Differentiable Functions

Definition:

Let $f: I \to \mathbb{R}$ be a real-valued function defined on open interval I ($I=(a,b)$), $x_0 \in I$, f is differentiable at x_0 if: $\exists\, \ell \in \mathbb{R}$ such that

$\forall\, \varepsilon>0, \exists\, \delta(x_0,\varepsilon)>0$ s.t. $\left| \frac{f(x)-f(x_0)}{x-x_0} - \ell \right| < \varepsilon$ whenever $|x-x_0|<\delta(x_0,\varepsilon)$

i.e.

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell = f'(x_0)$$

or f is differentiable at x_0 iff $\forall\, (x_n)$ sequence in I,
$\exists\, \ell = f'(x_0) \in \mathbb{R}$ s.t. $\frac{f(x_n)-f(x_0)}{x_n-x_0} \to \ell$ as $x_n \to x_0$,
$x_n \neq x_0, \forall n \in \mathbb{N}$.

If f is differentiable at each $x_0 \in (a,b)$ then we say that f is differentiable on $I=(a,b)$.

Examples: Is the following functions differentiable at $x_0 \in I=(a,b)$?

1. $f:(a,b)\to\mathbb{R}$, $f(x)=c$, $\forall x \in (a,b)$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{c-c}{x-x_0} = 0 = f'(x_0) = \ell$$

$\therefore f$ is differentiable at x_0 and $f'(x_0)=0$.

2. $f: I\to\mathbb{R}$, $f(x)=x$, $\forall x\in I$, $I=(a,b)$

Let (x_n) sequence in (a,b), $x_n \to x_0$, $x_n \neq x_0$, $\forall n \in \mathbb{N}$, $x_0 \in (a,b)$

$$\frac{f(x_n)-f(x_0)}{x_n-x_0} = \frac{x-x_0}{x-x_0} = 1 = f'(x_0) = \ell$$

$\therefore f$ is differentiable at x_0 and $f'(x_0)=1$.

3. $f:(a,b)\to\mathbb{R}$, $f(x)=x^2$, $\forall x\in(a,b)$.

Let (x_n) sequence in (a,b), $x_n\to x_0$, $x_n\neq x_0$, $\forall n\in\mathbb{N}$, $x_0\in(a,b)$

$$\frac{f(x_n)-f(x_0)}{x_n-x_0} = \frac{x_n^2-x_0^2}{x_n-x_0} = \frac{(x_n-x_0)(x_n+x_0)}{x_n-x_0} = x_n+x_0 = 2x_0 = f'(x_0)$$

(since $x_n \to x_0$)

$\therefore f$ is differentiable at x_0.

H.W.

4. $f, g, h: (a,b)\to\mathbb{R}$ defined as follows:

$f(x) = ax+b$, $\forall x\in(a,b)$, $a,b\in\mathbb{R}$

$g(x) = ax^2+bx+c$, $\forall x\in(a,b)$, $a,b,c\in\mathbb{R}$

$$h(x) = \begin{cases} x+2, & x\geq 0 \\ x^2+2, & x<0 \end{cases}$$

Is f, g differentiable at x_0?
Is h differentiable at 0?

5. We can concluded that if $f(x)=x^n$, $n\in\mathbb{N}$, then f is differentiable at x_0 and $f'(x_0)=n x_0^{n-1}$.

Theorem: Let $f: I \to \mathbb{R}$ be a real-valued function, (I any open interval in $\mathbb{R}$), $x_0 \in I$, if f is differentiable at x_0 then f has at most one derivative at x_0.

proof.

Suppose f has two derivatives l_1, l_2 s.t. $l_1 \neq l_2$,

Let $\varepsilon = |l_1 - l_2| > 0$,

$\exists\, \delta_1(x_0, \varepsilon) > 0$ s.t. $\left| \frac{f(x) - f(x_0)}{x - x_0} - l_1 \right| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta_1$

$\exists\, \delta_2(x_0, \varepsilon) > 0$ s.t. $\left| \frac{f(x) - f(x_0)}{x - x_0} - l_2 \right| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta_2$

choose $\delta = \min[\delta_1, \delta_2]$

$$
\begin{aligned}
0 < |l_1 - l_2| &= \left| \frac{f(x) - f(x_0)}{x - x_0} - l_2 - \left(\frac{f(x) - f(x_0)}{x - x_0} - l_1 \right) \right| \\
&\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - l_2 \right| + \left| \frac{f(x) - f(x_0)}{x - x_0} - l_1 \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = |l_1 - l_2| \quad C!
\end{aligned}
$$

$\therefore\ l_1 = l_2 \Rightarrow \therefore f$ has one derivative at x_0.

Theorem: Let $f: (a,b) \to \mathbb{R}$ be a real-valued function, $x_0 \in (a,b)$, if f is differentiable at x_0 then f is continuous at x_0.

proof: Since f is differentiable at x_0

we have $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$ exists

to prove that f is continuous at x_0

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right] = f'(x_0) \cdot 0 = 0$$

$\therefore \lim\limits_{x \to x_0} (f(x) - f(x_0)) = 0$ i.e. $\lim\limits_{x \to x_0} f(x) = f(x_0)$

$\therefore f$ is continuous at x_0.

Remark: The converse may not be true, for example:

Let $f: (-a, a) \to \mathbb{R}$, $f(x) = |x|$, $\forall x \in (-a, a)$, f is continuous on $(-a, a)$ but f is not differentiable at $0 \in (-a, a)$

Since, $\exists\, (x_n)$ sequence in $(-a, a)$, $l \in \mathbb{R}$ s.t.

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \not\to l \text{ as } x_n \to x_0,\ x_n \neq x_0,\ \forall n$$

$x_n = \frac{1}{n}, \frac{-1}{n}$ and $\frac{1}{n} \neq 0$, $\frac{-1}{n} \neq 0$, $\frac{1}{n} \to 0$, $\frac{-1}{n} \to 0$ (A.P.)

$$\lim_{x \to 0} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0} = \lim_{x \to 0} \frac{\frac{1}{n} - 0}{\frac{1}{n} - 0} = 1$$

$$\lim_{x \to 0} \frac{f(\frac{-1}{n}) - f(0)}{\frac{-1}{n} - 0} = \lim_{x \to 0} \frac{\frac{1}{n} - 0}{-\frac{1}{n} - 0} = 1$$

the limit does not exists at $0 \in (-a, a)$

$\therefore f = |x|$ is not differentiable at 0.