Riemann Integration

Definition:

Let $f:[a,b]\to\mathbb{R}$ be a bounded real-valued function defined on $[a,b]$, $J=[a,b]$, $|J|=b-a$ (length of J)

(i) Let $\Pi=\{a=x_0<x_1<x_2<\dots<x_n=b\}$ be a portion on $[a,b]$ called (Riemann partition)

$J_i=[x_{i-1},x_i]$ subintervals on $[a,b]$, $1\le i\le n$

$|J_i|=x_i-x_{i-1}$ length of subintervals, $1\le i\le n$

Define:

$M=\sup\{f(x):x\in J\}$, $m=\inf\{f(x):x\in J\}$

$M_i=\sup\{f(x):x\in J_i\}$, $m_i=\inf\{f(x):x\in J_i\}$

Notice that $m\le m_i\le M_i\le M$, $1\le i\le n$

Define:

$\overline{R}(f,\Pi)=\sum_{i=1}^{n}M_i\,|J_i|$ is called (Riemann Upper Sum)

$\underline{R}(f,\Pi)=\sum_{i=1}^{n}m_i\,|J_i|$ is called (Riemann Lower Sum)

Notice that $\underline{R}(f,\Pi)\le\overline{R}(f,\Pi)$ … (1)

Define:

$\overline{R}(f)=\{\overline{R}(f,\Pi):\Pi$ is any partition on $[a,b]\}$

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Notice that $\underline{R}(f)\le\overline{R}(f)$ … (2)

Since $\underline{R}(f)\le\overline{R}(f)$, so $\underline{R}(f)$ is bounded above by $\overline{R}(f)$ and $\overline{R}(f)$ is bounded below by $\underline{R}(f)$

Since, $\emptyset\ne\underline{R}(f)\subseteq\mathbb{R}$ and $\emptyset\ne\overline{R}(f)\subseteq\mathbb{R}$

and $\mathbb{R}$ is complete

So, $\underline{R}(f)$ has least upper bound

and $\overline{R}(f)$ has greatest lower bound

Define:

$\underline{\int}f=\sup(\underline{R}(f))$ is called (Riemann Lower Integral)

$\overline{\int}f=\inf(\overline{R}(f))$ is called (Riemann Upper Integral)

If $\overline{\int}f=\underline{\int}f$ then we say that f is Riemann integrable function on $[a,b]$.

and $\overline{\int}_{[a,b]}f=\underline{\int}_{[a,b]}f=\int_{[a,b]}f$.

Definition: (Refinement Partition)

Let $\pi = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition on $[a,b]$
and $\pi' = \{a = y_0 < y_1 < \dots < y_m = b\}$ be another partition on $[a,b]$
if $\pi \subseteq \pi'$ then we say that π' is a refinement for π.

Example:

Let $J = [1,2]$
$\pi = \{1, 1.5, 2\}$ be a partition and $\pi' = \{1, 1.25, 1.5, 1.75, 2\}$
be another partition on $[a,b]$
Since $\pi \subseteq \pi'$, So, π' is a refinement for π.

Remarks:

1. If π' is a refinement for π then
   $\underline{R}(f,\pi') \geq \underline{R}(f,\pi)$ and $\overline{R}(f,\pi') \leq \overline{R}(f,\pi)$.

2. For any partitions π, π' on $[a,b]$ we have $\underline{R}(f,\pi) \leq \overline{R}(f,\pi')$

Examples:

(1) Let $f : [a,b] \to \mathbb{R}$, $f(x) = c$, $\forall x \in [a,b]$. Check that f is Riemann integrable on $[a,b]$.

Answer:

Let $\pi = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition on $[a,b]$
$J_i = [x_{i-1}, x_i]$, $|J_i| = x_i - x_{i-1}$, $1 \leq i \leq n$
$M_i = \sup\{f(x) : x \in J_i\}$, $m_i = \inf\{f(x) : x \in J_i\}$

$$\begin{aligned}
\overline{R}(f,\pi) &= \sum_{i=1}^{n} M_i |J_i| \\
&= M_1 |J_1| + M_2 |J_2| + \dots + M_n |J_n| \\
&= c\,|J_1| + c\,|J_2| + \dots + c\,|J_n| \\
&= c\,(|J_1| + |J_2| + \dots + |J_n|) = c(b-a)
\end{aligned}$$

$$\begin{aligned}
\underline{R}(f,\pi) &= \sum_{i=1}^{n} m_i |J_i| \\
&= m_1 |J_1| + m_2 |J_2| + \dots + m_n |J_n| \\
&= c\,|J_1| + c\,|J_2| + \dots + c\,|J_n| \\
&= c\,(|J_1| + |J_2| + \dots + |J_n|) = c(b-a)
\end{aligned}$$

$\overline{R}(f) = \{\overline{R}(f,\pi) : \pi$ is any partition on $[a,b]\} = c(b-a)$
$\underline{R}(f) = \{\underline{R}(f,\pi) : \pi$ is any partition on $[a,b]\} = c(b-a)$

$\underline{\int} f = \sup(\underline{R}(f)) = c(b-a)$, $\overline{\int} f = \inf(\overline{R}(f)) = c(b-a)$

$\because \underline{\int} f = \overline{\int} f \Rightarrow f$ is Riemann integrable on $[a,b]$

and $\int_{[a,b]} f = c(b-a)$.

(2) Let $f:[0,2]\to\mathbb{R}$, $f(x)=3x$, $\forall x\in[0,2]$. Is f Riemann integrable on $[0,2]$?

Answer: Let $\Pi_n=\{a=0<\frac{2}{n}<\frac{4}{n}<\dots<\frac{2n}{n}=b\}$ be a partition (uniform partition) on $[0,2]$

$M_i=\sup\{f(x):x\in J_i\}=\sup\{3x:x\in J_i\}$

$m_i=\inf\{f(x):x\in J_i\}=\inf\{3x:x\in J_i\}$

$$\overline{R}(f,\Pi_n)=\sum_{i=1}^{n} M_i|J_i|$$
$$=M_1|J_1|+M_2|J_2|+\dots+M_n|J_n|$$
$$=(3\cdot\tfrac{2}{n})\cdot\tfrac{2}{n}+(3\cdot\tfrac{4}{n})\cdot\tfrac{2}{n}+\dots+(3\cdot\tfrac{2n}{n})\cdot\tfrac{2}{n}$$
$$=\frac{6}{n}\cdot\frac{2}{n}(1+2+\dots+n)$$
$$=\frac{12}{n^2}\cdot\frac{n(n+1)}{2}=6+\frac{6}{n}$$

$$\underline{R}(f,\Pi_n)=\sum_{i=1}^{n} m_i|J_i|$$
$$=m_1|J_1|+m_2|J_2|+\dots+m_n|J_n|$$
$$=(3\cdot0)\cdot\tfrac{2}{n}+(3\cdot\tfrac{2}{n})\cdot\tfrac{2}{n}+\dots+(3\cdot\tfrac{2(n-1)}{n})\cdot\tfrac{2}{n}$$
$$=6-\frac{6}{n}$$

$\overline{R}(f)=\{\overline{R}(f,\Pi):\Pi \text{ is any partition on } [0,2]\}=\{6+\frac{6}{n}:n\in\mathbb{N}\}$

$\underline{R}(f)=\{\underline{R}(f,\Pi):\Pi \text{ is any partition on } [0,2]\}=\{6-\frac{6}{n}:n\in\mathbb{N}\}$

$\underline{\int}f=\sup(\underline{R}(f))=6$ and $\overline{\int}f=\inf(\overline{R}(f))=6$

$\because \underline{\int}f=\overline{\int}f=6\Rightarrow\int_0^2 f=6$ i.e. f is Riemann integrable on $[0,2]$.

(3) Let $f:[a,b]\to\mathbb{R}$, $f(x)=\begin{cases}3, & x\in[a,b]\cap\mathbb{Q}\\ 7, & x\in[a,b]\cap\mathbb{Q}^c\end{cases}$

Is f Riemann integrable on $[a,b]$?

Answer: Let $\Pi=\{a=x_0<x_1<\dots<x_n=b\}$ be a partition on $[a,b]$

$J_i=[x_{i-1},x_i]$, $|J_i|=x_i-x_{i-1}$, $1\le i\le n$

$M_i=\sup\{f(x):x\in J_i\}$, $m_i=\inf\{f(x):x\in J_i\}$

$$\overline{R}(f,\Pi)=\sum_{i=1}^{n} M_i|J_i|$$
$$=M_1|J_1|+M_2|J_2|+\dots+M_n|J_n|$$
$$=7|J_1|+7|J_2|+\dots+7|J_n|$$
$$=7(|J_1|+|J_2|+\dots+|J_n|)=7(b-a)$$

$$\underline{R}(f,\Pi)=\sum_{i=1}^{n} m_i|J_i|$$
$$=m_1|J_1|+m_2|J_2|+\dots+m_n|J_n|$$
$$=3(|J_1|+|J_2|+\dots+|J_n|)=3(b-a)$$

$\overline{R}(f)=\{\overline{R}(f,\Pi):\Pi \text{ is any partition on } [a,b]\}=\{7(b-a)\}$

$\underline{R}(f)=\{\underline{R}(f,\Pi):\Pi \text{ is any partition on } [a,b]\}=\{3(b-a)\}$

$\underline{\int}f=\sup(\underline{R}(f))=3(b-a)$ and $\overline{\int}f=\inf(\overline{R}(f))=7(b-a)$

$\because \underline{\int}f\neq\overline{\int}f\Rightarrow\therefore f$ is not Riemann integrable on $[a,b]$

H.W.

Let $f:[a,b]\to\mathbb{R}$, $f(x)=\begin{cases}1, & x \text{ is rational in } [a,b]\\ -1, & x \text{ is irrational in } [a,b]\end{cases}$

Check that f is Riemann integrable on $[a,b]$?