Chapter One

The real numbers system

Definition (The Field):

Let F be a nonempty set and $+, .$ be two binary operations on F, then $(F,+,.)$ is called field if its satisfy the following conditions:

F1: (Closure Property), $\forall a,b \in F$ we have:

$a+b \in F$ and $a.b \in F$

F2: (Associative Property), $\forall a,b,c \in F$ we have:

$a+(b+c)=(a+b)+c \in F$ and $a.(b.c)=(a.b).c \in F$

F3: (Commutative Property), $\forall a,b \in F$ we have:

$a+b=b+a$ and $a.b=b.a$

F4: (Existence of identity element)

There is an element $0 \in F$ such that $a+0=0+a=a, \forall a \in F$, and

There is an element $1 \in F$ such that $a.1=1.a=a, \forall a \in F$

(Notice that: $1 \neq 0$).

F5: (Existence of inverse element)

$\forall a \in F, \exists -a \in F$ such that $a+(-a)=(-a)+a=0$

$\forall a \in F, \exists a^{-1} \in F$ such that $a.a^{-1}=a^{-1}.a=1$

F6: (Distributive Property), $\forall a,b,c \in F$ we have:

$a.(b+c)=a.b+a.c$ *and* $(a+b).c=a.c+b.c$

Note: The identity element for the binary operations + and . is unique.

Examples: $(\mathbb{R},+,.)$, $(\mathbb{Q},+,.)$ are fields.

Note:

$\mathbb{R}$ is the set of real numbers

$\mathbb{Q}$ is the set of rational numbers, where $\mathbb{Q}=\left\{\frac{a}{b}: a,b \text{ integers}, b \neq o \text{ and } g.c.d(a,b)=1\right\}$.

Definition (The Relation on A):

Let A be a nonempty set, R is called a relation on A if $R \subset A \times A$, where

$A \times A=\{(a,b): a,b \in A\}$, $(a,b) \in R$ i.e. $aRb, \forall a,b \in A$.

Definition (The Order Relation on A) or (Order Set):

Let A be a nonempty set, the relation $R: \leq$ on A is called order relation on A [$(A,\leq)$ order set] if its satisfy the following conditions:

i) $a \leq a, \forall a \in A$ (Reflexive).
ii) If $a \leq b$ and $b \leq a \Rightarrow a=b, \forall a,b \in A$ (Anti-symmetric).
iii) If $a \leq b$ and $b \leq c \Rightarrow a \leq c, \forall a,b,c \in A$ (Transitive).

Examples:

The relation $\leq$ on $\mathbb{R}$ ($\mathbb{Q}$) is order relation

i.e. $(\mathbb{R},\leq)$, $(\mathbb{Q},\leq)$ are order sets.

Definition (The Order Field):

Let $(F,+,.)$ be a field and $\leq$ be a relation on F, we say that $(F,+,.,\leq)$ is an order field if:

i) $a \leq a, \forall a \in F$ (Reflexive)
ii) If $a \leq b$ and $b \leq a \Rightarrow a=b, \forall a,b \in F$ (Anti-symmetric)
iii) If $a \leq b$ and $b \leq c \Rightarrow a \leq c, \forall a,b,c \in F$ (Transitive)
iv) Either $a \leq b$ or $b \leq a, \forall a,b \in F$
v) If $a \leq b$ and $c \leq d \Rightarrow a+c \leq b+d, \forall a,b,c,d \in F$
vi) If $a \leq b$ and $c > 0 \Rightarrow a.c \leq b.c, \forall a,b,c \in F$

The relation $\leq$ on $(F,+,.)$ is total order relation.

Examples:

$(\mathbb{R},+,.,\leq)$, $(\mathbb{Q},+,.,\leq)$ are order fields.

Bounded Set in Order Field $(F,+,.,\leq)$.

Definitions:

Let $(F,+,.,\leq)$ be an order field and $A\subseteq F$, then:

1) $u\in F$ is called **upper bound** for A [**u.b.(A)**] if $a\leq u, \forall a\in A$.
2) $\ell\in F$ is called **lower bound** for A [**ℓ.b.(A)**] if $\ell\leq a, \forall a\in A$.
3) A is called **bounded above** if it has upper bound.
4) A is called **bounded below** if it has lower bound.
5) A is called **bounded** if A has upper bound and lower bound
6) $u^*\in F$ is called **least upper bound** for A [**ℓ.u.b.(A) or $sup(A)$**] if
   - i) u^* is an upper bound for A i.e. $u^*\in F$ $s.t.$ $a\leq u^*, \forall a\in A$
   - ii) For each upper bound u for A we have $u^*\leq u$.
7) $\ell^*\in F$ is called **greatest lower bound** for A [**$g.\ell.b.(A)$ or $inf(A)$**] if
   - i) ℓ^* is a lower bound for A i.e. $\ell^*\in F$ $s.t.$ $\ell^*\leq a, \forall a\in A$
   - ii) For each lower bound ℓ for A we have $\ell\leq\ell^*$

Remarks:

1) $\ell-\alpha\leq\ell\leq a\leq u\leq u+\beta, \ \forall a\in A, \ \alpha,\beta>0.$
2) If the set A has least upper bound (greatest lower bound) then its unique.

Examples:

1. Let $A=(0,1]$. Find upper bound, lower bound, least upper bound and greatest lower bound.

Answer:

Since $1\in\mathbb{R}$ s.t. $a\leq 1$, $\forall a\in(0,1]$, $(1\in A)$
and $1.5\in\mathbb{R}$ s.t. $a<1.5$, $\forall a\in(0,1]$
$2\in\mathbb{R}$ s.t. $a<2$, $\forall a\in(0,1]$
$\vdots$

$\therefore$ u.b.$(A)=1,1.5,2,\cdots$ (upper bounds)

$\therefore$ $A=(0,1]$ is bounded above

ℓ.u.b.$(A)=1$ (least upper bound)

Now, since $0\in\mathbb{R}$ $s.t.$ $0<a$, $\forall a\in(0,1]$
and $-0.5\in\mathbb{R}$ $s.t.$ $-0.5<a$, $\forall a\in(0,1]$
$-1\in\mathbb{R}$ $s.t.$ $-1<a$, $\forall a\in(0,1]$
$\vdots$

$\therefore$ ℓ.b.$(A)=0,-0.5,-1,\cdots$ (lower bounds)

$\therefore$ $A=(0,1]$ is bounded below

$g.\ell.b.(A)=0$ (greatest lower bound)

i.e. $A=(0,1]$ is bounded (since A is bounded above and bounded below).

2. Let $B=\{3,4,5,6\}$. Find upper bound, lower bound, least upper bound and greatest lower bound.

Since $6\in\mathbb{R}$ s.t. $a\leq 6, \forall a\in B=\{3,4,5,6\}$

$\therefore$ u.b.$(B)=6,6.25,6.5,7,\cdots$

$\therefore$ $B=\{3,4,5,6\}$ is bounded above

ℓ.u.b.$(B)=6$

Now, since $3\in\mathbb{R}$ s.t. $3\leq a, \ \forall a\in B=\{3,4,5,6\}$

$\therefore$ ℓ.b.$(B)=3,2.5,2,1,\cdots$

$\therefore$ $B=\{3,4,5,6\}$ is bounded below

$g.\ell.b.(B)=3$

The set $B=\{3,4,5,6\}$ is bounded (since B is bounded above and bounded below).

3. $\mathbb{N}=\{1,2,3,\ldots\}$ is unbounded (since $\mathbb{N}$ is bounded below but unbounded from above)
4. $\mathbb{R}$ is unbounded (since $\mathbb{R}$ unbounded from above and from below).

H.W.

Check the $A_1=\{-n:n\in\mathbb{N}\}$, $A_2=(-1,1)$ and $A_3=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$ are bounded.