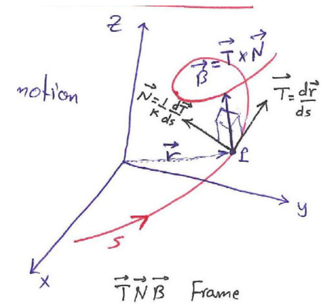
Definition:

If a particle is traveling along a space curve s, then we can describe the motion of the particle in terms of :



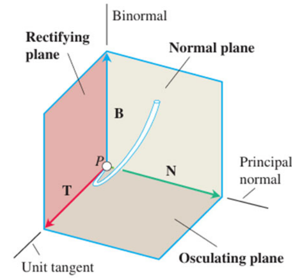
[1] The unit tangent vector $\vec{T}$ (forward direction)

[2] The unit normal vector $\vec{N}$ (the tendency of the motion)

(3) The unit binormal vector $\vec{B} = \vec{T} \times \vec{N}$ ($\perp$ to the plane created by $\vec{T}$ and $\vec{N}$)

$\vec{T}, \vec{N}, \vec{B}$ define **a right-handed frame** used to calculate the paths of particles moving through space. This frame is also called $\overrightarrow{TNB}$ frame.

Notes:



(1) The acceleration $\vec{a}$ always lies in the plane of $\vec{T}$ and $\vec{N}$

(2) $\vec{a} \perp \vec{B}$

(3) $a = a_T T + a_N N$ tells us how much of the acceleration takes place tangent to the motion (a_T) and how much takes place normal to the motion (a_N).

(4) (a_T) measures the rate of change of the length of $\vec{v}$ (the change in the speed)

(5) (a_N) measures the rate of change of the direction of $\vec{v}$.

(6) We can calculate a_N without finding K by: $|\vec{a}|^2 = a_T^2 + a_N^2 \Leftrightarrow a_N = \sqrt{|\vec{a}|^2 = a_T^2}$

Example:

Let $\vec{r}(t) = (1+3t)\vec{\imath} + (t-2)\vec{\jmath} - 3 + \vec{k}$ write $\vec{a}$ in the form $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$

$$\vec{v} = 3\vec{\imath} + \vec{\jmath} - 3\vec{k} \Rightarrow |\vec{v}| = \sqrt{9+1+9} = \sqrt{19}$$

$$a_T = \frac{d}{dt}|\vec{v}| = 0$$

$$\vec{a} = \vec{0} \Rightarrow a_N = \sqrt{|\vec{a}|^2 - a_T^2} = 0$$

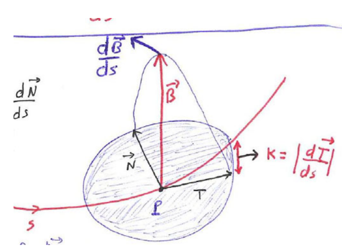
$$\vec{a} = (0)\vec{T} + (0)\vec{N} = \vec{0}$$

Torsion:

The Torsion function of the smooth curve is : $\tau = -\frac{dB}{ds} \cdot N$, where B is binormal vector .

$$\frac{d\vec{B}}{ds} = \frac{d}{ds}(\vec{T} \times \vec{N}) = \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} = \vec{0} + \vec{T} \times \frac{d\vec{N}}{ds} \quad \{since\ \vec{N} = \frac{1}{K}\frac{d\vec{T}}{ds}, \text{i.e.}\ \vec{N} \text{ is the direction of } \frac{d\vec{T}}{ds}\}$$

$$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$$



Now: $\frac{d\vec{B}}{ds} \perp \vec{T}$ "cross product" but $\frac{d\vec{B}}{ds} \perp \vec{B}$ "since $\vec{B}$ has constant length"

$then\ \frac{d\vec{B}}{ds} \perp$ plane of $\vec{B}$ and $\vec{T}$

Since $\vec{T} \times \vec{B} = -\vec{N}$ $then\ \frac{d\vec{B}}{ds} // \vec{N}$. Hence $\frac{d\vec{B}}{ds} = -\tau\,\vec{N}$ "scalar multiple of $\vec{N}$"

Note that now: $\frac{dB}{ds} \cdot N = -\tau\,\vec{N} \cdot \vec{N} = -\tau \Rightarrow$ hence $\tau = -\frac{dB}{ds} \cdot N$ is called the torsion.

Formulas for Computing Curvature and Torsion

Now we give easy-to-use formulas for computing the curvature and torsion of a smooth curve. (without proof)

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v}\times\vec{a}|^2} \quad \text{if } \vec{v}\times\vec{a} \neq \vec{0} \quad \text{and} \quad \dot{x} = \frac{dx}{dt}\ , \ddot{x} = \frac{d^2x}{dt^2}\ ,\ \ddot{x} = \frac{d^3x}{dt^3}$$

Computation formulas For Curves in Space

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration: $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2}$$

Example:

Find T, N, K, B and τ for the space curve $r(t) = (3\sin\ t)\mathrm{i} + (3\cos\ t)j + 4tk$

Sol.:

$$T = \frac{v}{|v|}$$

$$v = \frac{dr}{dt} = (3\cos t)i - (3\sin t)j + 4k$$

$$|v| = \sqrt{(3\cos\ t)^2 + (-3\sin\ t)^2 + (4)^2} = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t) + 16} = \sqrt{9+16} = \sqrt{25} = 5.$$

$$T = \frac{(3\cos t)i - (3\sin t)j + 4k}{5} = \frac{3\cos t}{5}i - \frac{3\sin t}{5}j + \frac{4}{5}k$$

$$N = \frac{dT/dt}{|dT/dt|}$$

$$\frac{dT}{dt} = \frac{-3\sin\ t}{5} i - \frac{3\cos\ t}{5} j + 0$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\frac{9}{25}\sin^2\ t + \frac{9}{25}\cos^2\ t} = \frac{3}{5}$$

$$N = \frac{-3/5\sin\ t}{3/5} i - \frac{3/5\cos\ t}{3/5} j$$

$$N = (-\sin\ t)i - (\cos\ t)j$$

$$K \quad = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right| = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$$

$$B = T \times N$$

$$B = \begin{vmatrix} i & j & k \\ 3/5\cos\ t & -3/5\sin\ t & 4/5 \\ -\sin\ t & -\cos\ t & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} -3/5\sin\ t & 4/5 \\ -\cos\ t & 0 \end{vmatrix} - j \begin{vmatrix} 3/5\cos\ t & 4/5 \\ -\sin\ t & 0 \end{vmatrix} + k \begin{vmatrix} 3/5\cos\ t & -3/5\sin\ t \\ -\sin\ t & -\cos\ t \end{vmatrix}$$

$$= (+4/5\cos\ t)i - (4/5\sin\ t)j + \left(\frac{3}{5}\cos^2\ t - \frac{3}{5}\sin^2\ t\right)k$$

$$\therefore B = (\frac{4}{5}\cos\ t)i - (\frac{4}{5}\sin\ t)j - \frac{3}{5}k$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v}\times\vec{a}|^2}$$

$$(3\cos t)i - (3\sin t)j + 4k$$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} 3\cos t & 3\sin t & 4 \\ -3\sin t & 3\cos t & 0 \\ -3\cos t & -3\sin t & 0 \end{vmatrix}$$

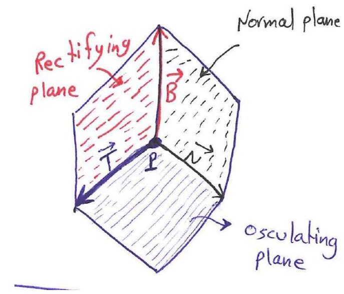
$$4(9\sin^2 t + 9\cos^2\ t = 36(\sin^2 + \cos^2\ t) = 36$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos\ t & 3\sin\ t & 4 \\ -3\sin\ t & 3\cos t & 0 \end{vmatrix} = -12\cos t\ i - 12 sint\ j + (9\cos^2 t + 9\sin^2 t)k = -12\cos t\ i - 12 sint\ j + 9k$$

$|v \times a|^2 = 144 + 81 = 225$

$\therefore \tau = \frac{36}{225} = \frac{4}{2.5}$

The planes determined by $\vec{T}, \vec{N}, \vec{B}$:



If P is the plane $\perp$ vector $\vec{r} = A\vec{\imath} + B\vec{\jmath} + C\vec{k}$ and the point (x_0, y_0, z_0) is

in the plane P then the equation of the plane is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Example:

1) Find $r(t), \vec{v}, \vec{T}, \vec{N}, \vec{B}$ at $t = 0$ for $\vec{r}(t) = (\cos\ t)\vec{\imath} + (\sin\ t)\vec{\jmath} + +t\vec{k}$

2) Find the torsion

(2) Find the equations of the osculating plane, normal plane and rectifying plane.

SOL

(1) $\vec{v} = (-\sin\ t)\vec{\imath} + (\cos\ t)\vec{\jmath} + \vec{k}$

$|\vec{v}| = \sqrt{\sin^2\ t + \cos^2\ t + 1} = \sqrt{2}$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{-1}{\sqrt{2}}\sin\ t\right)\vec{\imath} + \left(\frac{1}{\sqrt{2}}\cos\ t\right)\vec{\jmath} + \left(\frac{1}{\sqrt{2}}\right)\vec{k}$$

$$\vec{T}(0) = \left(\frac{1}{\sqrt{2}}\right)\vec{\jmath} + \left(\frac{1}{\sqrt{2}}\right)\vec{k}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{-1}{\sqrt{2}}\cos\ t\right)\vec{\imath} - \left(\frac{1}{\sqrt{2}}\sin\ t\right)\vec{\jmath} \Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = (-\cos\ t)\vec{\imath} - (\sin\ t)\vec{\jmath}$$

$$\vec{N}(0) = -\vec{\imath}$$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = \frac{-1}{\sqrt{2}}\vec{\jmath} + \frac{1}{\sqrt{2}}\vec{k}$$

$$\vec{r}(0) = \vec{\imath}$$

(2) $\vec{a} = (-\cos\ t)\vec{\imath} - (\sin\ t)\vec{\jmath}$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin\ t & \cos\ t & 1 \\ -\cos\ t & -\sin\ t & 0 \end{vmatrix} = (\sin\ t)\vec{\imath} - (\cos\ t)\vec{\jmath} + \vec{k}$$

$|\vec{v} \times \vec{a}| = \sqrt{\sin^2\ t + \cos^2\ t + 1} = \sqrt{2}$

$$\frac{d\vec{a}}{dt} = (\sin\ t)\vec{\imath} - (\cos\ t)j$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \dot{y} & \ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} -\sin\ t & \cos\ t & 1 \\ -\cos\ t & -\sin\ t & 0 \\ \sin\ t & -\cos\ t & 0 \end{vmatrix}}{2} = \frac{1}{2}$$

3) Since $\vec{r}(0) = \vec{\imath} \Rightarrow$ The point is $\boldsymbol{P}(1,0,0)$

$\vec{B}(0) = \frac{-1}{\sqrt{2}}\vec{\jmath} + \frac{1}{\sqrt{2}}\vec{k} \perp$ osculating plane

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

The equation for the osculating plane is $\frac{-1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0$

$y - z = 0$ is the osculating plane.

$\vec{T}(0) = \left(\frac{1}{\sqrt{2}}\right)\vec{\jmath} + \left(\frac{1}{\sqrt{2}}\right)\vec{k} \perp$ normal plane $\Rightarrow$ The equation of the normal plane is $\frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0$

$y + z = 0$ is the normal plane

$\vec{N}(0) = -\vec{\imath} \perp$ rectifying plane

$\Rightarrow$ The equation of the rectifying plane is $-(x - 1) = 0$

$x = 1$ is the rectifying plane

H.W.(5)

1 Find the Practical's velocity and acceleration vectors for the motion on the circle $r(t) = (\sin\ t)i + (\cos\ t)j$, where $t = \pi/4$ and $\pi/2$.

2 Find the Practice's velocity and acceleration vector for $r(t) = \left(4\cos\ \frac{t}{2}\right)i + \left(4\sin\ \frac{t}{2}\right)j$ where $t = \pi$ and $t = 3\pi/2$.

3 Find the velocity and speed and acceleration for $r(t) = (t + 1)i + (t^2 - 1)j + 2tk$, where $t = 1$.

4 Find the Continuity and Limit of

$$r(t) = (\sin\ t)i + (t^2 - \cos\ t)j + e^t k, \text{ where } t_0 = 0$$

5 Find the derivative of $r(t) = (\ln\ t)i + \left(\frac{t-1}{t+2}\right)\delta + (t\ln\ t)k$

6-Evaluate the integral

$$\int_0^1 [t^3 i + 7j + (t + 1)k]dt$$

7 Evaluate the integreel of

$$r(t) = (\sin t)i + (1 + \cos t)j + (\sec^2 - 1)k \quad \text{where} \quad -\frac{\pi}{4} \leqslant t \leq \frac{\pi}{4}.$$

8 Solve the initial value for r as avector function of t differential function. $\frac{dr}{dt} = -ti - tj - tk$ initial Condition: $r(0) = i + 2\hat{j} + 3k$.

9 Aspring gun at ground level fires a golf ball at an angle 45°. the ball lands 10 m away
a. What wass ball initial speed?
b. for the Same initial speed find the firing angle that make the range $6m$? Find flight time - and Maximum height.

10 Find the curve's unit tangent vector. Also, find the are length of the curve

$$r(t) = (t\cos\ t)i + (t\sin\ t)j + \left((2\sqrt{2}/3)t^3/2\right)1 < ,0 \leqslant t \leqslant \pi$$

11 Find the are length Parametrization of a curve at $t_o = a$ for the curve $r(t) = (\cos\ 4t)i + (\sin\ 4t)j + 4tk$

12 Find the T and N and K for the curves
a. $r(t) = (\ln\ \sec\ t)i + tj,\ -\pi/2 < t < \pi/2$
b. $r(t) = \frac{ti}{} + (\ln \cos\ t)_j,\ -\pi/2 < t < \pi/2$
c. $r(t) = (e^t \cos\ t)i + (e^t \sin\ t)_j + 2k$
13. Write acceleration in the form $a = a_T T + a_N N$ with out finding T and N $r(t) = (1 + 3t)i + (t - 2)j - 3tK$
14. Find c, T, N, and B at the given value of t Where $r(t) = (\cos\ t)i + (\sin\ t)j - k,\ t = \pi/4$