Example 3:
A projectile is fired at a speed of 840 m/s at an angle of 60°. We are tasked to calculate:
1. The time it will take to travel 21 km horizontally.
2. The total flight time of the projectile.

Given Data:
Initial velocity: v_0 = 840 m/s angle: θ = 60° Distance to travel: x = 21 km = 21,000 m Acceleration due to gravity: g = 9.81 m/s^2

The horizontal distance is related to time through the equation: the equation for time: $t = \frac{x}{v_{0x}} = \frac{x}{v_0 \cdot cos(\theta)}$ t = 21,000 / (840 · cos(60°)) =21,000 / (840 · 0.5) = 21,000 / 420 = 50 seconds
Total Flight Time: T = $\frac{2v_{0y}}{g}$ = $\frac{2v_0 \sin\theta}{g}$ = (2 · 840 · 0.866) / 9.81= 1454.88 / 9.81 ≈ 148.3 seconds

Arc Length in Space

Definition:
The length of a smooth curve in three-dimensional space: r(t) = x(t)i + y(t)j + z(t)k, for a ≤ t ≤ b, is given by the following formula:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\mathbf{v}|\, dt$$

Example:
A glider is soaring upward along the helix defined by: r(t) = (cos(t))i + (sin(t))j + t k Find the path length of the glider from t = 0 to t = 2π.

Solution:
The formula for arc length is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$r(t) = \underbrace{(\cos\ t)}_{x(t)}\, i + \underbrace{(\sin\ t)}_{y(t)}\, j + \underbrace{t}_{z(t)}\ k$$

$$L = \int_0^{2\pi} \sqrt{(-\sin\ t)^2 + (\cos\ t)^2 + (1)^2} dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2\ t + \cos^2\ t + 1} dt$$

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$$= \int_0^{2\pi} \sqrt{2} dt = \sqrt{2} t\Big]_0^{2\pi} = 2\pi\sqrt{2}.$$

Arc length parameter with base point $P(t_0)$.
Example:
Find the arc length parametrization of a curve if $t_0 = 0$ and the helix $r(t) = (\cos\ t)i + (\sin\ t)j + tk$

$$s(t) = \int_t^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\,d\tau$$
$$= \int_0^t \sqrt{(-\sin\ \tau)^2 + (\cos\ \tau)^2 + (1)^2}\,d\tau$$

$$s(t) = \int_0^t \sqrt{\sin^2\ \tau + \cos^2\ \tau + 1}\,d\tau$$
$$= \int_0^t \sqrt{2}\,d\tau = \sqrt{2}\tau\Big]_0^t = \sqrt{2}t$$
$$\therefore s = \sqrt{2}t \Rightarrow t = s/\sqrt{2}$$

Substituting $t = s/\sqrt{2}$ in the position vector r gives the arc length Parametrization for the helix.

$$r(t) = (\cos\ t)i + (\sin\ t)j + tk$$
$$r(t(s)) = \left(\cos\ \frac{s}{\sqrt{2}}\right)i + \left(\sin\ \frac{s}{\sqrt{2}}\right)j + \frac{s}{\sqrt{2}}k$$

Unit tangent vector

The unit tangent vector is $\mathrm{T} = \frac{v}{|v|}$ where v is the velocity vector $v = \frac{dr}{dt}$ is tangent to the curve $r(t)$.

$$\mathrm{T} = \frac{v}{|v|} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{r'(t)}{s'(t)} = \frac{dr}{ds}$$

Example:

Find the unit tangent vector of the carve $r(t) = (3\cos\ t)i + (3\sin\ t)j + t^2k$

Sol: $T = \frac{v}{|v|}$

$$v = -(3\sin\ t)i + (3\cos\ t)j + 2t''k$$
$$|v| = \sqrt{9\sin^2\ t + 9\cos^2\ t + 4t^2}$$
$$|v| = \sqrt{9\left(\frac{\sin^2\ t + \cos^2\ t}{''}\right) + 4t^2}$$
$$|v| = \sqrt{9 + 4t^2}$$
$$T = \frac{v}{|v|} = \frac{-3\sin\ t}{\sqrt{9 + 4t^2}}i + \frac{3\cos\ t}{\sqrt{9 + 4t^2}}j + \frac{2t}{\sqrt{9 + 4t^2}}$$

Curvature and Normal vectors of a curve

Definition:

If T is the unit tangent vector of a smooth curve, the curvature function of the curve is: K

Formula for Calculating Curvature:
If $(r(t)$ is a smooth curve, then the curvature is

$$K = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right| = \left|\frac{dt}{ds}\right|\left|\frac{dT}{dt}\right| = \left|\frac{1}{ds/dt}\right|\frac{dT}{dt}$$

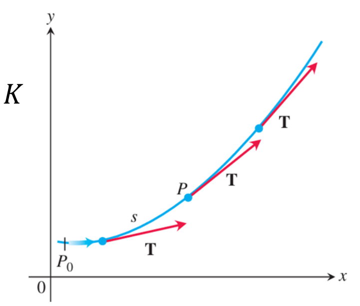


Figure1: As P moves along the curve in the direction of increasing arc, the unit tangent vector turns. value of $|dT/ds|$ at P is called the Curvature

$$K = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$$

Example:

A straight line is $r(t) = C + tv$ for constant vectors C and v. Thus, r′(t) = v, and the unit tangent vector $T = v/|v|$ is a constant vector that always points in the same direction and has derivative 0. It follows that, for any value of the parameter t, the curvature of the straight line is zero.

Example:

Find the curvature of a circle $r(t) = (a\cos\ t)i + (a\sin\ t)j$ of radius a.

Sol: $k = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$

$$\therefore v = \frac{dr}{dt} = -(a\sin\ t)i + (a\cos\ t)j$$

$$|v| = \sqrt{(-a\sin\ t)^2 + (a\cos\ t)^2} = \sqrt{a^2(\sin^2\ t + \cos^2\ t)} = |a| = a$$

$$T = \frac{v}{|v|} = \frac{-a\sin\ t}{a}i + \frac{a\cos\ t}{a}j = (-\sin\ t)i + (\cos\ t)j$$

$$\frac{dT}{dt} = (-\cos\ t)i - (\sin\ t)j$$

$$\left|\frac{dT}{dt}\right| = \sqrt{(-\cos\ t)^2 + (-\sin\ t)^2} = \sqrt{\cos^2\ t + \sin^2\ t} = 1$$

$$K = \frac{1}{|\ v|} \cdot \left|\frac{dT}{dt}\right| = \frac{1}{a} \cdot 1 = \frac{1}{a} = \frac{1}{radius}$$

The radius of curvature

To compute the radius of curvature of the curve at P is the radius of the circle of curvature, which is $r = \frac{1}{k}$.

Definition:

At a point where $k \neq 0$, the **principal unit normal vector** for a smooth curve in the plane is: $N = \frac{1}{K} \cdot \frac{dT}{ds}$

$$\text{N} = \frac{1}{K} \cdot \frac{dT}{ds} = \frac{1}{|dT/ds|} \cdot \frac{dT}{ds} = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} = \frac{\frac{dT}{dt} \cdot \frac{dt}{ds}}{\left|\frac{dT}{dt}\right| \cdot \left|\frac{dt}{ds}\right|} = \frac{dT/dt}{|dT/dt|} \qquad \left[sice\ \frac{dt}{ds} = \frac{1}{ds/dt} > 0\right]$$

∴ The formula for Calculating N where $r(t)$ is a smooth curve is $N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$ where $T = \frac{V}{|v|}$ is the unit tangent vector.

Example:

Find T and N for the Circular motion $r(t) = (\cos\ 2t)i + (\sin\ 2t)j$

Sol: (1) $T = \frac{v}{|v|}$

$$v = \frac{dr}{dt} = (-2\sin\ 2t)i + (2\cos\ 2t)j$$

$$|v| = \sqrt{4\sin^2\ 2t + 4\cos^2\ 2t} = \sqrt{4\left(\frac{\sin^2\ 2t + \cos^2\ 2t)}{11}\right.} = \sqrt{4} = 2$$

$$\therefore T = \frac{-2\sin\ 2t}{2} i + \frac{2\cos\ 2t}{2} j = -(\sin\ 2t)i + (\cos\ 2t)j$$

(2) $N = \frac{dT/dt}{|dT/dt|}$

$$\therefore dT/dt = (-2\cos\ 2t)i - (2\sin\ 2t)j$$
$$|dT/dt| = \sqrt{4\cos^2\ 2t + 4\sin^2\ 2t} = \sqrt{4} = 2$$
$$\therefore N = \frac{-2\cos\ 2t}{2} i + \frac{-2\sin\ 2t}{2} j$$
$$= (-\cos\ 2t)i + (-2\sin\ 2t)j$$

Example:

Find the curvature of the Parabola equation $r(t) = t_i + t^2 j$ at the origin

SOL:

$$K = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$$
$$v = \frac{dr}{dt} = i + 2tj$$
$$|v| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$
$$T = \frac{v}{|v|} = \frac{1}{\sqrt{1 + 4t^2}} i + \frac{2t}{\sqrt{1 + 4t^2}} j$$
$$T = (1 + 4t^2)^{\frac{-1}{2}} i + 2t(1 + 4t^2)^{\frac{-1}{2}} j$$
$$\frac{dT}{dt} = \left(\frac{-1}{2}(1 + 4t^2)^{\frac{-3}{2}} \cdot 8t\right) i + \left(2t \cdot \frac{-1}{2}(1 + 4t^2)^{\frac{-3}{2}} \cdot 8t + (1 + 4t^2)^{\frac{-1}{2}} \cdot 2\right) j$$

$$\frac{dT}{dt} = -4t(1 + 4t^2)^{\frac{-3}{2}} i + \left[-8t^2(1 + 4t^2)^{\frac{-3}{2}} + 2(1 + 4t^2)^{\frac{-3}{2}}\right] j$$

The curvature at the origin i.e. at $t = 0$

$$K(0) = \frac{1}{|v(0)|} \cdot \left|\frac{dT}{dt}(0)\right|$$

$$|v(0)| = \sqrt{1 + 4(0)^2} = 1$$
$$\left|\frac{dT}{dt}(0)\right| = |0i + 2j| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$$
$$\therefore K(0) = \frac{1}{1} \cdot 2 = 2$$

Example:

Find the curvature for the helix

$$r(t) = (a\cos t)i + (a\sin t)j + btk, \qquad a, b \geqslant 0,$$

Sol: $T = \frac{v}{|\mathrm{v}|}$

$$v = \frac{dr}{dt} = (-a\sin\ t)i + (a\cos\ t)j + bk$$
$$|v| = \sqrt{(-a\sin\ t)^2 + (a\cos\ t)^2 + (b)^2}$$
$$= \sqrt{a^2\sin^2\ t + a^2\cos^2\ t + b^2}$$
$$= \sqrt{a^2(\sin^2\ t + \cos^2\ t) + b^2} = \sqrt{a^2 + b^2}$$
$$T = \frac{1}{\sqrt{a^2 + b^2}}[(-a\sin\ t)i + (a\cos\ t)j + bk]$$
$$K = \frac{1}{|\mathrm{v}|} \cdot \left|\frac{dT}{dt}\right|$$
$$\frac{dT}{dt} = \frac{1}{\sqrt{a^2 + b^2}}[(-a\cos\ t)i - (a\sin\ t)j]$$
$$\frac{dT}{dt} = \frac{a}{\sqrt{a^2 + b^2}}[(-\cos\ t)i - (\sin\ t)j]$$
$$\left|\frac{dT}{dt}\right| = \left|\frac{a}{\sqrt{a^2 + b^2}}\right| |(-\cos\ t)i - (\sin\ t)j|$$
$$= \frac{a}{\sqrt{a^2 + b^2}}\sqrt{\underbrace{\cos^2\ t + \sin^2\ t}_{1}} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$K = \frac{1}{\sqrt{a^2 + b^2}} \cdot \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{a^2 + b^2}$$

Proof: The Tangent Vector is Orthogonal to the Normal Vector

1. The unit tangent vector is given by: T(t) = r'(t) / |r'(t)|
2. The principal normal vector is defined as: N(t) = T'(t) / |T'(t)|
3. We need to prove: T(t) · N(t) = 0, meaning they are orthogonal.

Since T(t) is a unit vector, its magnitude is always 1: $|T(t)| = 1$

Differentiating both sides with respect to t: d/dt(T(t)· T(t)) = 0 , By the Example we have 2 (T(t) · T'(t)) = 0

Dividing by 2: T(t) · T'(t) = 0

Since N(t) is just the normalized version of T'(t), i.e., N(t) = T'(t) / |T'(t)| we substitute:

T(t) · N(t) = T(t) · [T'(t) / |T'(t)|] = [T(t) · T'(t)] / |T'(t)| = 0 / |T'(t)| = 0

Since the dot product is zero, this proves that T(t) and N(t) are always orthogonal at every point along the curve:

H.W :

H.W. Arc Length Parameterization

1. Find the arc length of the curve: $\boldsymbol{r(t)} = (\boldsymbol{3t})\boldsymbol{i} + (\boldsymbol{2t^2})\boldsymbol{j} + (\boldsymbol{t^3})\boldsymbol{k}, \quad \boldsymbol{1 \leq t \leq 3}$

2. Find the arc length parameter for the curve: $\boldsymbol{r(t) = (t^2)i + (t^3)j + (t^4)k}$, ***with base point*** $\boldsymbol{P(1)}$

Compute s(t) Then, express t in terms of s

Homework 2: Unit Tangent Vector

1. Find the unit tangent vector **T(t)** for the curve: $\boldsymbol{r(t) = (t^2)i + (2t^3)j + (3t^4)k}$ and evaluate T(1).

2. Find a magnitude |T(t)|.

3. A particle moves along the curve: $\boldsymbol{r(t) = (e^t)i + (e^{-1})j + (t^2)k}$ Find the tangent vector at t = 0 and determine its direction.

Homework 3: Curvature & Normal Vectors of a Curve

1. Find the curvature of the curves: (1) $\boldsymbol{r(t) = (t, t^2, t^3)}$ **(2)** $\boldsymbol{r(t) = ln(sect)i + tj \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}}$

2. Find the principal normal vector N(t) of the curve: $\boldsymbol{r(t) = (\cos t, \sin t, t)}$

3. Determine the radius of curvature R at t = 0 for the curve: $\boldsymbol{r(t) = (t, t^2, t^3)}$ using the relationship:

$$\boldsymbol{R = \frac{1}{K}}$$

4-Find N for the helix $r(t) = (a\cos t)i + (a\sin t)j + btk,\ a, b \geqslant 0$ and $a^2 + b^2 \neq 0$

Tangential and Normal Components of Acceleration

Tangential and Normal Components of Acceleration

v = dr/dt Using the chain rule, it can be expressed as: $v = (\frac{dr}{ds})(\frac{ds}{dt})$

$\frac{dr}{ds} = T$, the unit tangent vector. ds/dt = |v|, the speed. Tnnnhen v = T (ds/dt)

Acceleration derivative of velocity: a = dv/dt   Substitute v = T (ds/dt):   a = d/dt[T (ds/dt)]

Using the product rule: a = (d^2s/dt^2)T + (ds/dt)(dT/dt)

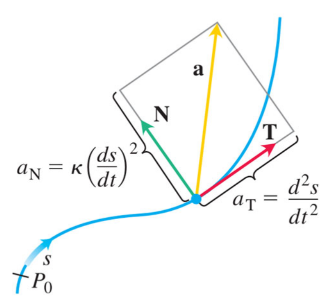
The first term (d^2s/dt^2)T: This represents the tangential component of acceleration, related to the rate of change of speed.

The second term (ds/dt)(**dT/dt**): - Rewrite dT/dt using the chain rule: dT/dt = (dT/ds)(ds/dt)

- Substituting, the second term becomes: (ds/dt)(dT/ds)(ds/dt) = $(ds/dt)^2$(dT/ds)

From the definition of Normal: dT/ds = κN ➔ $\mathbf{(ds/dt)^2\kappa N}$

$$a = (d^2s/dt^2)T + \kappa(ds/dt)^2 N$$



This shows that the acceleration vector has two components:

- Tangential acceleration: $(d^2s/dt^2)T = \frac{d}{dt}|v|T$, which changes the speed.

- Normal acceleration: $\kappa(ds/dt)^2N$, which changes the direction of motion.

Definition:

The acceleration vector is written as $a = a_T T + a_N N$ then $a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|v|$ and $a_N = K\left(\frac{ds}{dt}\right)^2 = \mathrm{k}|v|^2$

are the tangential and normal scalar Components of acceleration.

Remark:
The formula for Calculate the normal Component of Acceleration is $a_N = \sqrt{|a|^2 - a_T^2}$

Example:

Find the acceleration of the motion $r(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j,\ t > 0$

in the form $a = a_T \cdot T + a_N N$.

Sol:

1. find $a_T = \frac{d}{dt}|v|$

$$v = \frac{dr}{dt} = (-\sin\ t + t\cos\ t + \sin\ t)i + (\cos\ t + t\sin\ t - \cos\ t)j$$
$$= (t\cos\ t)i + (t\sin\ t)j$$
$$|v| = \sqrt{t^2\cos^2\ t + t^2\sin^2\ t} = \sqrt{t^2} = 1t\big) = t$$
$$a_T = \frac{d}{dt}|v| = \frac{d}{dt}(t) = 1$$

2. find $a_N = K|v|^2$ or $a_N = \sqrt{|a|^2 - a_T^2}$

$a = (\cos\ t + t\sin\ t)i + (\sin\ t - t\cos\ t)j$
$|a| = \sqrt{(\cos\ t + t\sin\ t)^2 + (\sin\ t - t\cos\ t)^2}$

$$\therefore |a| = \sqrt{t^2 + 1}$$
$$|a|^2 = t^2 + 1$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2}$$
$$= \sqrt{(t^2 + 1) - (1)} = \sqrt{t^2} = t.$$

We then use Equation (1) to find **a**:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$