Lecture 4

chapter 2

Solutions of Equations in One Variable

One of the fundamental problems of mathematics is that of solving equations of the form

$$f(x) = 0 \tag{4}$$

where f is a real valued function of a real variable x. Any number α satisfying equation (4) is called a **root** of the equation or a zero of f.

Most equations arising in practice are non-linear and are rarely of a form which allows the roots to be determined exactly. Consequently, numerical techniques must be used to find them.

Graphically, a solution, or a root, of Equation (4) refers to the point of intersection of $f(x)$ and the $x-$axis. Therefore, depending on the nature of the curve of $f(x)$ in relation to the $x-$axis, Equation (4) may have a unique solution, multiple solutions, or no solution. A root of an equation can sometimes be determined analytically resulting in an exact solution. For instance, the equation $e^{2x} - 3 = 0$ can be solved analytically to obtain a unique solution $x = \frac{1}{2}\ln 3$. In most situations, however, this is not possible and the root(s) must be found using a numerical procedure.

Bisection Technique

This technique based on the Intermediate Value Theorem. Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. The Intermediate Value Theorem implies that a number p exists in (a, b) with

$f(p) = 0$. The method calls for a repeated halving of subintervals of $[a, b]$ and, at each step, locating the half containing p. To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}$$

1. If $f(p_1) = 0$, then $p = p_1$, and we are done.
2. If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
   - If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
   - If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$. See Figure 2.

We can select a tolerance $\epsilon > 0$ and generate $p_1, p_2, \cdots, p_N$ until one of the following conditions is met:

- $|p_N - p_{N-1}| < \epsilon$,
- $\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0,$ or
- $f(p_N) < \epsilon$,

When using a computer to generate approximations, it is good practice to set an upper bound on the number of iterations. This eliminates the possibility of entering an infinite loop, a situation that can arise when the sequence diverges (and also when the program is incorrectly coded).

Example 0.6. *The function* $f(x) = x^3 + 4x^2 - 10$ *has a root in* $[1, 2]$, *because* $f(1) = -5$ *and and* $f(2) = 14$ *the Intermediate*

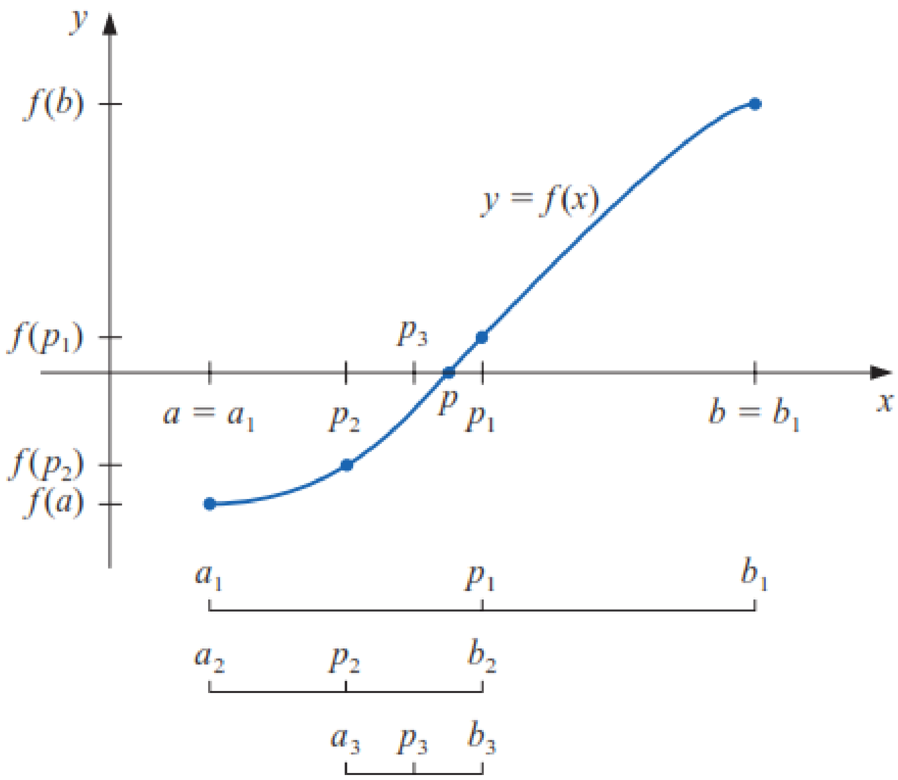

Figure 2: Produces of Bisection Technique

Value Theorem ensures that this continuous function has a root in $[1, 2]$*.*
Using Bisection method with the Matlab code to determine an approximation to the root.

Example 0.7. *The function* $f(x) = (x+1)^2 e^{(x^2-2)} - 1$ *has a root in* $[0, 1]$ *because* $f(0) < 0$ *and* $f(1) > 0$*. Use Bisection method to find the approximate root with* $\epsilon = 0.00001$*.*

MaTlab built-In Function fzero

The fzero function in MATLAB finds the roots of $f(x) = 0$ for a real function $f(x)$. FZERO Scalar nonlinear zero finding.

$X = FZERO(FUN, X_0)$ tries to find a zero of the function FUN near X_0, if X_0 is a scalar.
For example 0.6 use the following Matlab code:

```
clc
clear
fun = @(x) x.^3+4*x.^2-10; % function
x0 = 1; % initial point
x = fzero(fun,x0)
```

the resulte is:
$x = 1.365230013414097$

Theorem 0.8. *Suppose that $f \in C[a, b]$ and $f(a)f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with*

$$|p_n - p| < \frac{b-a}{2^n}, \qquad n \geq 1$$

Proof. For each $n \geq 1$, we have

$$b_1 - a_1 = \frac{1}{2}(b-a), \quad and \quad p_1 \in (a_1, b_1)$$

$$b_2 - a_2 = \frac{1}{2}\left[\frac{1}{2}(b-a)\right] = \frac{1}{2^2}(b-a), \quad and \quad p_2 \in (a_2, b_2)$$

$$b_3 - a_3 = \frac{1}{2}(b_2 - a_2) = \frac{1}{2^3}(b-a), \quad and \quad p_3 \in (a_3, b_3)$$

and so for the n step we can get

$$b_n - a_n = \frac{1}{2^n}(b-a), \quad and \quad p_n \in (a_n, b_n)$$

Since $p_n \in (a_n, b_n)$ and $|(a_n, b_n)| = b_n - a_n$ for all $n \geq 1$, it follows that

$$|p_n - p| < b_n - a_n = \frac{b-a}{2^n}$$

the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with rate of convergence of order $\frac{1}{2^n}$; that is

$$p_n = p + O\left(\frac{1}{2^n}\right)$$

$\square$

It is important to realize that Theorem 0.8 gives only a bound for approximation error and that this bound might be quite conservative. For example, this bound applied to the problem in Example 0.6 ensures only that

$$|p - p_9| < \frac{2-1}{2^9} = 0.001953125 \approx 2 \times 10^{-3}$$

but the actual error is much smaller:

$$\begin{aligned} |p - p_9| &\leq |1.365230013414097 - 1.365234375| \\ &\approx -0.000004361585903 \\ &\approx 4.4 \times 10^{-6} \end{aligned}$$

Example 0.9. *Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.*

Solution: *We we will use logarithms to find an integer N that satisfies*

$$\begin{aligned} |p - p_n| &< 2^{-N}(b_1 - a_1) \\ &= 2^{-N}(2-1) \\ &= 2^{-N} < 10^{-3} \end{aligned}$$

One can use logarithms to any base, but we will use base-10 logarithms because the tolerance is given as a power of 10. Since $2^{-N} < 10^{-3}$ implies that $\log_{10} 2^{-N} < \log_{10} 10^{-3} = -3$, we have

$$-N \log_{10} 2 < -3 \qquad and \qquad N > \frac{3}{\log_{10} 2} \approx 9.96$$

Hence, 10 iterations will ensure an approximation accurate to within 10^{-3}.