

Lecture 1

chapter 1

Introduction

Numerical analysis is concerned with the development and analysis of methods for the numerical solution of practical problems. Traditionally, these methods have been mainly used to solve problems in the physical sciences and engineering. However, they are finding increasing relevance in a much broader range of subjects including economics and business studies.

The first stage in the solution of a particular problem is the formulation of a mathematical model. Mathematical symbols are introduced to represent the variables involved and physical (or economic) principles are applied to derive equations which describe the behavior of these variables. Unfortunately, it is often impossible to find the exact solution of the resulting mathematical problem using standard techniques. In fact, there are very few problems for which an analytical solution can be determined. For example, there are formulas for solving quadratic, cubic and quartic polynomial equations, but no such formula exists for polynomial equations of degree greater than four or even for a simple equation such as

$$x = \cos(x)$$

Similarly, we can certainly evaluate the integral

$$A = \int_a^b e^x dx = e^x \Big|_a^b$$

as $e^b - e^a$, but we cannot find the exact value of

$$A = \int_a^b e^{x^2} dx$$

since no function exists which differentiates to e^{x^2} . Even when an analytical solution can be found it may be of more theoretical than practical use. For example, if the solution of a differential equation

$$y'' = f(x, y, y')$$

is expressed as an infinite sum of Bessel functions, then it is most unsuitable for calculating the numerical value of y corresponding to some numerical value of x .

Errors

Computations generally yield approximations as their output. This output may be an approximation to a true solution of an equation, or an approximation of a true value of some quantity. Errors are commonly measured in one of two ways: absolute error and relative error as the following definition.

Definition 1. If x_A is an approximation to x_T , the **error** is defined as

$$err(x_A) = x_T - x_A \quad (1)$$

The **absolute error** is defined as

$$Aerr(x_A) = |err(x_A)| = |x_T - x_A| \quad (2)$$

And the **relative error** is given by

$$rel(x_A) = \frac{\text{Absolute error}}{\text{True value}} = \frac{|x_T - x_A|}{|x_T|}, \quad x_T \neq 0 \quad (3)$$

Note that if the true value happens to be zero, $x = 0$, the relative error is regarded as undefined. The relative error is generally of more significance than the absolute error.

Let $x_T = \frac{19}{7} \approx 2.714285$ and $x_A = 2.718281$. Then

$$\text{err}(x_A) = x_T - x_A = \frac{19}{7} - 2.718281 \approx -0.003996$$

$$\text{Aerr}(x_A) = |\text{err}(x_A)| \approx 0.003996$$

$$\text{rel}(x_A) = \frac{\text{Aerr}(x_A)}{x_T} = \frac{0.003996}{2.7142857} \approx 0.00147$$

Example 0.1. Consider the following table

x_T	x_A	Absolute Error	Relative Error
1	0.99	0.01	0.01
1	1.1	0.01	0.01
-1.5	-1.2	0.3	0.2
100	99.99	0.01	0.0001
100	99	1	0.01

Example 0.2. Consider two different computations. In the first one, an estimate $x_A = 0.003$ is obtained for the true value $x_T = 0.004$. In the second one, $y_A = 1238$ for $y_T = 1258$. Therefore, the absolute errors are

$$\text{Aerr}(x_A) = |x_T - x_A| = 0.001, \quad \text{Aerr}(y_A) = |y_T - y_A| = 20$$

The corresponding relative errors are

$$\text{rel}(x_A) = \frac{\text{Aerr}(x_A)}{x_T} = \frac{0.001}{0.004} = 0.25,$$

$$\text{rel}(y_A) = \frac{\text{Aerr}(y_A)}{y_T} = \frac{20}{1258} = 0.0159$$

We notice that the absolute errors of 0.001 and 20 can be rather misleading, judging by their magnitudes. In other words, the fact that 0.001 is much smaller than 20 does not make the first error a smaller error relative to its corresponding computation.

In fact, looking at the relative errors, we see that 0.001 is associated with a 25% error, while 20 corresponds to 1.59% error, much smaller than the first. Because they convey a more specific type of information, relative errors are considered more significant than absolute errors.