

Lecture 3

Definition 3. A method is said to suffer from **induced instability** if small errors present at one stage of the method lead to bad effect in subsequent stages to such final results are totally inaccurate.

Nearly all numerical methods involve a repetitive sequence of calculations and so it is inevitable that small individual rounding errors accumulate as they proceed. However, the actual growth of these errors can occur in different ways. If, after n steps of the method, the total rounding error is approximately $Cn\epsilon$, where C is a positive constant and ϵ is the size of a typical rounding error, then the growth in rounding errors is usually acceptable. For example, if $C = 1$ and $\epsilon = 10^{-11}$, it takes about 50000 steps before the sixth decimal place is affected. On the other hand, if the total rounding error is approximately $Ca^n\epsilon$ or $Cn!\epsilon$, for some number $a > 1$, then the growth in rounding errors is usually unacceptable. For example, in the first case, if $C = 1$, $\epsilon = 10^{-11}$ and $a = 10$, it only takes about five steps before the sixth decimal place is affected. The second case is illustrated by the following example.

Example 0.5. Many successful algorithms are available for calculating individual real roots of polynomial equations of the form

$$p_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

Some of these are described later. An attractive idea would be to use these methods to estimate one of the real roots, α say, and then to divide $P_n(x)$ by $x - \alpha$ to produce a polynomial of degree $n - 1$ which contains the remaining roots. This process can then be repeated until all of the roots have been

located. This is usually referred to as the **method of deflation**. If α were an exact root of $P_n(x) = 0$, then the remaining $n - 1$ roots would, of course, be the zeros of the deflated polynomial of degree $n - 1$. However, in practice α might only be an approximate root and in this case the zeros of the deflated polynomial can be very different from those of $P_n(x)$. For example, consider the cubic

$$p_3(x) = x^3 - 13x^2 + 32x - 20 = (x - 1)(x - 2)(x - 10)$$

and suppose that an estimate of its largest zero is taken as 10.1. If we divide $p_3(x)$ by $x - 10.1$, the quotient is $x^2 - 2.9x + 2.71$ which has zeros $1.45 \pm 0.78i$. Clearly an error of 0.1 in the largest zero of $p_3(x)$ has induced a large error into the calculation of the remaining zeros.

It is interesting to note that if we divide $p_3(x)$ by $x - 1.1$, the corresponding quadratic has zeros 1.9 and 10.0 which are perfectly acceptable. The deflation process can be applied successfully provided that certain precautions are taken. In particular, the roots should be eliminated in increasing order of magnitude.

Of the two types of instability discussed, that of inherent instability is the most serious. Induced instability is a fault of the method and can be avoided either by modifying the existing method, as we did for some examples given in this section, or by using a completely different solution procedure. Inherent instability, however, is a fault of the problem so there is relatively little that we can do about it. The extent to which this property is potentially disastrous depends not only on the degree of ill-conditioning involved but also on the context from which the problem is taken.