Chapter Three

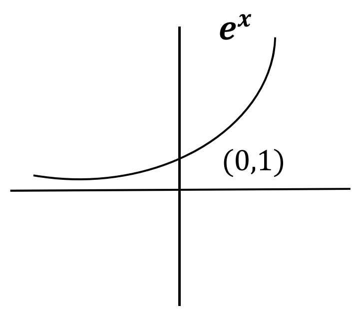
Elementary Functions

[1] The Exponential Functions

A real valued function $f(x) = e^x, f: \mathbb{R} \to \mathbb{R}^+$, is one-to-one and onto function, and

1. $e^{x_1}.e^{x_2} = e^{x_1 + x_2}$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$



Definition:

Let $z = x + iy$, define

$$\text{Exp}(z) = e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$$

If $f(z) = e^z = u + iv \to Re(z) = e^x \cos y$, $Im(z) = e^x \sin y$

If $y = 0 \to e^z = e^x$

If $x = 0 \to e^z = e^{iy} = \cos y + i \sin y$

Note: If $f(z) = e^z$, then

1. e^z is an analytic function, since

$$u = e^x \cos y \ , \qquad v = e^x \sin y$$

$$u_x = e^x \cos y = v_y \ , \qquad u_y = -e^x \sin y = -v_x$$

and u_x, u_y, v_y, v_x, u, v are continuous functions and satisfy C.R.E, therefore e^z is differentiable function $\forall z \in \mathbb{C}$.

2. $f'(z) = e^z$, since $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$

$$= e^x(\cos y + i \sin y) = e^z$$

3. $|e^z| = e^x$, since
   $|e^z| = |e^x e^{iy}| = |e^x||e^{iy}|$

$$= |e^x|\sqrt{\cos^2 y + \sin^2 y}$$

$$= |e^x|.1$$

$$= |e^x|$$

   But $e^x > 0, \forall x \in \mathbb{R}$, so $|e^z| = e^x$

4. $|e^z| \neq 0$, since $|e^z| = e^x \neq 0, \forall x \in \mathbb{R}$
   Note: $e^z = 0$ iff $|e^z| = 0$

5. $e^z: \mathbb{R} \to \mathbb{C} - \{0\}$

Example: Let $w \neq 0$ and $w = re^{i\theta}$, find z if $z = re^{i\theta} = w$

Solution:

$e^z = e^x . e^{iy} = re^{i\theta}$

$\to r = e^x \quad , \; y = \theta + 2n\pi \,, n = 0, \mp 1, \ldots$

$\to x = \log r \;, \; y = \theta + 2n\pi$

$\therefore z = \ln r + i(\theta + 2n\pi)$

Therefore $\forall\, w \in \mathbb{Z}, \exists$ infinity number of values of z such that $w = e^z$, therefore e^z is not one-to-one.

Note: e^z is periodic function with period 2π

$$e^z = e^{z+2\pi i}$$

Proof: Let $z = x + iy$, hence

$e^{z+2\pi i} = e^{x+iy+2\pi i} = e^{x+i(y+2\pi)}$

$$= e^x(\cos(y + 2\pi) + i\ \sin(y + 2\pi)) = e^x(\cos y + i \sin y) = e^z$$

In general: e^z is not one-to-one only if $-\pi < Im(z) < \pi$.

Properties of Exponential Function:

1. $e^{z_1}.e^{z_2} = e^{z_1+z_2}$

2. $e^{1/z} = e^{-z}$

3. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

4. $(e^z)^n = e^{nz},\ n \in \mathbb{Z}$

Proof:

1. Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

$$e^{z_1}.e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1).e^{x_2}(\cos y_2 + i\sin y_2)$$

$$= e^{x_1}.e^{x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$

$$= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$

$$= e^{x_1+x_2}.e^{i(y_1+y_2)}$$

$$= e^{(x_1+iy_1)+(x_2+iy_2)}$$

$$= e^{z_1+z_2}$$

By the same way, we can prove 2 and 3.

4. $(e^z)^n = (e^x\cos y + ie^x\sin y)^n$

$$= \left(e^x(\cos y + i\sin y)\right)^n$$

$$= e^{nx}(\cos y + i\sin y)^n$$

$$= e^{nx}(\cos ny + i\sin ny)$$

$$= e^{nx}e^{iny}$$

$$= e^{nx+iny}$$

$$= e^{n(x+iy)}$$

$$= e^{nz}$$

5. $e^0 = 1$

6. $\arg e^z = y + 2n\pi$

7. $\overline{(e^z)} = e^{\bar{z}}$

Proof:

$\overline{(e^z)} = e^x(\cos y - i\sin y)$

$= e^x(\cos(-y) + i\sin(-y))$

$= e^{x-iy}$

$= e^{\bar{z}}$

Polar Coordinates of Exponential Function:

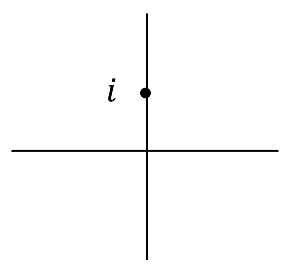
If $e^z = e^x(\cos y + i\sin y)$

$= r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$

Where $r = |e^z| = e^x, y = \theta + 2n\pi$

Example: Solve $e^z = i$

Solution: $z = \ln r + i(\theta + 2n\pi)$



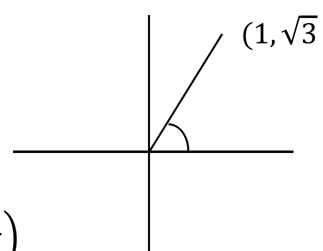
$r = |i| = 1$ and $\theta = \arg i = \frac{\pi}{2} + 2n\pi$

$\therefore z = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right), n = 0, \mp 1, \ldots$

$= i\left(\frac{\pi}{2} + 2n\pi\right)$

Example: Find the value of z such that

$$e^z = 1 + \sqrt{3}\,i$$



Solution: $z = \ln r + i(\theta + 2n\pi)$

$r = \sqrt{1+3} = 2$, $\theta = \frac{\pi}{3} + 2n\pi \rightarrow z = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

Example: Prove that

$$e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{e}\ \left(\frac{1+i}{\sqrt{2}}\right)$$

Proof: $e^{\left(\frac{2+\pi i}{4}\right)} = e^{\left(\frac{1}{2}+\frac{\pi}{4}i\right)}$

$$= e^{1/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= \sqrt{e}\left(\frac{1+i}{\sqrt{2}}\right)$$

Example: Prove that

$$e^{z+\pi i} = -e^{z}$$

Proof: $e^{z+\pi i} = e^{(x+iy)+\pi i}$

$$= e^{x+(y+\pi)i}$$

$$= e^{x}(\cos(y+\pi) + i\sin(y+\pi))$$

$$= e^{x}(-\cos y - i\sin y)$$

$$= -e^{x}(\cos y + i\sin y)$$

$$= -e^{z}$$

Example: Find all the complex solutions of

$$e^{z} = 1$$

Solution:

$e^{z} = 1 \rightarrow r = 1\ ,\ \theta = 0$

$\therefore z = \ln 1 + i(0 + 2n\pi) = i\ 2n\pi$

Example: Find all the complex solutions of

$$e^{4z} = i$$

Solution: $e^{4z} = i = e^{4x}(\cos 4y + i\ \sin 4y)$

$r = 1\ ,\ \theta = \frac{\pi}{2} + 2n\pi\ ,\ n = 0, \mp 1, \dots$

$e^{4z} = e^{4x}(\cos 4y + i \sin 4y)$

$\quad = 1.\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$

$\therefore e^{4x} = 1 \rightarrow 4x = ln1 \rightarrow x = 0$

$\&\ \cos 4y = \cos\frac{\pi}{2} \rightarrow 4y = \frac{\pi}{2} \rightarrow y = \frac{\pi}{8} + 2n\pi$

$\therefore z = x + iy = 0 + i\left(\frac{\pi}{8} + 2n\pi\right) = i\left(\frac{\pi}{8} + 2n\pi\right)$

Note:

1. $f(z) = e^{\bar{z}}$ is not analytic at any point (not analytic everywhere). (H.w)

2. $f(z) = e^{iz}$ is analytic function.

Proof:

$e^{iz} = e^{-y}(\cos x + i \sin x)$

i. $u_x = -e^{-y} \sin x\ ,\ u_y = -e^{-y} \cos x$

$\quad u_x = v_y \qquad ,\ u_y = -v_x \rightarrow$ C. R. E are satisfied.

ii. $u\ , v, u_x\ , u_y\ , v_y\ , v_x$ are continuous functions.

From (i) and (ii), we get e^{iz} is analytic function and

$\left(e^{iz}\right)' = u_x + iv_x$

$\qquad = -e^{-y} \sin x + ie^{-y} \cos x$

$\qquad = ie^{-y}(\cos x + i \sin x)$

$\qquad = ie^{iz}$

[2] Trigonometric Functions

Definition: Let $z = x + iy$, define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad , \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} \quad , \qquad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z} \quad , \qquad \csc z = \frac{1}{\sin z}$$

Note: $\sin z$ and $\cos z$ are analytic functions in the complex plane, hence they're entire functions, but $\tan z, \sec z$ are analytic only when $\cos z \neq 0$.

Note:

1. $(\sin z)' = \frac{1}{2i}\left[ie^{iz} + ie^{-iz}\right]$

$$= \frac{e^{iz} + e^{-iz}}{2i} = \cos z$$

2. $(\cos z)' = \frac{1}{2}\left[ie^{iz} - ie^{-iz}\right] = \frac{i}{2}\left[e^{iz} - e^{-iz}\right]$

$$= -\left[\frac{e^{iz} - e^{-iz}}{2i}\right] = -\sin z$$

Note:

1. $\cos^2 z + \sin^2 z = 1$

Proof:

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

2. $\cos z = \cos x \cosh y - i \sin x \sinh y$

   where $\cos iy = \cosh y$, $\sin iy = \sinh y$

3. $\sin z = \sin x \cosh y + i \cos x \sinh y$

4. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

5. $|\cos z|^2 = \cos^2 x + \sinh^2 y$

<u>Note:</u> $\sin z$ and $\cos z$ are periodic, since

1. $\sin(z + 2\pi) = \sin z$

2. $\cos(z + 2\pi) = \cos z$

But

3. $\tan(z + \pi) = \tan z$

<u>Proof</u>: 1. (H.w)

2. $\cos(z + 2\pi) = \cos(x + iy + 2\pi) = \cos(x + 2\pi + iy)$

$$= \cos(x + 2\pi)\cosh y - i \sin(x + 2\pi)\sinh y$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$= \cos z$$

3. (H.w)

<u>Note:</u> The zeros of $\sin z$ and $\cos z$ are real.

<u>Example:</u> The zero of $\cos z$ is $z = \frac{\pi}{2} + n\pi$.

<u>Solution</u>:

$\cos z = 0$

$\rightarrow \cos x \cosh y - i \sin x \sinh y = 0 + 0i$

$\therefore \cos x \cosh y = 0 \quad \ldots(1)$

& $\sin x \sinh y = 0 \quad \ldots(2)$

Since $\cos x \cosh y = 0 \rightarrow$ either $\cos x = 0$ or $\cosh y = 0$

If $\cos x = 0 \rightarrow x = \frac{\pi}{2} + n\pi$

Substituting in (2) we get

$\sinh y = 0 \rightarrow y = 0$

If $\cosh y = 0 \rightarrow$ this is not possible since ($\cosh y = \frac{e^{y} + e^{-y}}{2} \neq 0, \forall\, y$ and $\sinh y = \frac{e^{y} - e^{-y}}{2} = 0$ if $y = 0$).

$\therefore z = x + iy = \frac{\pi}{2} + n\pi + 0$

$\therefore z = \frac{\pi}{2} + n\pi$

Note: If we take equation (2) we get:

$\sin x \sinh y = 0 \rightarrow$ either $\sin x = 0$ or $\sinh y = 0$

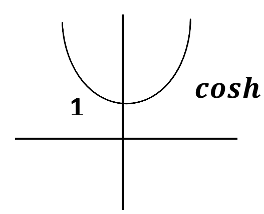
If $\sin x = 0 \rightarrow$ this is not possible since

$$\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$$

Then $\sinh y = 0 \rightarrow y = 0$

$\therefore z = \frac{\pi}{2} + n\pi + 0 = \frac{\pi}{2} + n\pi$

Note: Cosh$y$ (the range of cosh$y \geq 1$) is always positive.



Example: Find all the roots of

$$\sin z = 3$$

Solution:

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\sin z = 3 \rightarrow \sin x \cosh y + i \cos x \sinh y = 3 + 0i$

$\sin x \cosh y = 3 \quad \dots(1)$

$\cos x \sinh y = 0 \quad \dots(2)$

From (2) we get:

$\cos x \sinh y = 0$, then

Either $\cos x = 0 \rightarrow x = \frac{\pi}{2} + n\pi$

Or $\sinh y = 0 \rightarrow$ y=0 → this is not possible

Then $\cos x = 0 \rightarrow x = \frac{\pi}{2} + n\pi$

from (1) we get:

$$\sin\frac{\pi}{2} + n\pi = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

If n is even then $\cosh y = 3 \rightarrow y \cong 1.8$

If n is odd then $\cosh y = -3$ this is not possible

Example: Find all the roots of

$$\sin(\bar{z} + i) = 2i$$

Solution: $\sin(\bar{z} + i) = \sin(x - iy + i) = \sin(x + i(1 - y))$

$\rightarrow \sin\big(x + i(1 - y)\big) = 0 + 2i$

$\rightarrow \sin x \cosh(1 - y) + i \cos x \sinh(1 - y) = 0 + 2i$

$\sin x \cosh(1 - y) = 0 \quad \dots(1)$

$\cos x \sinh(1 - y) = 2 \quad \dots(2)$

From (1) we get:

$\sin x \cosh(1 - y) = 0$, then

Either $\cosh(1 - y) = 0 \rightarrow$ this is not possible

Or $\sin x = 0 \rightarrow x = n\pi$

$$\text{Then } \cos n\pi = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

From (2) we get:

If n is even then $\sinh(1-y) = 2$, then $\rightarrow\ 1 - y = \sinh^{-1}(2)$

$$\rightarrow \quad y = -\sinh^{-1}2 + 1$$

$$\rightarrow y = 0.4$$

If n is odd then $\sinh(1-y) = -2$, then $\rightarrow\ 1 - y = \sinh^{-1}(-2)$

$$\rightarrow \quad y = -\sinh^{-1}2 + 1$$

$$\rightarrow y = 2.4$$

Example: Prove that

$$\left|e^{2z+i} + e^{iz^2}\right| \le e^{2x} + e^{-2xy}$$

Proof:

$$\left|e^{2z+i} + e^{iz^2}\right| = \left|e^{2x+i(2y+1)} + e^{i(x^2-y^2+2ixy)}\right|$$

$$\le \left|e^{2x+i(2y+1)}\right| + \left|e^{i(x^2-y^2+2ixy)}\right|$$

$$= \left|e^{2x}\, e^{i(2y+1)}\right| + \left|e^{-2xy}\, e^{i(x^2-y^2)}\right|$$

$$= e^{2x} + e^{-2xy} \qquad \text{(Since } e^{i\cdots} = 1\text{)}$$

[3] Hyperbolic Functions

The hyperbolic Sine and Cosine of a complex variable defined as they are with a real variable; that is,

1. $\text{Sinh}\, z = \frac{e^z - e^{-z}}{2}$, $\text{Cosh}\, z = \frac{e^z + e^{-z}}{2}$

Since e^z and e^{-z} are entire functions, then it follows from definition (1) that $\sinh z$ and $\cosh z$ are entire functions, furthermore,

1. $\frac{d}{dz} \sinh z = \cosh z$

2. $\frac{d}{dz} \text{Cosh}\, z = \text{Sinh}\, z$

3. $\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$

$$= \frac{e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}}{4}$$

$$= 1$$

4. Sinh z and Cosh z are periodic functions with period $2\pi i$.

♦ Show that

$$\sinh(z + 2\pi i) = \sinh z$$

Proof:

$$\sinh(z + 2\pi i) = \frac{e^{z+2\pi i} - e^{-z-2\pi i}}{2}$$

$$= \frac{e^z . e^{2\pi i} - e^{-z} . e^{-2\pi i}}{2}$$

$$= \frac{e^z(\cos 2\pi i + i \sin 2\pi i) - e^{-z}(\cos(-2\pi i) + i \sin(-2\pi i))}{2}$$

$$= \frac{e^z - e^{-z}}{2} \qquad (\cos 2\pi i = 1 \,,\ \sin 2\pi i = 0)$$

$$= \sinh z$$

5. $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

Proof:

$$|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$

$$= \sinh^2 x(1 - \sin^2 y) + (1 + \sinh^2 x) \sin^2 y$$

$$= \sinh^2 x - \sinh^2 x \ \sin^2 y + \sin^2 y + \sinh^2 x \sin^2 y$$

$$= \sinh^2 x + \sin^2 y$$

6. $|\cosh z|^2 = \cos^2 y + \sinh^2 x$ (H.w)

7. The zeros of $\text{Sinh}\, z$ are $z = n\pi i$

Proof:

$\sinh z = \sinh x \cos y + i \cosh x \sin y$

$\sinh z = 0 \rightarrow \sinh x \cos y + i \cosh x \sin y = 0 + 0i$

$\sinh x \cos y = 0 \quad \dots(1)$

$\cosh x \sin y = 0 \quad \dots(2)$

From (1), we get:

$\sinh x \cos y = 0$, then

Either $\sinh x = 0$ or $\cos y = 0$

If $\sinh x = 0 \rightarrow x = 0$

Substituting in (2), we get:

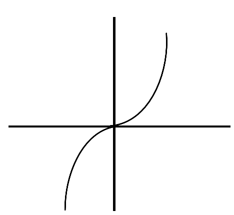
$\sin y = 0 \rightarrow y = n\pi$

If $\cos y = 0 \rightarrow$ this is not possible

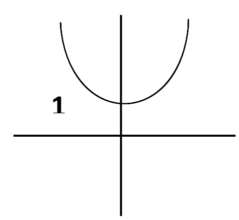
$\therefore z = x + iy = 0 + i(n\pi) = n\pi i$

Note: The *Cosh* cannot be negative in real numbers, but it can be in complex numbers.

sinhx



coshx



Example: Solve $e^{2z-1} = 1$

Solution:

$e^{2z-1} = e^{2(x+iy)-1} = e^{2x-1}.e^{2iy}$

$$= e^{2x-1}(\cos 2y + i \sin 2y)$$

$e^{2z-1} = 1 \rightarrow e^{2x-1}(\cos 2y + i \sin 2y) = \cos 0 + i \sin 0$

$e^{2x-1} \cos 2y = 1 \quad \dots (1)$

$e^{2x-1} \sin 2y = 0 \quad \dots (2)$

From (2), we get

Either $e^{2x-1} = 0$ or $\cos 2y = 0$

If $e^{2x-1} = 0 \rightarrow$ this is not possible

If $\sin 2y = 0 \rightarrow 2y = n\pi \rightarrow y = \frac{n\pi}{2}$ $, n = 0, \mp 1, \dots$

Substituting in (1), we get:

$e^{2x-1} = 1 \rightarrow 2x - 1 = 0 \rightarrow x = \frac{1}{2}$

$\therefore z = x + iy = \frac{1}{2} + i\frac{n\pi}{2} = \frac{1}{2}\ (1 + n\pi i)$