**Definition (Complete Property):**

The ordered field $(F,+,.,\leq)$ is said to be complete if every nonempty subset $A$ of $F$ which is bounded above has least upper bound.

**Examples:**

1. The real numbers system $(\mathbb{R},+,.,\leq)$ is complete order field.
2. The order field of rational numbers $(\mathbb{Q},+,.,\leq)$ is not complete. Since

   Let $S=\{x\in\mathbb{Q}^{+}$ such that $x^2<2\}\subseteq\mathbb{Q}$ and $1\in S\neq\emptyset$

   $S$ is bounded above but has no least upper bound in $\mathbb{Q}$ because $\sqrt{2}\notin\mathbb{Q}$

i.e. $\exists$ a nonempty subset in $\mathbb{Q}$ which is bounded from above but has no least upper bound.

**Theorem:**

The equation $x^2=2$ has no root in $\mathbb{Q}$ ($\sqrt{2}\notin\mathbb{Q}$).

**Proof:**

Assume that $x^2=2$ has a root in $\mathbb{Q}$, so there is $x=\frac{a}{b}\in\mathbb{Q}$ such that $x^2=\left(\frac{a}{b}\right)^2=2$

$\left(\frac{a}{b}\right)^2=\frac{a^2}{b^2}=2\Rightarrow a^2=2b^2$

$\because b\neq0\Rightarrow a\neq0$

Suppose $a,b$ are positive numbers such that $g.c.d\,(a,b)=1$

1. If $a,b$ are odd numbers $\Rightarrow a^2$ is odd $\Rightarrow 2b^2$ is odd C! ($2b^2$ is even)

2. If $a$ is odd number and $b$ is even number

   $\Rightarrow b=2d\Rightarrow a^2=8d^2\Rightarrow a^2$ is even C! ($a$ is odd)

3. If $a$ is even number and $b$ is odd number

   $\Rightarrow a=2c\Rightarrow 4c^2=2b^2\Rightarrow 2c^2=b^2\Rightarrow b^2$ is even C! ($b$ is odd)

4. If $a,b$ are even numbers impossible since $g.c.d\,(a,b)=1$

$\therefore$ there is no rational number satisfy $x^2=2$. i.e. $\sqrt{2}\notin\mathbb{Q}$.



**Theorem: (Archimedean Property):**

For all $x,y\in\mathbb{R}$ and $x>0$, then $\exists\, n\in\mathbb{N}$ such that $nx>y$.

**Proof:**

Assume that $\forall n\in\mathbb{N},\ \exists\, x,y\in\mathbb{R}\,(x>0)\ s.t.\ nx\leq y$

Let $S=\{nx: n\in\mathbb{N}\}\subseteq\mathbb{R}$ and $x\in S\neq\emptyset$

$y$ is an upper bound of $S$

Since $\mathbb{R}$ is complete $\Rightarrow S$ has least upper bound say $\alpha$

$\alpha=\ell.\text{u.b.}(S)$

$\because x>0\Rightarrow -x<0\Rightarrow \alpha-x<\alpha$

i.e. $\alpha-x$ can not be upper bound of $S$

$\therefore \exists\, mx\in S\ s.t.\ \alpha-x<mx\Rightarrow \alpha<x(m+1)$

But $x(m+1)\in S$ and this is contradiction that $\alpha=\ell.\text{u.b}(S)$

$\therefore \forall\, x,y\in\mathbb{R}$ and $x>0, \exists\, n\in\mathbb{N}$ s.t $nx>y$.

**Corollary:**

$\forall\varepsilon>0, \exists n\in\mathbb{N}$ such that $0<\frac{1}{n}<\varepsilon$.

**Proof:**

Given $\varepsilon>0$, by using A.P. (Archimedean Property), $\forall\, x,y\in\mathbb{R}$ and $x>0$, $\exists\, n\in\mathbb{N}$ s.t. $nx>y$

Let $x=\varepsilon>0$ and $y=1\Rightarrow n\varepsilon>1\Rightarrow 0<\frac{1}{n}<\varepsilon$.

**Theorem: (Density of Rational Numbers in $\mathbb{R}$):**

If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\, r \in \mathbb{Q}$ such that $x < r < y$.

**Proof:**

Let $x, y \in \mathbb{R}$ and $x < y$

If $x < 0 < y \Rightarrow 0 \in \mathbb{Q}$ result holds.

If $x > 0$ $(y > 0)$ we have $y - x > 0$ $\quad (x < y)$

By A.P. $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y - x$.

$\Rightarrow 1 < n(y - x) = ny - nx$

$1 < ny - nx \Rightarrow 1 + nx < ny \cdots (1)$

$nx > 0 \Rightarrow \exists m \in \mathbb{N}$ such that $m - 1 \leq nx < m \cdots (2)$

From (1) and (2) we get $nx < m \leq nx + 1 < ny$

$\Rightarrow nx < m < ny$

$\therefore x < \frac{m}{n} < y \quad (n \neq 0$ since $n \in \mathbb{N})$.

**Theorem: (Density of Irrational Numbers in $\mathbb{R}$):**

If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\, s \in \mathbb{Q}'$ (irrational number) such that $x < s < y$.

**Proof:**

Let $x, y \in \mathbb{R}$ and $x < y$, $\sqrt{2} \in \mathbb{Q}' \subseteq \mathbb{R} \Rightarrow \sqrt{2} \in \mathbb{R}$

$\sqrt{2}\, x < \sqrt{2}\, y \in \mathbb{R}$

By (D. $\mathbb{Q}$ in $\mathbb{R}$), $\exists\, r \in \mathbb{Q}$ such that

$\sqrt{2}x < r < \sqrt{2}y$

$x < \frac{r}{\sqrt{2}} < y \quad (\frac{r}{\sqrt{2}} \in \mathbb{Q}')$.

**H.W.**

Prove that if $x, y \in \mathbb{Q}$, then $\exists\, r \in \mathbb{Q}'$ such that $x < r < y$.