5. Double and Iterated Integrals over Rectangles

How to construct double integral?

- Assume $f(x,y)$ is defined on a rectangular region R

$$R: a \le x \le b, c \le y \le d$$

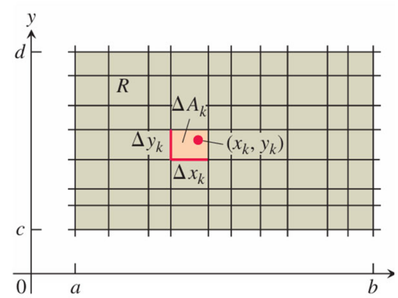


Figure 1: Rectangular grid partitioning the region R into small rectangles of area ΔAk = Δxk Δyk

- Divide R into n small rectangulars with width Δx and height Δy
- Each small rectangle has area $\Delta A = \Delta x \Delta y$
- These n small rectangles form a partition of R and the number n gets large as Δx and Δy become smaller.
- If we order the areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_k, \ldots, \Delta A_n$ and in each ΔA_K "small rectangle" we choose a point (x_k, y_k) and evaluate $f(x_k, y_k)$ "hieght", then the Riemann sum over R is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

- As $\Delta x \to 0$ and $\Delta y \to 0$ the norm of the partition $\|p\| \to 0$ Hence, $n \to \infty$, where $\|p\| = \max\{\Delta x, \Delta y\}$ for any rectangle.
- Therefore, $\lim_{\|p\| \to 0} S_n = \lim_{\|p\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$
- If the limit exists, then its called double integral:

$$\iint_R f(x,y)dA \text{ or } \iint_R f(x,y)dxdy$$

  and the function f is said to be integrable.

- The volume of the resulting solid is

$$V = \iint_R f(x,y)dA \text{ if } f(x,y) \text{ is continuous.}$$

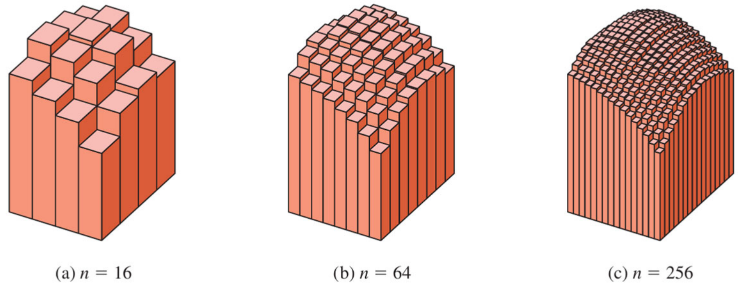


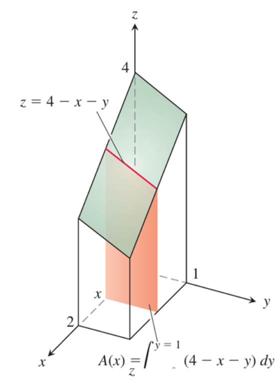
Figure 2: As n increases, the Riemann sum approximations approach the total volume of the solid

Fubini's Theorem for Calculating Double integrals :

Example:

Find the volume under the plane $z = 4 - x - y$ over the region $R: 0 \leq x \leq 2, 0 \leq y \leq 1$.

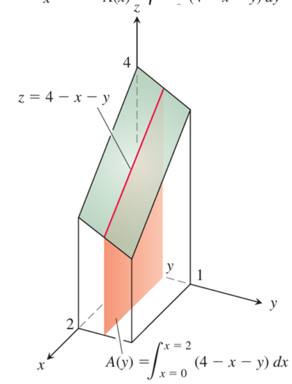
If cross section $\perp$ x-axis is taken, then the volume is $V = \int_{x=0}^{x=2} A(x)dx$ where



$$A(x) = \int_{y=0}^{y=1} (4 - x - y)dy$$

$$= \int_0^2 \int_0^1 (4 - x - y)dydx = 5$$

If cross section $\perp$ y-axis is taken, then the volume is $V = \int^{y=1} A\,(y)dy$ where



$$= \int_0^{y=0} \int_0^2 (4 - x - y)dxdy = 5$$

Fubini's Theorem -First form

If $f(x, y)$ is continous on rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

Example:

Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1, 0 \le y \le 1$.

$$V = \int_0^1 \int_0^1 (2 - x - y)dxdy = \int_0^1 \left(2x - \frac{x^2}{2} - yx\right)\Big|_{x=0}^{x} dy$$

$$= \int_0^1 \left(\frac{3}{2} - y\right) dy = \frac{3}{2}y - \frac{y^2}{2}\Big|_0^1 = 1$$

Example:

Calculate the double integral: $\iint_R f(x,y)\, dA$ Given: f(x,y) = 100 - $6x^2y$ Over the region: $0 \le x \le 2$, $-1 \le y \le 1$

Solution:

$$\iint_R f(x,y)\, dA = \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1}\Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400.$$

Example:

Find the volume of the region bounded above by the surface $z = 2\sin\ x\cos\ y$ and below by the rectangle $R: 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{4}$.

$$V=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} 2\sin\ x\cos\ y dy dx=\int_0^{\frac{\pi}{2}}\left(2\sin\ x\sin\ y|_{y=0}^{y=\frac{\pi}{4}}\right)dx$$
$$=\int_0^{\frac{\pi}{2}}\sqrt{2}\sin\ x dx=-\sqrt{2}\cos\ x|_0^{\frac{\pi}{2}}=\sqrt{2}$$

H.W. (6)

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $\int_1^2\int_0^4 2xy\,dy\,dx$

2. $\int_0^2\int_{-1}^1 (x-y)\,dy\,dx$

3. $\int_{-1}^0\int_{-1}^1 (x+y+1)\,dx\,dy$

4. $\int_0^1\int_0^1 \left(1-\frac{x^2+y^2}{2}\right)dx\,dy$

5. $\int_0^3\int_0^2 (4-y^2)\,dy\,dx$

6. $\int_0^3\int_{-2}^0 (x^2y-2xy)\,dy\,dx$

7. $\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$

8. $\int_1^4\int_0^4 \left(\frac{x}{2}+\sqrt{y}\right)dx\,dy$

9. $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$

10. $\int_0^1\int_1^2 xye^x\,dy\,dx$

11. $\int_{-1}^2\int_0^{\pi/2} y\sin x\,dx\,dy$

12. $\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)\,dx\,dy$

13. $\int_1^4\int_1^e \frac{\ln x}{xy}\,dx\,dy$

14. $\int_{-1}^2\int_1^2 x\ln y\,dy\,dx$

Evaluating Double Integrals over Rectangles

In Exercises 15–22, evaluate the double integral over the given region R.

15. $\iint_R (6y^2-2x)\,dA,\quad R:\ 0\le x\le 1,\ 0\le y\le 2$

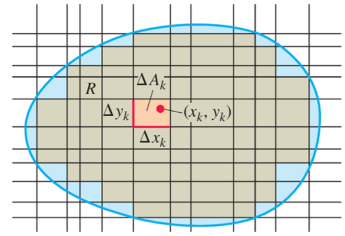
16. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right)dA,\quad R:\ 0\le x\le 4,\ 1\le y\le 2$

17. $\iint_R xy\cos y\,dA,\quad R:\ -1\le x\le 1,\ 0\le y\le \pi$

18. $\iint_R y\sin(x+y)\,dA,\quad R:\ -\pi\le x\le 0,\ 0\le y\le \pi$

6. Double Integrals over General Regions (Bounded, Nonrectangular Regions)

In this section we define and evaluate double integrals over bounded regions in the plane which are more general than rectangles. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants. To define the double integral of a function f(x,y) over a bounded, nonrectangular region R, such as the Figure.



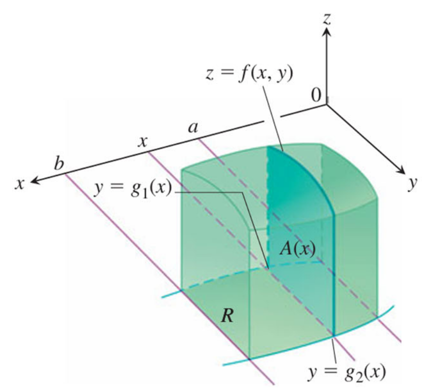
By covering R with a grid of small rectangular cells whose union contains all points of R. However, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved, and some of the small rectangles in the grid lie partly outside R.

We takes the rectangles that lie completely inside R, not using any that are either partly or completely outside. To get approximately R by a grid of very small rectangles that are completely inside R.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum.

$$\iint_R f(x,y)dA = \lim_{\|p\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

- If $z = f(x,y)$ is positive continuous, If R is a region like the one shown in the xy-plane in Figure bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines x = a, x = b, we may again calculate the volume by the method of slicing. We first calculate the cross sectional area

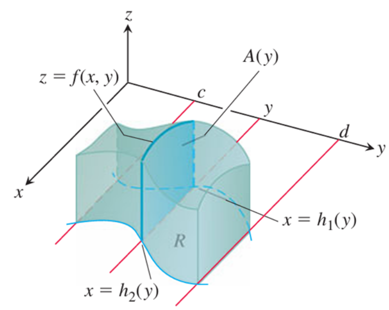


$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Similarly, if R is a region like the one shown in Figure, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines y=c and y=d, then the volume calculated by slicing is given by the iterated integral



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Fubini's Theorem - stronger form

Let $f(x,y)$ be a continuous on region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

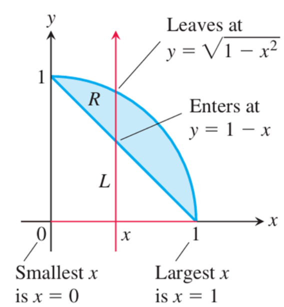
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Finding Limits of Integration

Using Vertical Cross-Sections

When $\iint_R f(x,y)dA$ integrating first with respect to y and then with respect to x:



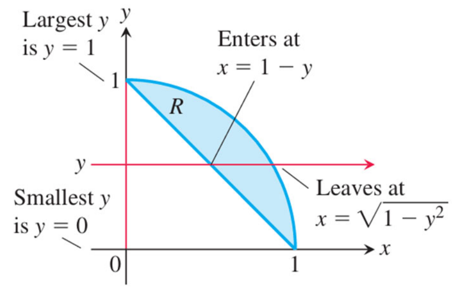
1. Sketch the region of integration and label the bounding curves.

2. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants

3. Find the x-limits of integration. Choose x-limits that include all the vertical lines through R.

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

Using Horizontal Cross-Sections



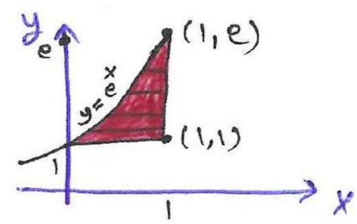
To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3

$$\iint_R f(x,y)\,dA = \int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

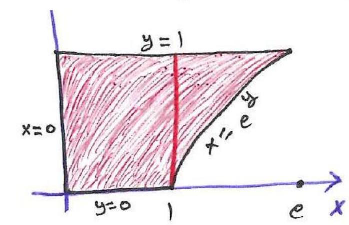
Example:

sketch the region of integration and write an equivalent double integral with the order of integration reversed for

(1) $\int_0^1 \int_1^{e^x} dydx = \int_1^e \int_{\ln y}^1 dxdy$

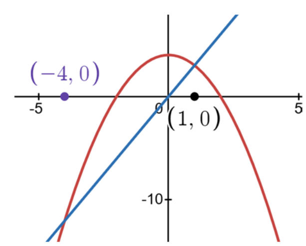


(2) $\int_0^1 \int_0^{e^y} dxdy = \int_0^1 \int_0^1 dydx + \int_0^e \int_{\ln x}^1 dydx$



Example:

Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$



$$4 - x^2 = 3x \Leftrightarrow x^2 + 3x - 4 = 0 \Leftrightarrow x = 1 \text{ or } x = -4$$

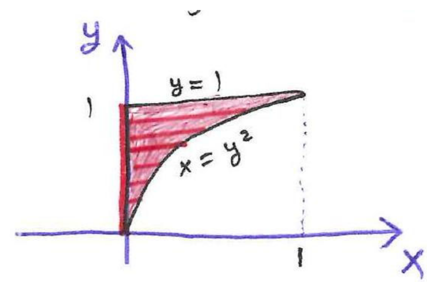
$$V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)dydx = \int_{-4}^{1} (x+4)(4-3x-x^2)dx = \frac{625}{12}$$

Example:

(1) Sketch the region of integration and evaluate the integral

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy = \int_0^1 3y^2 e^{xy}\Big|_{x=0}^{x=y^2} dy$$

$$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2\right)dy = e^{y^3} - y^3\big|_0^1 = e - 2$$



.

(2) reverse the order of integration: $x = y^2$ ➔ then in first quarter $y = \sqrt{x}$

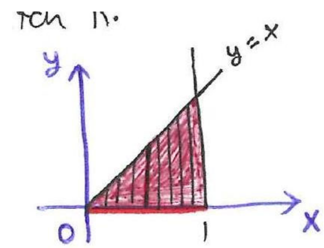
$$\int_0^1 \int_{\sqrt{x}}^1 3y^3 e^{xy} dydx$$

Example:

Evaluate $\iint_R \frac{\sin x}{x} dA$ where R is triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$. sketch R.

$$\int_0^1 \int_0^x \frac{\sin x}{x} dydx = \int_0^1 \mathrm{y}\frac{\sin x}{x}\Big|_0^x dx = \int_0^1 \sin x \, dx$$

$$= 1 - \cos(1)$$



(2) reverse the order of integration $\int_0 \int_y^1 \frac{\sin x}{x} dxdy$ (difficult to integrate)

Properties of Double Integrals:

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following is true:

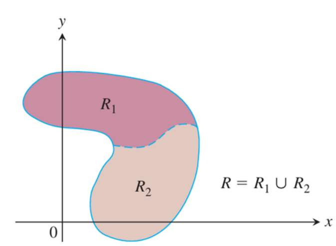
(1) Constant Multiple: $\iint_R cf(x, y)dA = c \iint_R f(x, y)dA, c \in \mathbb{R}$.

(2) Sum and Difference: $\iint_R (f(x, y) \pm g(x, y))dA = \iint_R f(x, y)dA \pm \iint_R g(x, y)dA$

(3) Domination: [a] If $f(x, y) \geqslant 0$ on R, then $\iint_R f(x, y)dA \geqslant 0$.

[b] If $f(x, y) \geq g(x, y)$ on R, then $\iint_R f(x, y)dA \geq \iint_R g(x, y)dA$

(4) Additivity: $\iint_R f(x, y)dA = \iint_{R_1} f(x, y)dA + \iint_{R_2} f(x, y)dA$ if R is the union of two nonover lapping regions R_1 and R_2. $R = R_1 \cup R_2$



Example:

Evaluate the improper integral over the unbounded region R: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdy}{(x^2+1)(y^2+1)}$

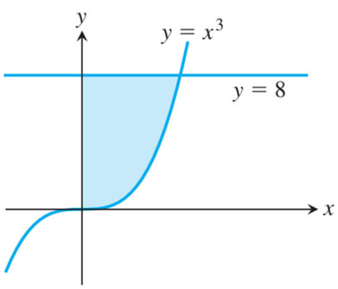
$$=\int_{-\infty}^{\infty}2\int_{0}^{\infty}\frac{dxdy}{(x^2+1)(y^2+1)}=2\int_{-\infty}^{\infty}\left(\lim_{b\to\infty}\tan^{-1}\ x\Big|_{x=0}^{x=b}\right)\frac{dy}{y^2+1}\frac{dx}{(x^2+1)(y^2+1)}\quad y=\frac{1}{x^2+1}x$$

$$=2\int_{-\infty}^{\infty}\frac{\pi}{x}\frac{dy}{y^2+1}=2\pi\int_{0}^{\infty}\frac{dy}{y^2+1}=2\pi\lim_{b\to\infty}\tan^{-1}\ y\Big|_{0}^{\infty}=2\pi\left(\frac{\pi}{2}\right)=\pi^z$$

H.W. (7)

1) Finding Limits of Integration: write an iterated integral for 4R dA over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

9.



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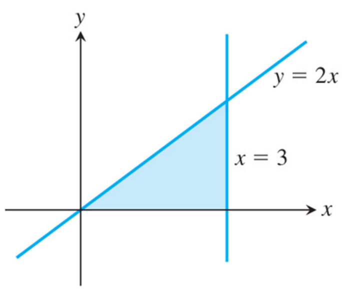

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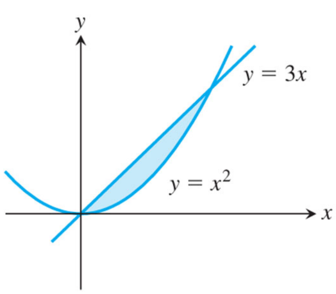

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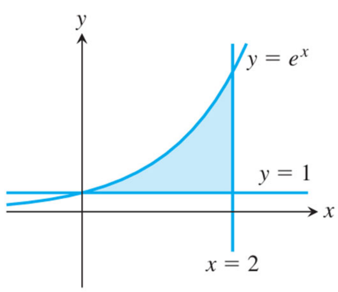


13. Bounded by y= tanx, x=0, and y=1

2) Sketch the region of integration and evaluate the integral.

21. $\int_{1}^{\ln 8}\int_{0}^{\ln y} e^{x+y}\,dx\,dy$

22. $\int_{1}^{2}\int_{y}^{y^2} dx\,dy$

23. $\int_{0}^{1}\int_{0}^{y^2} 3y^3 e^{xy}\,dx\,dy$

24. $\int_{1}^{4}\int_{0}^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx$

3) Reversing the Order of Integration

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$$

$$\int_1^e \int_0^{\ln x} xy\, dy\, dx$$

4) Integrals over Unbounded Regions

$$\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\, dy\, dx$$

70. $$\int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1)\, dy\, dx$$

5) Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

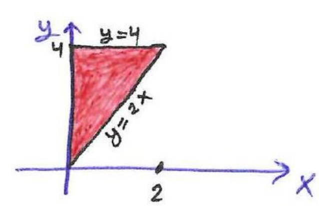
7. Area by Double Integration

- Recall the Riemann sum in the definition of a double integral $S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$.
- If we Let $f(x, y) = 1$, then the Riemann sum becomes $S_n = \sum_{k=1}^{n} \Delta A_k$ which is the sum of the areas of small rectangles * We define the area of a closed, bounded plane region R by:

$$A = \lim_{\|\rho\|\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA$$

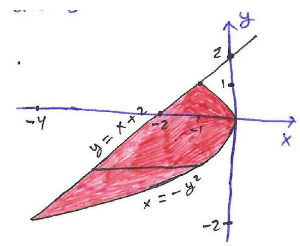
Example:

Find the area of the region R bounded by $x = 0, y = 2x$ and $y = 4$. (1)



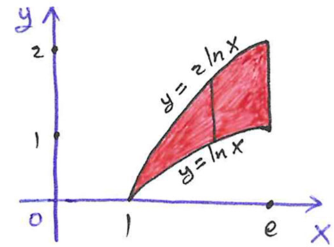
$$A = \int_0^2 \int_{2x}^4 dydx = \int_0^2 (y - 2x)dx = 4 = \int_0^4 \int_0^{\frac{y}{2}} dxdy$$

(2) The parabola $x=-y^2$ and the line $y=x+2$.



$$A=\int_{-2}^{1}\int_{y-2}^{-y^2}dxdy=\int_{-2}^{1}(2-y-y^2)dy$$
$$=\frac{9}{2}$$

(3) The curve $y=\ln\ x$ and $y=2\ln\ x$ and the line $x=e$



$$A=\int_{1}^{e}\int_{\ln\ x}^{2\ln\ x}dydx=\int_{1}^{e}\ln\ xdx$$
$$=(x\ln\ x-x)|_1^e=1$$

Average Value:

- Remember that the average value of an integrable function of one variable, $f(x)$, on closed interval $[a,b]$ is $\text{av}(f)=\frac{1}{b-a}\int_a^b f(x)dx$.
- We now define the average value of an integrable function f over a region R as follows:

$$\text{av}(f)=\frac{1}{A}\iint_R f dA, \text{where } A \text{ is the area } R.$$

Example:

Find the average value of $f(x,y)=\frac{1}{xy}$ over the square

$$\ln 2\le x\le 2\ln 2\,,\qquad \ln\ 2\le y\le 2\ln\ 2.$$
$$A=\int_{\ln\ 2}^{2\ln\ 2}\int_{\ln\ 2}^{2\ln\ 2}dydx=\int_{\ln\ 2}^{2\ln\ 2}\ln\ 2dx=(\ln\ 2)^2$$
$$av(f)=\frac{1}{(\ln\ 2)^2}\int_{\ln\ 2}^{2\ln\ 2}\int_{\ln\ 2}^{2\ln\ 2}\frac{dydx}{xy}=\frac{1}{(\ln\ 2)^2}\int_{\ln\ 2}^{2\ln\ 2}\frac{1}{x}\left[\ln\ y|_{y=\ln\ 2}^{y=2\ln\ 2}\right]dx$$
$$=\frac{1}{(\ln\ 2)^2}\int_{\ln\ 2}^{2\ln\ 2}\frac{\ln\ 2}{x}dx=\frac{1}{\ln\ 2}\int_{\ln\ 2}^{\ln\ 2}\frac{dx}{x}$$
$$=\frac{1}{\ln\ 2}\left[\ln\ x|_{\ln\ 2}^{\ln\ 2}\right]=\frac{\ln\ 2}{\ln\ 2}=1$$

Example:

Find the average value of $f(x,y)=x\cos\ xy$ over the rectangle R ! $0\leqslant x\leqslant\pi,\ 0\leqslant y\leqslant 1$.

SoI.

Average value of f over $R = \frac{1}{\text{area of } R} \iint_R f dA$

(1)

$$\begin{aligned} \text{area of } R \quad &= A = \iint_R dA \\ &= \int_0^{\pi} \int_0^{1} dy\, dx = \int_0^{\pi} [y]_0^1 dx \\ &= \int_0^{\pi} dx = [x]_0^{\pi} = \pi \end{aligned}$$

(2)

$$\begin{aligned} \iint_R f dA \quad &= \int_0^{\pi} \left(\int_0^{1} x_0^x \cos\ xy^{\cos} dy \right) dx \\ &= \int_0^{\pi} [\sin\ xy]_0^1 dx = \\ &= \int_0^{\pi} (\sin\ x - \sin\ 0) dx = \int_0^{\pi} \sin\ x dx \\ &= [-\cos\ x]_0^{\pi} = -\cos_{-1}^{\pi} - (-\cos\ 0_{\pi}^{1}) = 1 + 1 = 2 \end{aligned}$$

$\therefore$ A verage value of f over $R = \frac{1}{\pi} \cdot 2 = \frac{2}{\pi}$