Definitions:

Let $f: S \to \mathbb{R}$, $S \subseteq \mathbb{R}$, be a real-valued function, $x_0 \in S, y_0 \in S$, then:

(1) $x_0 \in S$ is called local maximum point if $\exists\, V$ open interval about x_0 s.t. $f(x) \leq f(x_0)$, $\forall x \in S \cap V$.

(2) $y_0 \in S$ is called locall minimum point if $\exists\, U$ open interval about y_0 s.t. $f(y_0) \leq f(y)$, $\forall y \in S \cap V$.

Proposition:

Let $f: I \to \mathbb{R}$ be a differentiable function at $x_0 \in I$, (I any open interval), if x_0 is local maximum point or local minimum point then $f'(x_0)=0$.

Proof:

$$\text{Let } f'(x_0) \neq 0 \Rightarrow f'(x_0) = \begin{cases} f'(x_0) > 0 \Rightarrow f \text{ is increasing function at } x_0 \\ \quad \text{i.e. } x_0 \text{ is not local maximum point} \\ f'(x_0) < 0 \Rightarrow f \text{ is decreasing function at } x_0 \\ \quad \text{i.e. } x_0 \text{ is not local minimum point} \end{cases}$$

$\therefore f'(x_0) = 0$.

Remark: The converse of the above proposition may not be true

for example: $f: (-1,1) \to \mathbb{R}$, $f(x) = x^3$, $\forall x \in (-1,1)$, $f'(x) = 3x^2$, $f'(0) = 0$ and f is increasing for all $x \in \mathbb{R}$, but $x_0 = 0$ is not (local maximum point or local minimum point).

Theorem: Roll's Theorem

Let f be a continuous function on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$ then $\exists\, c \in (a,b)$ such that $f'(c) = 0$.

Proof:

Case 1: If $f(x) = c$ (constant function), $\forall x \in (a,b)$ then $f'(x) = 0$.

Case 2: If $f(x)$ is not constant function, $\forall x \in (a,b)$

$\because f$ is continuous on $[a,b]$ and $[a,b]$ is compact set

$\therefore f$ has minimum point and maximum point say x_0, y_0

i.e. $\exists\, x_0, y_0 \in [a,b]$ s.t. $f(x_0) \leq f(x) \leq f(y_0)$

If $f(x_0) = f(y_0) \Rightarrow f$ is constant C!

$\therefore f(x_0) \neq f(y_0) \Rightarrow x_0 \neq y_0$

If $x_0 = a$ or b and $y_0 = a$ or b

$\Rightarrow f(x_0) = f(a) = f(b) = f(y_0) \Rightarrow f$ is constant C!

$\therefore$ at least one of (x_0 or y_0) $\neq$ (a or b)

i.e. at least one of x_0 or y_0 belongs to $[a,b]$

$\because f$ is differentiable on (a,b)

$\therefore f$ has (local minimum point or local maximum point) at x_0 or y_0 and $f'(x_0) = 0$ or $f'(y_0) = 0$

Put $x_0 = c$ or $y_0 = c$.

Theorem: Roll's Theorem

Let f be a continuous function on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$ then $\exists\ c \in (a,b)$ such that $f'(c)=0$.

Proof:

Case 1: If $f(x)=C$ (Constant function), $\forall x \in (a,b)$ then $f'(x)=0$.

Case 2: If $f(x)$ is not constant function, $\forall x \in (a,b)$

$\because f$ is continuous on $[a,b]$ and $[a,b]$ is compact set

$\therefore f$ has minimum point and maximum point say x_0, y_0

i.e. $\exists\ x_0, y_0 \in [a,b]$ s.t. $f(x_0) \leq f(x) \leq f(y_0)$

If $f(x_0)=f(y_0) \Rightarrow f$ is constant C!

$\therefore f(x_0) \neq f(y_0) \Rightarrow x_0 \neq y_0$

If $x_0 = a$ or b and $y_0 = a$ or b

$\Rightarrow f(x_0)=f(a)=f(b)=f(y_0) \Rightarrow f$ is constant C!

$\therefore$ at least one of $(x_0$ or $y_0) \neq (a$ or $b)$

i.e. at least one of x_0 or y_0 belongs to $[a,b]$

$\because f$ is differentiable on (a,b)

$\therefore f$ has (local minimum point or local maximum point) at x_0 or y_0 and $f'(x_0)=0$ or $f'(y_0)=0$

Put $x_0 = c$ or $y_0 = c$.

Theorem (Mean Value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b) then $\exists\ c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof:

Define $g: [a,b] \to \mathbb{R}$ as follows

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$\because f$ is continuous on $[a,b]$ and differentiable on (a,b)

$$g(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$$

$\therefore g(a) = g(b)$

By (Roll's Th.), $\exists\ c \in (a,b)$ s.t. $g'(c)=0$

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$\therefore f'(c) = \frac{f(b)-f(c)}{b-a}.$$