Chapter Two

Sequences of Real Numbers

Definition (Sequence of Real Numbers):

The sequence of real numbers S_n is a function from $\mathbb{N}$ into $\mathbb{R}$

i.e. $S: \mathbb{N} \to \mathbb{R}$ defined as $S(n) = S_n \in \mathbb{R}, \forall n \in \mathbb{N}$, denoted as $S_n, (S_n), < S_n >, \{S_n\}$.

$\{S_n : n \in \mathbb{N}\}$ the range of the sequence.

Examples: 1) $S_n = n$ 2) $S_n = 1$ 3) $S_n = (-1)^n$ 4) $S_n = \frac{1}{n}$

Definition (Convergent of Sequence of Real Numbers):

Let a_n be a sequence of real numbers, $a \in \mathbb{R}$ we say a_n convergence to a if:

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

a is called convergence point of a_n, write $a_n \to a_n$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = a$.

Theorem (Uniqueness of Convergence Point):

If the sequence of real numbers a_n convergent then it has unique limit point.

Proof:

Assume that $a_n \to a$, $a_n \to b$ such that $a \neq b \Rightarrow |b - a| > 0$

$a_n \to a \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$a_n \to b \Rightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|a_n - b| < \varepsilon, \forall n > n_1(\varepsilon)$.

Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$|b - a| = |b - a_n + a_n - a| \leq |a_n - a| + |a_n - b| < \varepsilon + \varepsilon = 2\varepsilon$

Let $\varepsilon = \frac{|b-a|}{2} > 0 \Rightarrow |b - a| < 2\frac{|b-a|}{2} = |b - a|$ C! $\Rightarrow \therefore a = b$.

Examples:

1) Is the sequence of real numbers $a_n = C$ is convergent?

Answer:

We have to prove that $a_n = C \to C$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon$, $\forall n > n_0(\varepsilon)$.

$|a_n - a| = |C - C| = 0 < \varepsilon, \forall n > n_0(\varepsilon)$.

$\therefore a_n = C \to C$

2) The sequence of real numbers $a_n = (-1)^n$ does not convergent (divergent $a_n = (-1)^n \nrightarrow$).

Since $a_n = (-1)^n = -1, 1, -1, 1, \cdots$ has two convergence points which are -1 & 1

and $-1 \neq 1$ so $a_n = (-1)^n \nrightarrow$ divergent.

3) Show that the sequence of real numbers $a_n = \frac{1}{n}$ is convergent (convergence to 0).

Proof:

We have to prove that $a_n = \frac{1}{n} \to 0$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$$|a_n - a| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$$

By A. P. $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $0 < \frac{1}{n_0(\varepsilon)} < \varepsilon$.

$$\forall n > n_0(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{n_0(\varepsilon)} < \varepsilon \Rightarrow \therefore \frac{1}{n} < \varepsilon, \forall n > n_0(\varepsilon)$$

i.e $|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$, $\forall n > n_0(\varepsilon) \Rightarrow \therefore a_n = \frac{1}{n} \to 0$.

4) Discuss the convergent of the sequence of real numbers $S_n = \frac{n}{n+1}$.

Answer:

We have to prove that $S_n = \frac{1}{n+1} \to 1$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \forall n > n_0(\varepsilon)$.

$$|S_n - S| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-n-1}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

By A. P. $\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon) > 0$ such that $0 < \frac{1}{n_0(\varepsilon)} < \varepsilon$.

$\forall n > n_0(\varepsilon) \Rightarrow n + 1 > n_0(\varepsilon) + 1 > n_0(\varepsilon)$

$$\Rightarrow \frac{1}{n+1} < \frac{1}{n_0(\varepsilon)+1} < \frac{1}{n_0(\varepsilon)} < \varepsilon$$

$$\Rightarrow \therefore \frac{1}{n+1} < \varepsilon$$

i.e. $|S_n - S| = \frac{1}{n+1} < \varepsilon, \forall n > n_0(\varepsilon)$

$$\therefore S_n = \frac{1}{n+1} \to 1$$

Definition (Bounded Sequence of Real Numbers):

Let a_n be a sequence of real numbers, we say that a_n is bounded iff $\exists M > 0$, $(M \in \mathbb{R})$, such that $|a_n| < M, \forall n \in \mathbb{N}$.

Theorem:

Every convergent sequence of real numbers a_n is bounded.

Remark:

The converse may not be true, for example $a_n = (-1)^n$ is bounded sequence but not convergent.

Definition (Monotone Sequence of Real Numbers):

Let (a_n) be a sequence of real numbers, then:

(a_n) is called increasing sequence ($\uparrow$) if $a_n \le a_{n+1}, \forall n \in \mathbb{N}$.

(a_n) is called decreasing sequence ($\downarrow$) if $a_n \ge a_{n+1}, \forall n \in \mathbb{N}$.

(a_n) is called monotone equence ($\updownarrow$) if a_n increasing ($\uparrow$) or a_n decreasing ($\downarrow$).

For example:

$a_n = n\ (\uparrow)$, $a_n = \frac{1}{n}(\downarrow)$, $a_n = k(\leftrightarrow)$.

Theorem:

Every bounded and monotone sequence of real numbers (a_n) is convergent.

Proof:

Let $S = \{a_n : n \in \mathbb{N}\}, \emptyset \ne S \subseteq \mathbb{R}$, S is bounded (since range is bounded set)

By completeness of $\mathbb{R} \Rightarrow S$ has least upper bound say a

We claim $a_n \to a$

$\forall \varepsilon > 0$, $a - \varepsilon < a$

$a - \varepsilon$ is not upper bound for $S \Rightarrow \exists\, a_{n0}(\varepsilon) > 0$ such that $a - \varepsilon < a_{n0}(\varepsilon)$

Since (a_n) monotone (increasing) $\Rightarrow a_{n0}(\varepsilon) \le a_n, \forall n > n_0(\varepsilon)$

$\Rightarrow a - \varepsilon < a_n \Rightarrow |a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$ i.e. $a_n \to a$

Example: Let $a_1 = 1$, $a_{n+1} = \frac{1}{4}(2a_n + 3)$, $\forall n \ge 1$

1) a_n is bounded

$\because a_1 = 1 < 2, a_2 = \frac{5}{4} < 2, \ldots \Rightarrow a_n \le 2, \forall n \ge 1$ i.e. a_n is bounded

(we can prove that $a_n \le 2, \forall n \ge 1$ by using mathematical induction)

2) a_n monotone (increasing)

$\because a_1 = 1, a_2 = \frac{1}{4}(2.1 + 3) = \frac{5}{4}, \ldots \Rightarrow a_n = \left(1, \frac{5}{4}, \ldots\right)$ is increasing

(we can prove that $a_n \le a_{n+1}$ by using mathematical induction)

$\therefore a_n$ is convergent (by the above theorem).