(Algebra of Convergent Sequence of Real Numbers)

Theorem: Let $a_n \to a, b_n \to b$ be two convergent sequence in $\mathbb{R}$, then:

i) $a_n + b_n \to a + b$
ii) $a_n - b_n \to a - b$
iii) $a_n . b_n \to a.b$
iv) $Ca_n \to Ca, \quad \forall C \in \mathbb{R}$
v) $\frac{a_n}{b_n} \to \frac{a}{b}, \quad b_n \neq 0$ and $b \neq 0$.

Proof: (i) To prove $a_n + b_n \to a + b$

Since $a_n \to a \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

Since $b_n \to b \Rightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \forall n > n_1(\varepsilon)$

Let $\varepsilon = \frac{\varepsilon}{2} > 0$

We choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n > n_2(\varepsilon)$

$\therefore a_n + b_n \to a + b.$

Proof: (iii) To prove $a_n . b_n \to a.b$

1) Since a_n converges to a, so a_n is bounded $\Rightarrow \exists M_1 > 0$ such that $|a_n| < M_1, \forall n \in \mathbb{N}$

2) $a_n \to a \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

Let $\varepsilon = \frac{\varepsilon}{2|b|} > 0 \Longrightarrow |a_n - a| < \frac{\varepsilon}{2|b|}, \forall n > n_0(\varepsilon)$

$b_n \to b \Longrightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \forall n > n_1(\varepsilon)$.

Let $\varepsilon = \frac{\varepsilon}{2M_1} > 0 \Longrightarrow |b_n - b| < \frac{\varepsilon}{2M_1}, \forall n > n_1(\varepsilon)$.

3) Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$$\begin{aligned}|a_n . b_n - a.b| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |(a_n)(b_n - b) + (a_n - a)(b)| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \\ &< M_1 \frac{\varepsilon}{2M_1} + |b| \frac{\varepsilon}{2|b|} = \varepsilon, \forall n > n_2(\varepsilon)\end{aligned}$$

$\therefore a_n . b_n \to a.b.$

Proof: (v) To prove $\frac{a_n}{b_n} \to \frac{a}{b}, \; b_n \neq 0, b \neq 0$

1) To prove $\frac{1}{b_n} \to \frac{1}{b}, b_n \neq 0, b \neq 0$

$\because b_n \to b \Longrightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \forall n > n_0(\varepsilon)$

$\because b \neq 0 \Longrightarrow b > 0 \; (-b > 0)$.

Let $\varepsilon = \frac{b}{2} > 0$

$|b_n - b| < \varepsilon$ means

$-\varepsilon < b_n - b < \varepsilon$

$b - \varepsilon < b_n < b + \varepsilon$

$b - \frac{b}{2} < b_n < b + \frac{b}{2} \Longrightarrow 0 < \frac{b}{2} < b_n < \frac{3b}{2} \Longrightarrow 0 < \frac{2}{3b} < \frac{1}{b_n} < \frac{2}{b}$

$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{1}{|b_n||b|}|b_n - b| < \frac{2}{b^2} . \varepsilon = \frac{2}{b^2} . \frac{b^2 \varepsilon}{2} = \varepsilon$, (we choose $\varepsilon = \frac{b^2 \varepsilon}{2}$)

$\therefore \frac{1}{b_n} \to \frac{1}{b}$

2) By using part (iii) we get $\frac{a_n}{b_n} \to \frac{a}{b}, \; b_n \neq 0, b \neq 0$

Theorem: Let a_n be a sequence of real numbers if $a_n \to a$ then $|a_n| \to |a|$.

Remark: The converse may not be true.

For example:

$a_n = (-1)^n, \; |a_n| = |(-1)^n| = 1 \to 1$. But a_n does not converge.

Theorem (Sandwich Theorem):

If $a_n \to a, b_n \to a,$ (c_n) be a sequence of real numbers such that $a_n \leq c_n \leq b_n$, then $c_n \to a$.

Proof:

$a_n \to a, \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$b_n \to a \Rightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|b_n - a| < \varepsilon, \forall n > n_1(\varepsilon)$.

Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$-\varepsilon < a_n - a \leq c_n - a \leq b_n - a < \varepsilon$

$\Rightarrow -\varepsilon < c_n - a < \varepsilon$

i.e. $|c_n - a| < \varepsilon, \forall n > n_2(\varepsilon)$

$\therefore c_n \to a.$

Example:

Discuss the convergent of $a_n = \dfrac{\sin(n)}{n}$.

Answer:

$-1 \leq \sin(n) \leq 1$

$-\dfrac{1}{n} \leq \dfrac{\sin(n)}{n} \leq \dfrac{1}{n}$

By Archimedean property $\dfrac{1}{n} \to 0$ *and* $-\dfrac{1}{n} \to 0$

By Sandwich theorem $a_n = \dfrac{sin(n)}{n} \to 0.$

Definition (Cauchy Sequence):

Let (a_n) be a sequence of real numbers. (a_n) is called Cauchy sequence if $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a_m| < \varepsilon,\ \forall n, m > n_0(\varepsilon)$.

Remark:

i) If (a_n) convergence to a, then (a_n) is Cauchy.
ii) The converse of (i) is not true.

Proof: (i) If $(a_n) \to a$, then (a_n) is Cauchy.

$a_n \to a$ means $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \frac{\varepsilon}{2}, \forall n > n_0(\varepsilon)$

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a_m - a| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall n, m > n_0(\varepsilon) \end{aligned}$$

(ii) The converse of (i) is not true.

For example: Let $X = \mathbb{R}\backslash\{0\}, (a_n) = \frac{1}{n}$

$(a_n) = \dfrac{1}{n} \to 0$ in $\mathbb{R}$ (By Archimedean Property) $\Rightarrow \therefore (a_n) = \dfrac{1}{n}$ is Cauchy

But does not convergent in $\mathbb{R}\backslash\{0\}$.