CHAPTER ONE

MATHEMATICAL LOGIC

المنطق الرياضياتي

علم المنطق هو أحد فروع علم الرياضيات الصرفة وهو علم حديث نسبياً، وقد أخذت أهميته تتزايد يوماً بعد يوم. يفهم من إسم هذا العلم أنه يشارك اللغات في وظائفها ومدلولاتها وتعبيراتها، فعلم المنطق يرتكز على مبادىء واضحة متفق عليها عالمياً، وله رموز خاصة به، ومن الجدير بالذكر أن كل علم أو بالاحرى كل فرع من فروع المعرفة له ألفاظ ومصطلحات خاصة به، إلا ان هذه الالفاظ ربما لا تستخدم في حديثنا اليومي، وقد يستخدم بعضها بمعنى مقارب لما تعنيه في حديثنا اليومي، وفي حالات يختلف معناها تماماً عن مقصودنا وذلك ربما يرجع الى عدم دقة التعبير عندنا وليس معناه قصوراً في اللغة المستخدمة في التعبير. ولما كان هذا الابهام غير مرغوب فيه وبخاصة في الرياضيات، فلم يترك الامر للإجتهاد في المنطق الرياضياتي، بل اتفق على رموز و أدوات لربط الجمل واعطيت معاني محددة تماماً لاتقبل اللبس والغموض وهذا يقودنا الى القول بان المنطق الرياضياتي لغة علمية متفق عليها بين الرياضياتيين، ولاغنى للرياضيات عن المنطق. فالرياضيات تحتاج الى تفكير منطقي ولا يكون برهان صيغة او مبرهنة رياضياتية مثلاً سهلاً ومقبولاً ما لم يستند في خطواته على سلسلة من الافكار مرتبط بعضها ببعض.

العبارات (التقارير): Statements

أن الجمل في اللغة العربية منها ماهي فعلية ومنها ماهي إسمية ومنها ماهي إستفهامية أو طلبية...الخ. وفي المنطق الرياضياتي نقسم الجمل الى قسمين هما:

(أ) جمل خبرية وهي التي تحمل إلينا خبراً ما.

(ب) جمل غير خبرية (إنشائية) وهي التي لا تحمل خبراً معيناً.

<u>Definition (1.1):</u>

A ***statement*** is a declarative sentence which is either (True: T) or (False: F), but not both. We use the letters ***p, q, r, s, …,*** **ect** to denote a proposition.

كل جملة تحمل خبراً ما ويمكن الحكم بأنها إما صائبة (True) وإما خاطئة (False)، ولا تكون صائبة وخاطئة في آن واحد تسمى **عبارة (او تقرير)**.

<u>Definition (1.2):</u>

A statement which has truth news sentence is called ***true statement*** and a statement which has false news sentence is called ***false statement***.

كل جملة خبرية صائبة تسمى **عبارة صائبة** وكل جملة خبرية خاطئة تسمى **عبارة خاطئة**.

Example (1.3):

(1) The sun is rises from the east. *News sentence (truth statement).*
(2) Baghdad is the capital of Iraq. *News sentence (truth statement).*
(3) 17 < 14. *News sentence (false statement).*
(4) Wow, this grove is very beautiful. *Sentence is not news (Wonder).*
(5) Please Fawaz, be wishful doing a good. *News sentence.*
(6) $3 + x = 7$, where x is an integer. *News sentence.*
(We can't judge it as true or false unless we know the value of the variable x). Statements from this type called (***Propositional Functions***).

Negation of Statements:

If we want to negation the statement "*it's raining today*", we will say "*it doesn't rain today*". If the statement that we want to negation it is true, then the negation statement will be false. So, contrariwise.

نفي العبارة (التقرير): إذا أردنا أن ننفي العبارة "السماء تمطر اليوم" فإننا نقول "السماء لا تمطر اليوم". وإذا كانت العبارة المراد نفيها صائبة فإن نفيها تكون عبارة خاطئة والعكس بالعكس.

Example (1.4):

(1) $2+3=8$. *false statement.* ***Negation:*** $2+3\neq8$. *true statement.*
(2) Baghdad is a capital of Iraq. *true statement.* ***Negation***: Baghdad is not a capital of Iraq. *false statement.*

Often, we symbolized the statement as the letter of the alphabet to ease. In Example (1.4), if we symbol for the statement contained in paragraph (1) under the symbol "***p***", we will symbolize to the negation of this statement by "***~p***" (read negation of ***p*** or not ***p***). The two statements ***p*** and ***~p*** are impossible to be true or false at the same time. We will use the letter **T** to symbolize the word (True) and the letter **F** to symbolize the word (False). Then, we will generate a table which is called ***the right table*** (***truth table***) that describes the ***p*** and ***~p*** together as shown in the following table:

كثيراً ما نرمز لعبارة ما بحرف من حروف الهجاء للسهولة ففي المثال (1.4) إذا رمزنا للعبارة الواردة في الفقرة (1) بالرمز P فإننا نرمز لنفي هذه العبارة بالرمز P ~ (تقرأ نفي P) وحيث ان العبارتين P و P~ يستحيل ان يكونا صائبتين معاً أو خاطئتين معاً، فإننا لو جعلنا الحرف T يرمز لكلمة صائب (True) والحرف F يرمز لكلمة خاطىء (False)، لأمكننا تكوين جدول يدعى جدول الصواب (جدول الحقيقة) يصف P و P~ معاً كما هو موضح في الجدول الآتي:

p	$\sim p$
T	F
F	T

Table 1

ملاحظات:

(1) تسمى T و F بقيمتي الصواب (أو الحقيقة) ويستعاض عن كل منهما أحياناً بـ 1 و 0 على الترتيب.

(2) لاحظ أنه مهما كانت العبارة P فإنها إما أن تاخذ القيمة T أو القيمة F، أما قيمة صواب العبارة P ~ فيجب أن تخالف قيمة صواب P كما أشرنا الى ذلك آنفاً.

Definition (1.5):

A statement that carrying one news is called a ***simple statement*** (***primitive***), and if a statement carried two (or more than one) news then it is called ***compound statement***.

In other words, a proposition is said to be ***primitive statement***, if it cannot be divided into simpler proposition. And a proposition is called ***compound statement***, if it is compound of one or more primitive propositions using logical connective operators.

Example (1.6):

(1) Water freezes at zero degrees and boils at 100 degrees. (compound statement).

(2) Nawaf studies mathematics or geograph (compound statement).

(3) If $3 + 1 = 4$, then $6 + 7 = 13$. (compound statement).

(4) Equilateral triangle abc if and only if it was Equiangular. (compound statement).

Basic Logical Connective Operators:

There are some basic logical operators that connect simple propositions to produce composite proposition. These operators are:

1. Conjunction operator (and):

Let p and q are two primitive propositions. The conjunction of p and q is denoted by **"$p \wedge q$"** and read as "p and q". If both p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

Below is the truth table for the conjunction of two propositions:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2

Example (1.7):

Assuming that *p, q, s* and *t* are respectively the following statements:
2 + 2 = 4, the moon orbits around Mars, passing the Euphrates River in Iraq, 3 = 0.

We find that *p*, *r* true statements, while *q*, *t* false statements. Thus by reference to the Table (2) conclude that:

$p \wedge s$ is a true statement, but the $p \wedge q$, $p \wedge t$, $q\wedge t$, $q \wedge t$, $(p \wedge t) \wedge s$ are all false statements.

2. Disjunction operator (or):

Let *p* and *q* are two primitive propositions. The disjunction of *p* and *q* is denoted by "$p \vee q$" and read as "p or q". We say that "$p \vee q$" is true when p is true or q is true or both are true. If both p and q are false, then $p \vee q$ is false.

Below is the truth table for the disjunction of two propositions:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3

Example (1.8):

(1) 5 + 1 = 6 or 3×4 = 12. *correct statement.*

(2) 9 an even number or 9 an odd number . *correct statement.*

(3) Riyadh, the capital of Syria or Delhi, the capital of Algeria. *false statement.*

3. Conditional operator (If…then…):

Let *p* and *q* are two primitive propositions. The conditional statement "$p \to q$" is the proposition "if *p* then *q*". The conditional statement "$p \to q$" is false if *p* is true and *q* is false, otherwise "$p \to q$" is true.

Below is the truth table for the conditional operator of two propositions:

P	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4

Example (1.9):

(1) $5 + 7 = 12 \longrightarrow 2 + 6 = 8$ **T**
(2) $5 + 7 = 11 \longrightarrow 2 + 6 = 8$ **T**
(3) $5 + 7 = 11 \longrightarrow 2 + 6 \neq 8$ **T**
(4) $5 + 7 = 12 \longrightarrow 2 + 6 = 7$ **F**

4. Bi-conditional operator (if and only if):

Let p and q are two primitive propositions. The bi-conditional statement "$p \leftrightarrow q$" is the proposition "p if and only if q". The bi-conditional statement "$p \leftrightarrow q$" is true when p and q have the same true value, otherwise "$p \leftrightarrow q$" is false.

Below is the truth table for the bi-conditional operator of two propositions:

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5

For that we set true table in terms of tables (2) and (4) as follows:

A	B	$A \to B$	$B \to A$	$(A \to B) \wedge (B \to A)$	$A \longleftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Table 6

Notes from the table (5) that the phrase $A \leftrightarrow B$ are truth when they are two statements A and B true together or false together.

Example (1.10):

(1) $5 + 3 = 8 \leftrightarrow 5 \times 3 = 15$. T
(2) Iraq is located in Europe, $\leftrightarrow 5 + 3 = 8$. F
(3) $5\times3 = 15 \leftrightarrow$ Fatima man's name. F
(4) Sanaa, the capital of Russia's $\leftrightarrow$ sugar tastes bitter. T

Definition (1.11): Logical Equivalence

Two statements that have the same truth values are called logically equivalent. The notation $p \equiv q$ or $p= q$ denotes that p and q are logically equivalent.

Example (1.12):

(1) $(p \to q) \leftrightarrow (q \to p) \equiv$ (Table 6).
(2) $p \equiv p \wedge p \equiv p \vee p \equiv \sim (\sim p)$ as in the following table.

P	p	~p	$p \wedge p$	$p \vee p$	~(~p)
T	T	F	T	T	T
F	F	T	F	F	F

Table 7

Theorem (1.13): De Morgan's Laws

Let p and q are two statements, define the following logical equivalence:

(A) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
(B) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

Proof:
(A) Consider the following truth table for $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$

P	q	~p	~q	p∧ q	~(p ∧ q)	(~p) ∨ (~q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 8

From columns sixth and seventh we see the equally of the right values and thus was required.
(B) To prove it is required in the same way (A) (leave to the student).

Note:
We can prove the validity of paragraph (B) from Theorem (1.13) in another way as follows:

Negation the right end of the relationship (B), we find that:

$\sim[(\sim p) \wedge (\sim q)] \equiv \sim(\sim p) \vee \sim(\sim q)$ *According to paragraph (A) of this Theorem.*

$\equiv p \vee q$ *Table (7)*

$\sim[(\sim p) \wedge (\sim q)] \equiv p \vee q$ (*)

By the negation of the relationship (*) we get the required proved which

$\sim(\sim[(\sim p) \wedge (\sim q)] \equiv \sim(p \vee q)$

Therefore; $(\sim p) \wedge \sim q) \equiv \sim(p \vee q)$

Theorem (1.14):
If p and q are any two statements, then: $p \rightarrow q \equiv \sim(p \wedge \sim q)$.

Proof: According from Definition (1.11) is enough to create table (9), in which we see that the fifth and sixth columns are equal in truth values so desired proved.

p	Q	~ q	p ∧ ~ q	p → q	~(p ∧ ~ q)
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Table 9

<u>Definition (1.15):</u>

A compound statement that is all it's values are true is called a ***logically truth***. And it is ***logically false*** if all it's values are false.

<u>Example (1.16):</u>

(1) A statement $p \vee \sim p$. *logically truth.*

(2) A statement $p \wedge \sim p$. *logically false.*

Consider the following table;

p	**~ p**	**p ∨ ~ p**	**p ∧ ~p**
T	F	T	F
F	T	T	F

Table 10

<u>Note:</u>

Some statements may be not logically truth and not logically false, as in the phrases $p \to q$, $p \leftrightarrow q$, for example.

<u>Definition (1.17):</u>

A statement p lead to a statement q, and represented that by symbol $p \Rightarrow q$, if the statement $p \to q$ is logically truth. As sometimes we say that p is the introduction and q is the result.

<u>Example (1.18):</u>

1) For any statement p, then $p \Rightarrow p \vee \sim p$. Since, $p \to p \vee \sim p$ is a logically truth statement.
2) For any two statements p and q, then $p \Rightarrow p \vee q$. Since, the statement $p \to p \vee q$ is a logically truth statement.
3) For any two statements p and q, then $p \wedge q \Rightarrow p \vee q$. Since, the statement $p \wedge q \to p \vee q$ is a logically truth statement.

<u>Notes:</u> Let p and q are two statements;

1) If $p \Rightarrow q$, then from the truth table for the compound statement $p \to q$, we show that:
   (A) The statement q is truth whenever the statement p is truth.
   (B) The statement p is false whenever the statement q is false.
2) If $p \Rightarrow q$ we express it by saying that, if the statement p is true, then it is enough to lead that the statement q is true too).

3) By the symbol $p \not\Rightarrow q$ we mean that the statement p does not lead to the statement q.
4) $p \Rightarrow q$ does not have a correct table, since the symbol "$\Rightarrow$" is not logical connective operator between the two statements p and q.

Definition (1.19):

Both statements p and q are lead to the other, in other words, the statement p is lead to the statement q and the statement q is lead to the statement p, and symbolized $p \Leftrightarrow q$, if the statement $p \leftrightarrow q$ is logically truth.

The symbol "$\Leftrightarrow$" is not logical connective operator between the two statements p and q. Therefore, $p \Leftrightarrow q$ dose not have truth table. Sometimes we will express the symbol "$\Leftrightarrow$" by saying ***"the necessary and sufficient condition"***. It also means equivalent to the word. And sometimes it can be used instead of the symbol "$\equiv$" as illustrated by the following example.

Example (1.20):

Let p and q are any two statements, then $\sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$.

Solution: Following is the truth table for the statement $\sim(p \rightarrow q) \leftrightarrow p \wedge \sim q$;

p	**q**	**$\sim q$**	**$p \rightarrow q$**	**$p \wedge \sim q$**	**$\sim(p \rightarrow q)$**	**$\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$**
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Table 11

Note; from the above table the statement

$\sim(p \rightarrow q) \leftrightarrow p \wedge \sim q$ is logically truth as shown in the seventh column. And since the truth values in columns fifth and sixth in that table are equal, which is consistent with the definition of parity, that is mean $\sim(p \rightarrow q) \equiv p \wedge \sim q$. we would consider that the symbols $\Leftrightarrow$ or $\equiv$ have the same meaning.

Here it should be noted that if it is not considered $p \Leftrightarrow q$ accrued, we symbolized by the symbol $p \not\Leftrightarrow q$.

Example (1.21):

Verification the relationship between $x = 3 \Rightarrow x^2 = 9$ or not, taking advantage of the comments received after the Example (1.18).

Solution:

(A) ***The first method***: From our knowledge of mathematical. We know that if the statement $x = 3$ is true, then it is necessarily lead to that statement $x^2 = 9$ is true, also. Since it cannot be $x = 3$ while the $x^2 \neq 9$. Thus, $x = 3 \Rightarrow x^2 = 9$ verification.

(**B**) ***The second method:*** From our information also. We know that if the statement $x^2 = 9$ is false, then the statement $x = 3$ be false, also. Which means that, $x \neq 3$. Hence, $x = 3 \Rightarrow x^2 = 9$ is verification.

Note:

If we have one statement, then the number of possible truth values of that statement is two. And if we have two different statements, then the number of possible truth values of the two statements is four. And if we have three different statements, then the number of possible truth values is eight. That is lead that we can proof that, if we have n different statements, then the number of possible truth values equal to 2^n that $B(n) = 2^n; \quad n \in \mathbb{N}$.

ملاحظة:

إذا كانت لدينا عبارة واحدة فإن عدد قيم صوابها الممكنة إثنان، وإذا كانت لدينا عبارتان مختلفتان فإن عدد قيم صوابهما الممكنة أربع، وإذا كانت لدينا ثلاث عبارات مختلفة فإن عدد قيم صوابها الممكنة ثمان، هذا ويمكن البرهان أنه إذا كان لدينا n من العبارات المختلفة فإن عدد قيم صوابها الممكنة يساوي 2^n أي أن $B(n) = 2^n; \quad n \in \mathbb{N}$

Theorem (1.22):

Consider the statements p, q and r. Then;

(1) $p \wedge p \equiv p$ as well as $p \vee p \equiv p$.

(2) $p \wedge q \equiv q \wedge p$ as well as $p \vee q \equiv q \vee p$ (substitution property)

(3) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ as well as $(p \vee q) \vee r \equiv p \vee (q \vee r)$ property (respectively) the merger.

(4) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ distribution of property.

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ distribution of property.

proof: We will prove that "$\wedge$" is distributed on "$\vee$", (leaving the rest of the proofs on the health properties mentioned in the theorem on the student) for that creating the following table and conclude that it's health is required, as shown in the two columns seventh and eighth.

p	**q**	**r**	**p∧q**	**p∧r**	**q∨r**	**p∧ (q∨ r)**	**(p∧q)∨(p∧r)**
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

Table 12

Exercises

1) If p representation for the statement "*the rain came down*" and q representation for the statement "*the ground is grow*". Write the verbal translator for each of the following:

(a) $p \wedge q$ (b) $p \vee q$ (c) $\sim p \wedge q$ (d) $p \rightarrow q$
(e) $q \leftrightarrow p$ (f) $\sim p$ (g) $\sim p \rightarrow \sim q$

2) If p and q are two statements proving that:

$$p \rightarrow q \equiv (\sim p) \vee q \equiv \sim(p \wedge \sim q) \equiv \sim q \rightarrow \sim p$$

3) Proved that the following statements truth logically:

(a) $p \rightarrow p \vee p$ (b) $p \rightarrow p \vee q$ (c) $p \wedge q \longrightarrow p$
(d) $p \wedge q \rightarrow q \wedge p$ (e) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

4) Prove that the following statement is not truth logically and not false logically

$$p \rightarrow p \wedge q.$$

5) If p, q and r are three statements imposed, prove that:

$$p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$$