[4] Logarithmic Functions

The real logarithm function lnx is defined as the inverse of the exponential function. $y = lnx$ is the unique solution of the equation $x = e^y$. In order to define the complex logarithm $\log z$, as the inverse of e^w , one must solve the complex equation:

$$z = e^w ,$$

for w, where z is any non-zero complex number. If w = u + vi be the Cartesian form for w and $z = re^{i\theta}$ be the exponential form for z.

Then the above equation can be written as

$$e^u e^{vi} = re^{i\theta} .$$

$e^u = r$, $v = \theta$ ➔ $u = \ln r$ and $v = \arg z$ ➔ $w = \ln r + i \arg z$

Therefore the logarithm of a complex number z is defined by:

$\log z = \ln|z| + i\, argz , z \neq 0$

$\log z = \ln r + i(\theta + 2n\pi), n = 0, \mp 1, \mp 2, \ldots$

<u>Definition:</u> (Principal value)

The principal branch (Principal value) of the complex logarithmic function which is given by:

$$\text{Log}\, z = \ln|z| + i\,\text{Arg}\, z = \ln r + i\theta$$

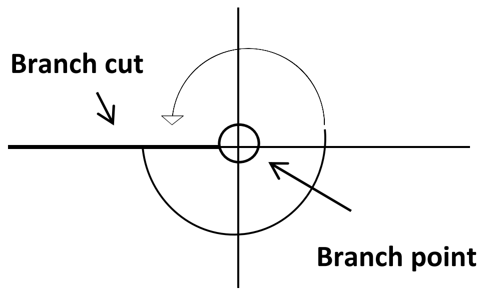
is continuous in the domain $\{r > 0, -\pi < \theta < \pi\}$.

<u>Remarks:</u>

1. The function $\log z = \ln r + i(\theta + 2n\pi)$ is a multiple-valued function. The values of $\log z$ have the same real part, but their imaginary parts differ by interval multiple of 2π. These function is not continuous and therefore not analytic in branch points.
2. For each fixed real number α , the single- value function $\log z = \ln r + i\theta$, $(r > 0\ and\ \alpha < \theta < \alpha + 2\pi)$ is a branch of the multiple- valued function. The function $\text{Log}\, z = \ln r + i\theta$ is a single-valued function called the **principal branch**.

Note:

The principal branch of $\operatorname{Log} z$ is discontinuous at $z = 0$, since this function is not defined at $z = 0$. Also it is not continuous at every point in the negative real axis. The nonpositive real axis is called a branch cut for $\operatorname{Log} z$ and the point 0 is called a branch point.



$$-\pi < \theta < \pi$$

To verify that,

Let $z_0 \in$ branch cut, then

$\operatorname{Arg} z \to \pi$ when $z \to z_0$ from the 2^{nd} quarter

And

$\operatorname{Arg} z \to -\pi$ when $z \to z_0$ from the 3^{rd} quarter

Thus $\lim_{z \to z_0} \log z$ is not exist.

Examples:

1. Find $\log(1 + \sqrt{3}\, i)$ and $\operatorname{Log}(1 + \sqrt{3}\, i)$

Solution:

$z = 1 + \sqrt{3}\, i \to x = 1, \; y = \sqrt{3}$

$r = |z| = \sqrt{1 + 3} = 2$, and

$$\left.\begin{aligned} 1 &= 2\cos\theta \to \cos\theta = \tfrac{1}{2} \\ \sqrt{3} &= 2\sin\theta \to \sin\theta = \tfrac{\sqrt{3}}{2} \end{aligned}\right\} \to \theta = \frac{\pi}{3}$$

Thus:

$$\log\left(1+\sqrt{3}\,i\right)=\ln 2+i\left(\frac{\pi}{3}+2n\pi\right)$$

And:

$$\mathrm{Log}\left(1+\sqrt{3}\,i\right)=\ln 2+i\ \frac{\pi}{3}$$

2. $\log(1+i)=\ln\sqrt{2}+i\left(\frac{\pi}{4}+2n\pi\right)$

   $\mathrm{Log}(1+i)=\ln 2+i\ \frac{\pi}{3}$

3. $\log(1)=\ln 1+i(0+2n\pi)=2n\pi\, i$

   $\mathrm{Log}(1)=\ln 1+i0=0$

4. $\log(3i)=\ln 3+i\left(\frac{\pi}{2}+2n\pi\right)$

   $\mathrm{Log}(3i)=\ln 3+i\ \frac{\pi}{2}$

5. $\log(-3i)=\ln 3+i\left(\frac{-\pi}{2}+2n\pi\right)$

   $\mathrm{Log}(-3i)=\ln 3-i\ \frac{\pi}{2}$

Properties:

Let z_1 , $z_2\neq 0$, $n\in\mathbb{N}$, and $\alpha\in\mathbb{R}$, then

1. $\log(z_1z_2)=\log z_1+\log z_2$
2. $\log\left(\frac{z_1}{z_2}\right)=\log z_1-\log z_2$
3. $e^{\log z}=z\ ,\forall\, z\neq 0$
4. a) $z^n=e^{n\log z}$, $n=1,2,3,\ldots$

   b) $z^{1/n}=e^{1/n\,\log z}$
5. $\log e^z=z+2n\pi i$
6. $\mathrm{Log}(e^z)=z$

Example: Find all the roots of the equation

$$\log z=\frac{\pi}{2}i$$

Solution:

1. Taking the e for both sides

$$e^{\log z} = e^{\frac{\pi}{2}i} \rightarrow z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\rightarrow\ z = i$$

2. We can find the roots in another way as follows:

$$\log z = \frac{\pi}{2}i \rightarrow \ln r + i(\theta + 2n\pi) = 0 + \frac{\pi}{2}i$$

$$\rightarrow \ln r = 0 \rightarrow r = 1 \text{ and}$$

$$\rightarrow \theta + 2n\pi = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{2} - 2n\pi$$

$$\therefore z = re^{i\theta} = e^{i\left(\frac{\pi}{2} - 2n\pi\right)}$$

$$= e^{i\frac{\pi}{2}}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= i$$

Example: Show that when $n = 0, \mp1, \mp2, \dots$

$$\log\left(i^{1/2}\right) = \left(n + \frac{1}{4}\right)\pi i$$

Solution: $\left(i^{1/2}\right) = e^{\frac{1}{2}\log i}$

$$\rightarrow \log\left(i^{1/2}\right) = \log e^{\frac{1}{2}\log i} = \frac{1}{2}\log i \quad \dots 1$$

Since $\log i = i\left(\frac{\pi}{2} + 2n\pi\right)$, then

$$\rightarrow \log\left(i^{1/2}\right) = \frac{1}{2}\, i\left(\frac{\pi}{2} + 2n\pi\right) \quad \text{(By 1)}$$

$$= \left(\frac{1}{4} + n\right)\pi i$$

Example: Show that if $Re(z_1) > 0$ and $Re(z_2) > 0$, then:

$$\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2$$

Proof: Suppose that $Re(z_1) > 0$, $Re(z_2) > 0$, then

$$z_1 = r_1 e^{i\theta_1} \rightarrow \frac{-\pi}{2} < \theta_1 < \frac{\pi}{2}$$

$$z_2 = r_2 e^{i\theta_2} \rightarrow \frac{-\pi}{2} < \theta_2 < \frac{\pi}{2}$$

$\rightarrow -\pi < \theta_1 + \theta_2 < \pi$, which enables us to write

$$\begin{aligned} \operatorname{Log}(z_1 z_2) &= \ln|z_1 z_2| + i \operatorname{Arg}(z_1 z_2) \\ &= \ln(r_1 r_2) + i(\theta_1 + \theta_2) \\ &= \ln z_1 + \ln z_2 + i\theta_1 + i\theta_2 \\ &= \ln z_1 + i\theta_1 + \ln z_2 + i\theta_2 \\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2 \end{aligned}$$

In general

1- $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$
2- $\operatorname{Log}\left(\frac{z_1}{z_2}\right) \neq \operatorname{Log} z_1 - \operatorname{Log} z_2$

Example: Take $z_1 = z_2 = -1$

$\rightarrow \operatorname{Log}(z_1 z_2) = \operatorname{Log}(1) = \ln(1) + 0i = 0$

$\rightarrow \operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(-1) + \operatorname{Log}(-1) = 2\pi i$

$\rightarrow \operatorname{Log}(1) \neq \operatorname{Log}(-1) + \operatorname{Log}(-1)$

Hence

$$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$$

In general:

$\operatorname{Log} z^n \neq n \log z$

Example: Show that:

a) If $\log z = \ln r + i \arg z$, $\left(r > 0\,,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right)$, then

$$\log i^2 = 2 \log i$$

b) If $\log z = \ln r + i \arg z$, $\left(r > 0\,,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$, then

$$\log i^2 \neq 2\log i$$

<u>*Solution*</u>:

a) $\log i^2 = \log(-1)$ $\qquad (z = -1 + 0i)$

$$= \ln(1) + i\pi$$

$$= i\pi \ , \ \text{where} \ \ \pi \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$$

And

$2\log i = 2\left(\ln(1) + i\frac{\pi}{2}\right) = i\pi \qquad (z = 0 + i)$

$\therefore \log i^2 = 2\log i$

b) $\log i^2 = i\pi$, where $\pi \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right)$, and

$2\log i = 2(\ln(1) + i\theta)$

$$= 2i\theta$$

$$= 2i\left(\frac{\pi}{2}\right),\ \frac{\pi}{2} \notin \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right)$$

$\rightarrow \theta = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2} \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right)$

$\rightarrow 2\log i = 2i\left(\frac{5\pi}{2}\right) = 5\pi i$

$$\therefore \log i^2 \neq 2\log i$$

<u>**Example:**</u> Show that

$$\text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$$

<u>*Solution*</u>:

$\rightarrow \text{Log}(1+i)^2 = \text{Log}(1 + 2i + i^2)$

$$= \text{Log}(1 + 2i - 1)$$

$$= \text{Log}\,2i$$

$$= \ln 2 + i\frac{\pi}{2}$$

$$\rightarrow 2\,\text{Log}(1+i) = 2\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]$$

$$= 2\ln(2)^{1/2} + i\frac{\pi}{2}$$

$$= \ln 2 + i\frac{\pi}{2}$$

$$\therefore \text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$$

Example: Show that

$$\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i)$$

Solution:

$$\rightarrow \text{Log}(-1+i)^2 = \text{Log}(-2i)$$

$$= \ln 2 - i\frac{\pi}{2}$$

$$\rightarrow 2\,\text{Log}(-1+i) = 2\left[\ln\sqrt{2} + i\frac{3\pi}{4}\right]$$

$$= \ln 2 + i\frac{3\pi}{2}$$

Hence

$$\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i)$$

Derivative of logarithm function :

For each fixed real number α , The single valued logarithmic function log z $\log z = \ln r + i\theta$, $(r > 0\ and\ \alpha < \theta < \alpha + 2\pi)$ is differentiable on its domain and

$$\frac{d}{dz}(\log z) = \frac{1}{z} \quad , \quad r > 0\ \&\ \alpha < \theta < \alpha + 2\pi$$

Proof:

$$\log z = \ln r + i\theta\ ,\ \ r > 0\ \&\ \alpha < \theta < \alpha + 2\pi$$

Let $u = \ln r$, $v = \theta$, then

$$\left.\begin{array}{ll} u_r = \frac{1}{r} \ , & v_r = 0 \\ u_\theta = 0 \ , & v_\theta = 1 \end{array}\right\} \Rightarrow \begin{array}{l} u_r = \frac{1}{r} v_\theta \\ u_\theta = -r\, v_r \end{array}$$

$\therefore$ C. R. Eqs are satisfied and since $u_r, u_\theta, v_r, v_\theta, u, v$ are continuous functions, then $\log z$ is differentiable on its domain and

$$\frac{d}{dz}(\log z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i0\right)$$

$$= \frac{1}{re^{i\theta}} = \frac{1}{z}$$

In particular

$\frac{d}{dz}(\text{Log}\, z) = \frac{1}{z}$, $r > 0$ & $-\pi < \theta < \pi$. i.e. Log z analytic for all z except when $Re(z) \leq 0$, and $Im(z) = 0$.

Note: $(\text{Log}\, f(z)) = \frac{f'(z)}{f(z)}$.

Example: Find $\frac{d}{dz}(\text{Log}\, 3z^2)$

Solution: $f(z) = 3z^2 \rightarrow \frac{d}{dz}(\text{Log}\, f(z)) = \frac{f'(z)}{f(z)} = \frac{6z}{3z^2} = \frac{2}{z}$.

Example: Determine the domain of analyticity for the function

$$f(z) = \text{Log}(3z - i)$$

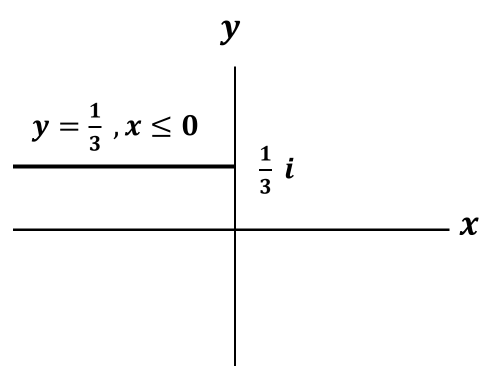
Solution:

The function $\text{Log}(3z - i)$ is analytic everywhere with $Re(3z - i) \leq 0$, and $Im(3z - i) = 0$, must be removed, i.e.

$$Re(3z - i) \leq 0 \rightarrow Re\big(3x + i(3y - 1)\big) = 3x \leq 0 \rightarrow x \leq 0$$

$$Im(3z - i) = 0 \rightarrow Im\big(3x + i(3y - 1)\big) = 3y - 1 = 0 \rightarrow y = \frac{1}{3}$$

Thus f is analytic everywhere except the horizontal line $x \leq 0,\ y = \frac{1}{3}$



Example: Show that the function

$$f(z) = \frac{\mathrm{Log}(z+4)}{z^2+i}$$

is analytic everywhere except for the point $\left(\frac{-(1-i)}{\sqrt{2}}, \frac{(1-i)}{\sqrt{2}}\right)$ and the portion $x \le -4$ of the real axis.

Solution: $\mathrm{Log}(z+4)$ is analytic everywhere except for the points that satisfy the condition

$$Re(z+4) \le 0 \text{ and } Im(z+4) = 0$$

$$\left.\begin{array}{r}\to x+4 \le 0 \\ x \le -4\end{array}\right\}, y = 0 \text{ and } z^2 + i = 0 \to z^2 = -i \to z = \mp (-i)^{1/2}$$

$$z = re^{i\theta} = \mp \left(e^{-i\frac{\pi}{2}}\right)^{1/2}$$

$$= \mp e^{-i\frac{\pi}{4}}$$

$$= \mp \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$$= \mp \left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$= \mp \frac{(1-i)}{\sqrt{2}}$$

Hence f is not analytic at the point $\mp \frac{(1-i)}{\sqrt{2}}$ and the line $x \le -4, \quad y = 0$.

[5] Complex Exponents

We define z^c, where $z, c \in \mathbb{C}$ and $z \ne 0$, by

$$z^c = e^{c \log z} \qquad \dots (1)$$

And

$$c^z = e^{z \log c} \quad (c \ne 0)$$

Example: Find i^{-2i}

<u>Solution</u>: $i^{-2i} = e^{-2i\log i}$

$$= e^{-2i\left(\frac{\pi}{2}+2n\pi\right)i}$$

$$= e^{(4n+1)\pi}\ ,\ n = 0, \mp1, \mp2, \dots$$

Which is multiple valued.

<u>Note:</u> In a view of the property $e^{-z} = \frac{1}{e^z}$, we have $z^{-c} = \frac{1}{z^c}\ (z \neq 0)$ and so

$$(i)^{-2i} = \frac{1}{i^{2i}} = e^{(4n+1)\pi}\ ,\ n = 0, \mp1, \mp2, \dots$$

We notice that the function $\log z = \ln r + i\theta,\ r > 0\ ,\ \alpha < \theta < \alpha + 2\pi$, is a single-valued and analytic function in the domain, thus when the branch of $\log z$ is used, it follows that

$$z^c = e^{c\log z}$$

is also single-valued and analytic in the same domain, and

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\left(e^{c\log z}\right) = \frac{c}{z}e^{c\log z}$$

Since $z = e^{\log z}$, then

$$\frac{d}{dz}(z^c) = c\ \frac{e^{c\log z}}{e^{\log z}} = ce^{c\log z}e^{-\log z}$$

$$= ce^{c\log z - \log z}$$

$$= ce^{(c-1)\log z}$$

$$= cz^{c-1}$$

$$\therefore \frac{d}{dz}(z^c) = cz^{c-1}\ \ (r > 0\ ,\ \alpha < \arg z < \alpha + 2\pi)$$

When $\alpha = -\pi$ then $-\pi < \arg z < \pi$, the function

$$z^c = e^{c\,\mathrm{Log}\,z}\ , \qquad z \neq 0$$

Is called principal value of z^c.

<u>Example:</u> Find the principal value of the following:

a) $(i)^i$

<u>*Solution*</u>: p.v. $(i)^i = e^{i\,\text{Log}\,i} = e^{i\left(\ln 1 + i\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}}$

b) $\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i}$

<u>*Solution*</u>:

$$\text{p.v.} \left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i} = e^{3\pi i\,\text{Log}\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]}$$

$$= e^{3\pi i\left[\ln\left|\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right| - i\frac{2\pi}{3}\right]}$$

$$= e^{3\pi i\left(\ln e - i\frac{2\pi}{3}\right)}$$

$$= e^{3\pi i\left(1 - i\frac{2\pi}{3}\right)}$$

$$= e^{3\pi i + 2\pi^2}$$

$$= e^{2\pi^2}.e^{3\pi i}$$

$$= -e^{2\pi^2} \quad \left(e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1\right)$$

c) $z^{2/3}$

<u>*Solution*</u>: p.v $z^{2/3} = e^{\frac{2}{3}\text{Log}\,z} = e^{\frac{2}{3}(\ln|z| + i\theta)}$

$$= e^{\frac{2}{3}\ln r + \frac{2}{3}\theta i}$$

$$= e^{\ln r^{2/3}}.e^{\frac{2}{3}\theta i}$$

$$= \sqrt[3]{r^2}\,e^{\frac{2}{3}\theta i}$$

<u>**Note:**</u> One can show that the above p.v. is analytic in the domain $r > 0\,, -\pi < \theta < \pi$.

Finally,

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\left(e^{z\log c}\right) = e^{z\log c}.\log c = c^z\log c$$

Which is analytic when the value of $\log c$ is specified, i.e.: it is analytic everywhere.

[6] Inverse of Trigonometric and Hyperbolic Functions

In this section, we shall show the following identities:

1. $\sin^{-1} z = -i\log\left(iz + \sqrt{1-z^2}\right)$

2. $\cos^{-1} z = -i\log\left(z + i\sqrt{1-z^2}\right)$

3. $\tan^{-1} z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$

4. $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$

5. $\cosh^{-1} z = \log\left(z + \sqrt{z^2-1}\right)$

6. $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

7. $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$

8. $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1-z^2}}$

9. $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$

10. $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$

11. $\frac{d}{dz}\cosh^{-1} z = \frac{1}{\sqrt{z^2-1}}$

12. $\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$

Example: Find the values of the following:

1) $\sin^{-1}(-i)$ 2) $\tan^{-1} 2i$ 3) $\cosh^{-1}(-1)$ 4) $\tanh^{-1}(0)$

Solution:

1) $\sin^{-1}(-i) = -i\log\left[i(-i) + \sqrt{1-(-i)^2}\right]$

$$= -i\log\left[1+\sqrt{2}\right] \qquad \dots(1)$$

Now: $\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + i2n\pi$

And:

$$\log(1-\sqrt{2}) = \ln|1-\sqrt{2}| + i(\pi + 2n\pi)$$

$$= -\ln|1-\sqrt{2}| + i(2n+1)\pi \quad \dots (2)$$

Since $(-1)^n \ln(1+\sqrt{2}) + n\pi i$, constitute the set of values of $\ln(1 \mp \sqrt{2})$ and $n\pi i$ is the same as $2k\pi i$ when n is even and $(2k+1)\pi i$ when n is odd, so

$$\sin^{-1}(-i) = -i[(-1)^n \ln(1+\sqrt{2}) + n\pi i]$$

$$= n\pi + i(-1)^{n+1} \ln(1+\sqrt{2})$$

2) $\tan^{-1} 2i = \frac{i}{2}\log\left(\frac{i+2i}{i-2i}\right)$

$$= \frac{i}{2}\log(-3)$$

$$= \frac{i}{2}[\ln 3 + i(\pi + 2n\pi)]$$

$$= \frac{-1}{2}(2n+1)\pi + \frac{i}{2}\ln 3$$

3) $\cosh^{-1}(-1) = \log\left[-1 \mp \sqrt{(-1)^2 - 1}\right] = \log(-1)$

$$= \ln 1 + i(\pi + 2n\pi)$$

$$= (2n+1)\pi i\ , \ \ n = 0, \mp 1, \mp 2, \dots$$

4) $\tanh^{-1}(0) = \frac{1}{2}\log\left(\frac{i+0}{i-0}\right)$

$$= \ln 1 + 2n\pi i$$

$$= 2n\pi i\ , \ \ n = 0, \mp 1, \mp 2, \dots$$

<u>Example:</u> Solve

$$\sin z = 2$$

<u>*Solution*</u>: $\sin z = 2 \to z = \sin^{-1} 2$

$$= -i\log\left(2i + \sqrt{1-4}\right)$$

$$= -i\log\left(2i + \sqrt{3}\, i\right)$$

$$= -i\log\left(\left(2 + \sqrt{3}\,\right)i\right)$$

$$\to -i\log\left(\left(2 + \sqrt{3}\,\right)i\right) = -i\left[\log i + \log\left(2 + \sqrt{3}\,\right)\right]$$

$$= -i\left[\left(\ln 1 + \left(\frac{\pi}{2} + 2n\pi\right)i\right) + \log\left(2 + \sqrt{3}\,\right)\right]$$

$$= \frac{\pi}{2} + 2n\pi - i\log\left(2 + \sqrt{3}\,\right)$$

$$= \pi(1 + 2n) - i\log\left(2 + \sqrt{3}\,\right)$$

<u>Example:</u> Solve

$$\cos z = \sqrt{2}$$

<u>*Solution*</u>: $\cos z = \sqrt{2} \to z = \cos^{-1}\sqrt{2}$

$$\cos^{-1} z = -i\log\left(z + i\sqrt{1 - z^2}\right)$$

$$\cos^{-1}\sqrt{2} = -i\log\left(\sqrt{2} + i\sqrt{1 - \left(\sqrt{2}\right)^2}\right)$$

$$= -i\log\left(\sqrt{2} + i\sqrt{1-2}\right)$$

$$= -i\log\left(\sqrt{2} - 1\right)$$

$$= -i\log\left(\sqrt{2} - 1\right) + 2$$