H.W.(8)

1) Area by Double Integrals sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$

2. The lines $x = 0$, $y = 2x$, and $y = 4$

3. The parabola $x = -y^2$ and the line $y = x + 2$

4. The parabola $x = y - y^2$ and the line $y = -x$

2) Give the areas of regions in the xy-plane:

$$\int_0^6 \int_{y^2/3}^{2y} dx\, dy \qquad \textbf{14.} \int_0^3 \int_{-x}^{x(2-x)} dy\, dx$$

3) Finding Average Values :

1) Find the average value of ƒ(x,y)=sin(x+y) over

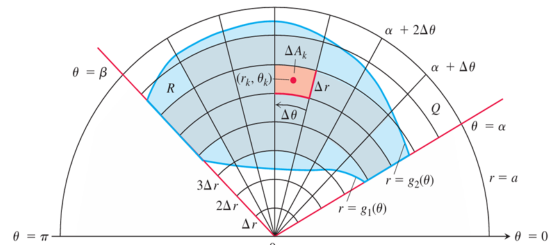
a. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$.

b. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$.

2) Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

8. Double Integrals in Polar Form

How to construct double integral?



Assume $f(r, \theta)$ is defined on region

R: $g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \leqslant \beta$ contained in the region

Q: $0 \le r \le a, \alpha \le \theta \le \beta$

where $g_1(\theta)$ and $g_2(\theta)$ are continuous curves: $0 \le g_1(\theta) \le g_2(\theta) \le a$.

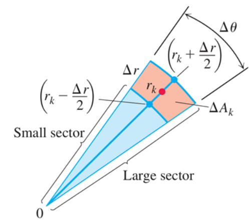
- We cover Q by a grid of circular ares and rays.

  The arcs are cut from circles centered at origin with radii $\Delta r, 2\Delta r, 3\Delta r, \dots, m\Delta r$ where $\Delta r = \frac{a}{m}$

  The rays are: $\theta = \alpha, \theta = \alpha + \Delta\theta, \theta = \alpha + 2\Delta\theta, \dots, \theta = \alpha + \text{m}'\Delta\theta = B$
- The arcs and rays partition Q into small polar rectangles.
- We number the polar rectangles that lie inside R with areas:

$$\Delta A_1, \Delta A_2, \dots, \Delta A_n$$

- Let (r_k, θ_k) be any point in the k^{th} polar rectangle whose area is ΔA_k. Then $S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$
  If f is continuer, then this sum will approach a limit and as $\Delta r \to 0$ and $\Delta\theta \to 0$, the limit is called the double integral of f over R: $\lim_{n\to\infty} S_n = \iint_R f(r,\theta)dA$.
- To find ΔA_k : Recall that
  the area of a circular **sector** is $A = \frac{1}{2}\theta r^2$



- We choose r_k to be the average of the radii of the inner and outer arcs bounding the k^{th} polar rectangle.

Inner radius: $\frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$

Outer radius: $\frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$

- $\Delta A_k = A_L - A_s = \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\Delta r) = r_k\Delta r\Delta\theta$

Hence, $S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta\theta$.

- As $n \to \infty$: $\lim_{n\to\infty} s_n = \iint_R f(r,\theta) r dr d\theta$

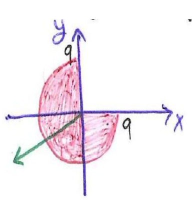
Thus $\int_R f(r,\theta)dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta) r dr d\theta$

- Note that if *f(r, θ) is the constant function whose value is 1* then the integral of f over R is the area in Polar coordinate. That is The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r dr d\theta$$

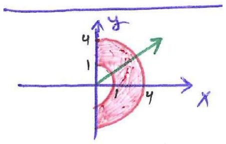
Example:

Describe the given region in polar coordinates:

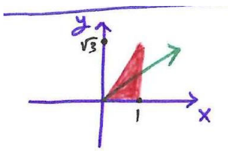


$$x^2 + y^2 = 9^2 \Leftrightarrow r^2 = 9^2 \Leftrightarrow r = 9$$

$$sol: 0 \leqslant r \leqslant 9, \quad \frac{\pi}{2} \leqslant \theta \leqslant 2\pi$$



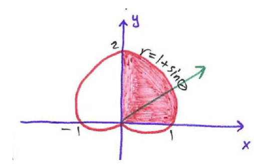
$$x^2 + y^2 = 1^2 \quad \Leftrightarrow r = 1$$
$$x^2 + y^2 = 4^2 \quad \Leftrightarrow r = 4$$
$$sol: 1 \leqslant r \leqslant 4 \quad , \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$



$$x = 1 \Leftrightarrow r\cos\ \theta = 1 \Leftrightarrow r = \sec\ \theta$$
$$y = \sqrt{3}, x = 1 \Leftrightarrow \tan\ \theta = \sqrt{3} \ \Leftrightarrow \theta = \tfrac{\pi}{3}$$
$$0 \leqslant r \leq \sec\theta\,, \quad 0 \leq \theta \leq \tfrac{\pi}{3}$$

Example:

Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\ \theta$



$$A = \int_0^{\frac{\pi}{2}} \int_0^{1+\sin\ \theta} r dr d\theta = \frac{3\pi}{8} + 1$$

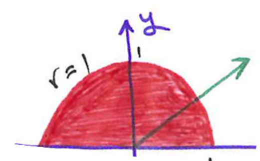
Changing Cartesian Integrals into Polar Integrals:

$$\iint_R f(x,y)dxdy = \iint_G f(r\cos\ \theta, r\sin\ \theta) r dr d\theta$$

That is replace (I) x by $r\cos\ \theta$
(2) y by $\mathrm{r}\sin\theta$
(3) $dxdy$ by $rdrd\theta$

Example:

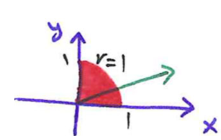
Find $\iint_R e^{x^2+y^2} dydx$, where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.



$$\iint_R e^{x^2+y^2} dydx = \int_0^{\pi}\int_0^1 e^{r^2} r dr d\theta = \frac{\pi(e-1)}{2}$$

Example:

Find $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dydx = \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3} \right) dx$



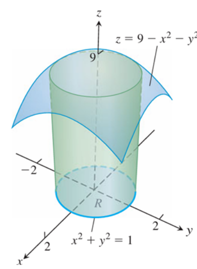
we can integrate this by trigonometric substitution $x = \sin\theta$ .. but it will take some times. However if we change to polar:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dydx \quad = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 r dr d\theta \quad \text{since} \quad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{matrix}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{4} = \frac{\pi}{8}$$

Example:

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



Solution:

The region of integration R is the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by r=1,$0 \le \theta \le 2\pi$. The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

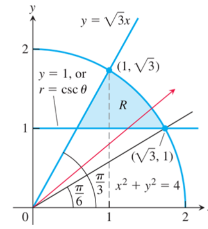
Example:

Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line y=1, and below the line $y = \sqrt{3}\, x$.

Sol:

The given circle equation is: $x^2 + y^2 = 4$

The given lines are:



1. y = 1, which in polar form is r = 1/sin(θ).

2. y = √3x, which corresponds to θ = π/3.

Intersection of y = 1 with the circle ➔ $x^2 = 3$➔ $2\cos\theta = \sqrt{3}$➔ θ = π/6.

Then the angle θ varies from θ = π/6 to θ = π/3 (line y = √3x).

The radial limit r varies from 1/sin(θ)=$\csc\theta$ to 2.

$$\iint_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{1}{2} r^2\right]_{r=\csc\theta}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}[4 - \csc^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.$$

-The average value of f over R is $\text{av}(f) = \frac{1}{\text{Area }(R)} \iint_R f(r,\theta) r dr d\theta$

Example:

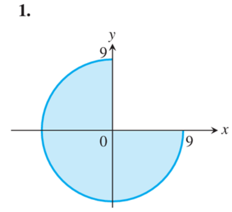
Find the average height of the surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane.

averge height $= \frac{4}{a^2\pi} \int_0^{\frac{\pi}{2}} \int_0^a r\sqrt{a^2 - r^2} dr d\theta = \frac{4}{3\pi a^2} \int_0^{\frac{\pi}{2}} a^3 d\theta = \frac{2a}{3} R$

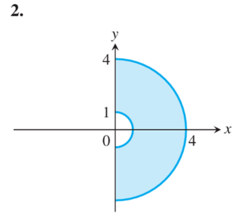
H.W. (9)

1. Describe the given region in polar coordinates.:

1.



2.



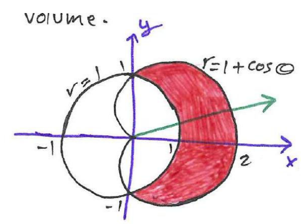
2. Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$$

$$\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,d$$

3,Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

4. The region that lies inside the cardioid $r = 1 + \cos\ \theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.



5. Find the average distance from a point P(x,y) in the disk $x^2 + y^2 \leq a^2$ to the origin.

6. Evaluate the integral $\iint_R (x^2 + y^2)^{-2}\,dA$, where R is the region inside the circle $x^2 + y^2 = 2$ for $x \leq -1$.