Algebra of Differentiable Real-Valued Functions

Theorem: Let $f, g: I \to \mathbb{R}$ be two real-valued functions defined on open intervals I, if f, g are differentiable at $x_0 \in I$, then:

1. $f \pm g$ is differentiable at x_0 and $(f \pm g)'(x_0) = f'(x_0) + g'(x_0)$.
2. $f \cdot g$ is differentiable at x_0 and $(f \cdot g)'(x_0) = f(x_0)\, g'(x_0) + g(x_0)\, f'(x_0)$.
3. kf is differentiable at x_0, $k \in \mathbb{R}$ and $(kf)'(x_0) = k f'(x_0)$.
4. $\frac{f}{g}$ is differentiable at x_0, $g(x_0) \neq 0$ and $\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{(g(x_0))^2}$.

proof (4):

$$\lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\, g(x_0) - f(x_0)\, g(x)}{(x - x_0)\, g(x)\, g(x_0)}$$

$$= \lim_{x \to x_0} \frac{f(x)\, g(x_0) + f(x_0) g(x_0) - f(x_0) g(x_0) - g(x) f(x_0)}{(x - x_0)\ g(x)\, g(x_0)}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right] \frac{1}{g(x)\, g(x_0)}$$

$$= \left[f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0) \right] \frac{1}{g(x_0)\, g(x_0)}$$

$$= \frac{g(x_0)\, f'(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)} = \left(\frac{f}{g}\right)'(x_0)$$

$\therefore \frac{f}{g}$ is differentiable at x_0 and $\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{(g(x_0))^2}$

Note: since g is differentiable at $x_0 \Rightarrow g$ is continuous at x_0, and $g(x_0) \neq 0$, then $\frac{1}{g}$ is continuous at x_0 i.e. $\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{g(x_0)}$.

Theorem:

Let $f: (a,b) \to \mathbb{R}$ be a real-valued function and $x_0 \in (a,b)$, then f is differentiable at x_0 iff there exists a real number α and a continuous function $w: (a,b) \to \mathbb{R}$ with $w(x_0) = 0$ satisfies $f(x) = f(x_0) + [(x - x_0)\alpha + (x - x_0)\, w(x)]$.

Proof: $(\Rightarrow)$

Let f be a differentiable at x_0, $x_0 \in (a,b)$

$\exists\ (x_n)$ in (a,b) and $\alpha \in \mathbb{R}$ s.t. $\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \alpha$,

as $x_n \to x_0$, $x_n \neq x_0$, $\forall n$

Define $w(x): (a,b) \to \mathbb{R}$ as follows

$$w(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} - \alpha, & x \neq x_0 \\ 0, & x = x_0 \end{cases}$$

$\because x_n \to x_0 \Rightarrow w(x_n) = \frac{f(x_n) - f(x_0)}{x_n - x_0} - \alpha \to 0 = w(x_0)$

$\therefore w(x_n) \to w(x_0) \Rightarrow \therefore f$ is continuous (by Th.(3) in ch. 5)

$\because x \neq x_0 \Rightarrow w(x) = \frac{f(x) - f(x_0)}{x - x_0} - \alpha = \frac{f(x) - f(x_0) - \alpha(x - x_0)}{x - x_0}$

$\Rightarrow (x - x_0)\, w(x) = f(x) - f(x_0) - \alpha(x - x_0)$

$\therefore f(x) = f(x_0) + [(x - x_0)\alpha + (x - x_0)\, w(x)]$.

$\Leftarrow$) T.P. f is differentiable at x_0, $x_0 \in (a,b)$

We have $\alpha \in \mathbb{R}$ and $w: (a,b) \to \mathbb{R}$ continuous, $w(x_0) = 0$ and $f(x) = f(x_0) + [(x - x_0)\alpha + (x - x_0)\, w(x)]$

Let (x_n) sequence in (a,b) s.t. $x_n \to x_0$, $x_n \neq x_0$, $\forall n$

T.P. $\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \alpha$

$\because x_n \to x_0$ and w continuous $\Rightarrow w(x_n) \to w(x_0)$ (by Th.(3) in ch. 5)

$(x - x_0)\, w(x) = f(x) - f(x_0) - \alpha(x - x_0)$

Increasing Function and Decreasing Function

Definitions: Let $f: I \to \mathbb{R}$ be a real-valued function on open interval I and $x_0 \in I$, then

(1) f is increasing function at x_0 if $\exists V$ open interval, $x_0 \in V$ s.t. if $x < x_0 \Rightarrow f(x) < f(x_0)$
and if $x > x_0 \Rightarrow f(x) > f(x_0)$, $\forall x \in V$.

(2) f is decreasing function at x_0 if $\exists V$ open interval, $x_0 \in V$ s.t. if $x < x_0 \Rightarrow f(x) > f(x_0)$
and if $x > x_0 \Rightarrow f(x) < f(x_0)$, $\forall x \in V$.

(3) If f is increasing function at each $x_0 \in I$ then we say that f is increasing function.

(4) If f is decreasing function at each $x_0 \in I$ then we say that f is decreasing function.

Theorem:
Let $f: I \to \mathbb{R}$ be a differentiable function at x_0, $x_0 \in I$, (I any open interval in $\mathbb{R}$). If $f'(x_0) > 0$ then f is increasing at x_0 and if $f'(x_0) < 0$ then f is decreasing at x_0, then if $f'(x_0) \neq 0$ then there exists V open interval about x_0 such that f is 1-1 on V.

proof:
Let f be a differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$

i.e. $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0 = \begin{cases} f'(x_0) > 0 \\ f'(x_0) < 0 \end{cases}$

Case (1): If $f'(x_0) > 0$, $\exists V$ open interval about x_0 ($x_0 \in V$)

for $x < x_0$ ($(x - x_0) < 0$)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) > 0 \Rightarrow f(x) - f(x_0) < 0 \Rightarrow f(x) < f(x_0)$$

for $x > x_0$ ($(x - x_0) > 0$)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) > 0 \Rightarrow f(x) - f(x_0) > 0 \Rightarrow f(x) > f(x_0)$$

$\therefore$ f is increasing function at x_0

Case (2): If $f'(x_0) < 0$, $\exists V$ open interval about x_0 ($x_0 \in V$)

for $x < x_0$ ($(x - x_0) < 0$)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) < 0 \Rightarrow f(x) - f(x_0) > 0 \Rightarrow f(x) > f(x_0)$$

for $x > x_0$ ($(x - x_0) > 0$)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) < 0 \Rightarrow f(x) - f(x_0) < 0 \Rightarrow f(x) < f(x_0)$$

$\therefore$ f is decreasing function at x_0

$\therefore$ f is 1-1 on V.

Remark: The converse of the above theorem may not be true, for example: $f: (-1, 1) \to \mathbb{R}$, $f(x) = x^3$ is increasing at $x_0 \in (-1, 1)$ and $f'(x) = 3x^2$ but if $x_0 = 0 \in (-1,$
$f'(0) = 0$.