"Derivatives"

Def: The function f' defined by the formula

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

is called the derivative of f with respect to x. when the limit exist, f is said to be differentiable.

Remark: The derivative of the function f (or y) w.r.t. x is denoted by

$$f'(x) = \frac{d f(x)}{dx}, \quad y' = \frac{dy}{dx}$$

Differentiation Rules: Let f, g be a differentiable functions at x, and c be any constant, then:

1. $\frac{dx}{dx} = 1$
2. $\frac{d}{dx} c = 0$
3. $\frac{d}{dx}(x^n) = n x^{n-1}$
4. $\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot f'(x)$.
5. $\frac{d}{dx}[c f(x)] = c f'(x)$
6. $\frac{d}{dx}[f(x) \mp g(x)] = f'(x) \mp g'(x)$
7. $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
8. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$.

Examples: Find $\frac{dy}{dx}$ for the following functions.

① $y = 2x^3 + \sqrt{5x^2+2}$

Sol:

$$y' = 6x^2 + \frac{1}{2}(5x^2+2)^{-\frac{1}{2}} \cdot 10x$$

$$= 6x^2 + \frac{10x}{2\sqrt{5x^2+2}}$$

$$= 6x^2 + \frac{5x}{\sqrt{5x^2+2}}$$

② $y = (4x^2-1)(7x^3+x)$

Sol:

$$y' = (4x^2-1) \cdot (21x^2+1) + (7x^3+x) \cdot (8x)$$

$$= 140x^4 - 9x^2 - 1$$

③ $y = \frac{x^2-1}{x^4+1}$

Sol:

$$y' = \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{2x^5 + 2x - 4x^5 + 4x^3}{(x^4+1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2} = \frac{-2x(x^4 - 4x^2 - 1)}{(x^4+1)^2}$$

④ $y = \left(3x^5 + 2x + \frac{1}{x}\right)^3$

Sol:

$$y' = 3\left(3x^5 + 2x + \frac{1}{x}\right)^2 \cdot \left(15x^4 + 2 - \frac{1}{x^2}\right).$$

"Higher Derivatives"

1. $y' = \frac{dy}{dx}$ first derivative
2. $y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ second derivative
3. $y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ third derivative

In general to find n-derivative

$$y^{(n)} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}.$$

Examples:

① Let $y = 3x^4 - 2x^3 + x^2 - 4x + 2$. Find $y^{(5)}$ (or $\frac{d^5y}{dx^5}$).

Sol:

$y' = 12x^3 - 6x^2 + 2x - 4$
$y'' = 36x^2 - 12x + 2$
$y''' = 72x - 12$
$y^{(4)} = 72$
$y^{(5)} = 0$

② Show that $y = x^3 + 3x + 1$ satisfies the equation
$y''' + xy'' - 2y' = 0$.

Sol.

$y = x^3 + 3x + 1$
$y' = 3x^2 + 3$
$y'' = 6x$
$y''' = 6$

نعوض عن y و y' و y'' و y''' نحصل على

$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3)$
$= 6 + 6x^2 - 6x^2 - 6 = 0.$

"Exponential Functions"

الدوال الرئيسية

Def.(1): A function of the form $y = a^{u(x)}$, $0 < a \neq 1$ is called an exponential function with bassis a.

Def.(2): (The Natural Exponential Function)

If $a = e$ in Def (1) $\Rightarrow y = e^{u(x)}$ is called natural exponential function.

من الدوال الرئيسية المهمة هي الدالة e وتسمى الدالة الرئيسية الطبيعية
وهي قيمة ثابتة $e = 2.7182828 \simeq 2.7$

Def.(3): (Logarithum Function) الدالة اللوغاريتمية

A function of the form $y = \log_a u(x)$ is called logarithum function of $u(x)$ with bassis a where $0 < a \neq 1$.

Def.(4): (The Natural Logarithum Function)

If $a = e$ in Def.(3) $\Rightarrow y = \log_e u(x) = \ln u(x)$ is called natural logarithum function.

Remarks:

1. $\ln e = 1$
2. $\ln e^{u(x)} = u(x)$
3. $e^{\ln u(x)} = u(x)$
4. $\ln 1 = 0$
5. $e^0 = 1$

6. $\log_a (bc) = \log_a b + \log_a c$
7. $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$
8. $\log_a \left(\frac{1}{c}\right) = -\log_a c$
9. $\log_a b^n = n \log_a b$
10. $\log_a 1 = 0$
11. $\log_a a = 1$

ملاحظة : عندما لا يكتب الاساس a في Log(u) فذلك يعني ان $a = 10$.

"Derivative of Exponential and Logarithm Functions"

1. $\frac{d}{dx}\left[a^{u(x)}\right] = a^{u(x)} \cdot \ln a \cdot \frac{du}{dx}$

2. If $a = e$

$$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)} \cdot \frac{du}{dx} \quad (\text{since } \ln e = 1)$$

3. $\frac{d}{dx}\left[\ln u(x)\right] = \frac{1}{u(x)} \cdot \frac{du}{dx}$, (i.e. $\frac{u'(x)}{u(x)}$).

Examples: Find $\frac{dy}{dx}$, ($= y'$).

1. $y = 5^{x^2} \Rightarrow y' = 5^{x^2} \cdot \ln 5 \cdot 2x = 2x\, 5^{x^2} \ln 5$.
2. $y = e^{4x^2} \Rightarrow y' = e^{4x^2} \cdot 8x = 8x\, e^{4x^2}$.
3. $y = \ln 2x^3 \Rightarrow y' = \frac{1}{2x^3} \cdot 6x^2 = \frac{3}{x}$.
4. $y = e^{2x} + \ln(5x^2 + 7x + 1)$

$$y' = e^{2x} \cdot 2 + \frac{1}{5x^2 + 7x + 1} \cdot (10x + 7)$$

$$= 2e^{2x} + \frac{10x + 7}{5x^2 + 7x + 1}.$$

H.W.: $y = e^{3x^3} \ln(x^4 + 4)$.

"Trigonometric Functions"

الدوال المثلثية

① $y = \sin\theta$ ② $y = \cos\theta$ ③ $y = \tan\theta$

④ $y = \sec\theta$ ⑤ $y = \csc\theta$ ⑥ $y = \cot\theta$.

Some Relation:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
2. $\sec\theta = \frac{1}{\cos\theta}$
3. $\csc\theta = \frac{1}{\sin\theta}$
4. $\sin^2\theta + \cos^2\theta = 1$
5. $1 + \cot^2\theta = \csc^2\theta$ (by devided (4) by $\sin^2\theta$).
6. $\tan^2\theta + 1 = \sec^2\theta$
7. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
8. $\sin(A - B) = \sin A \cos B - \sin B \cos A$
9. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
10. $\sin(A + B) = \sin A \cos B + \sin B \cos A$
11. $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
12. $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

"Derivatives of Trigonometric Function"

Let $\theta = \theta(x)$, then,

1. $\frac{d}{dx}(\sin\theta) = \cos\theta \cdot \frac{d\theta}{dx}$
2. $\frac{d}{dx}(\cos\theta) = -\sin\theta \cdot \frac{d\theta}{dx}$
3. $\frac{d}{dx}(\tan\theta) = \sec^2\theta \cdot \frac{d\theta}{dx}$
4. $\frac{d}{dx}(\cot\theta) = -\csc^2\theta \cdot \frac{d\theta}{dx}$
5. $\frac{d}{dx}(\sec\theta) = \sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dx}$
6. $\frac{d}{dx}(\csc\theta) = -\csc\theta \cdot \cot\theta \cdot \frac{d\theta}{dx}$

Examples: Find $\frac{dy}{dx}$ $(= y')$.

1. $y = 2\cos x - 3\sin(x^2+1)$

$y' = 2(-\sin x) - 3\cos(x^2+1) \cdot 2x$
$= -2\sin x - 6x\cos(x^2+1)$.

2. $y = x^3\tan(2x)$

$y' = x^3 \cdot \sec^2(2x) \cdot 2 + \tan(2x) \cdot 3x^2$
$= 2x^3\sec^2(2x) + 3x^2\tan(2x)$.

3. $y = \ln(\sin(3x))$

$y' = \frac{1}{\sin(3x)} \cdot \cos(3x) \cdot 3 = 3\,\frac{\cos(3x)}{\sin(3x)} = 3\cot(3x)$

4. If $y = x^{x\sin(x)}$. Find $\frac{dy}{dx}$ $(= y')$.

sol.

نطبق الدالة ln على الطرفين

$\ln(y) = \ln(x^{x\sin(x)})$

$\ln(y) = (x\sin(x))\ln(x)$

نأخذ المشتقة للطرفين بالنسبة إلى x

$\frac{d}{dx}\ln(y) = \frac{d}{dx}[x\sin(x)\ln(x)]$

$\frac{1}{y}\cdot y' = x\sin(x)\cdot\frac{1}{x} + \ln(x)\cdot[x\cos(x) + \sin(x)]$

$\Rightarrow y' = \sin(x) + x\ln(x)\cos(x) + \ln(x)\sin(x)]\cdot y.$

5. $y = \sqrt{\cos(x+1)}$

$\Rightarrow y = (\cos(x+1))^{\frac{1}{2}}$

$y' = \frac{1}{2}(\cos(x+1))^{-\frac{1}{2}}\cdot(-\sin(x+1))$

$= \frac{-\sin(x+1)}{2\sqrt{\cos(x+1)}}.$

H.W. Find $\frac{dy}{dx}$ for the following.

(1) $y = e^{x^5+2}\cot(3x) + \ln(3x)$, (3) $y = \frac{e^x}{\ln x}$

(2) $y = \sin(\sqrt{1+\cos x})$, (4) $y = x^2\cos(e^x)$.

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Example: Solve (Find the value of x).

$2\ln\sqrt{x+1} + \ln(x-3) = \ln 5$

sol.

$2\ln(x+1)^{\frac{1}{2}} + \ln(x-3) = \ln 5$

$\cancel{2}\cdot\frac{1}{\cancel{2}}\ln(x+1) + \ln(x-3) = \ln 5$

$\ln[(x+1)\cdot(x-3)] = \ln 5$

$\ln(x^2 - 2x - 3) = \ln 5$

نطبق الدالة e على الطرفين

$e^{\ln(x^2-2x-3)} = e^{\ln 5}$

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$x^2 - 2x - 3 = 5$ (since $e^{\ln u(x)} = u(x)$)

$\Rightarrow x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

either $x - 4 = 0 \Rightarrow x = 4$

or $x + 2 = 0 \Rightarrow x = -2$ خطأ

$\therefore \boxed{x = 4}$