_1}$

$\frac{3}{4} = \frac{y - 5}{x - 2}$

$4(y - 5) = 3(x - 2)$
$4y - 20 = 3x - 6$
$4y = 3x - 6 + 20$
$4y = 3x + 14$
$y = \frac{3}{4}x + \frac{14}{4}$

$y = \frac{3}{4}x + \frac{7}{2}$

② Find the equation of the line passing through the point $P(-2,3)$, with slope 9.

Sol:

$$m = \frac{y - y_1}{x - x_1}$$

$$9 = \frac{y-3}{x+2}$$

$$y-3 = 9(x+2)$$
$$y-3 = 9x+18$$
$$y = 9x+18+3$$
$$y = 9x+21$$

③ $P(3,4)$, slope $\frac{1}{2}$.

Sol:

$$m = \frac{y-y_1}{x-x_1}$$

$$\frac{1}{2} = \frac{y-4}{x-3}$$

$$2(y-4) = x-3$$
$$2y-8 = x-3$$
$$2y = x-3+8$$
$$2y = x+5$$
$$y = \frac{1}{2}x + \frac{5}{2}.$$

"Derivatives"

Def: The function f' defined by the formula

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

is called the derivative of f with respect to x. when the limit exist, f is said to be differentiable.

Remark: The derivative of the function f (or y) w.r.t. x is denoted by

$$f'(x) = \frac{d f(x)}{dx}, \quad y' = \frac{dy}{dx}$$

Differentiation Rules: Let f, g be a differentiable functions at x, and c be any constant, then:

1. $\frac{dx}{dx} = 1$
2. $\frac{d}{dx} c = 0$
3. $\frac{d}{dx}(x^n) = n x^{n-1}$
4. $\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot f'(x)$.
5. $\frac{d}{dx}[c f(x)] = c f'(x)$
6. $\frac{d}{dx}[f(x) \mp g(x)] = f'(x) \mp g'(x)$
7. $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
8. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$.