طريقة النقطة الصامدة

2 Fixed-Point Iteration

A fixed point for a function is a number at which the value of the function does not change when the function is applied.

Definition . *The number p is a fixed point for a given function g if $g(p) = p$.*

Suppose that the equation $f(x) = 0$ can be rearranged as

$$x = g(x) \tag{2.2}$$

Any solution of this equation is called a fixed point of g. An obvious iteration to try for the calculation of fixed points is

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, \cdots \tag{2.3}$$

The value of x_0 is chosen arbitrarily and the hope is that the sequence $x_0, x_1, x_2, \cdots$ converges to a number α which will automatically satisfy equation (2.2).

Moreover, since equation (2.2) is a rearrangement of (2.1), α is guaranteed to be a zero of f.

$\boxed{f(x) = 0}$

In general, there are many different ways of rearranging $f(x) = 0$ in the form (2.2). However, only some of these are likely to give rise to successful iterations, as the following example demonstrates.

Example 1 . *Consider the quadratic equation*

$$x^2 - 2x - 8 = 0$$

with roots -2 and 4. Three possible rearrangements of this equation are

(a) $x_{n+1} = \sqrt{2x_n + 8}$ — g_1 ① $x^2 = 2x + 8 \Rightarrow x = \sqrt{2x+8}$

(b) $x_{n+1} = \frac{2x_n + 8}{x}$ — g_2 ② $\frac{x^2}{x} = \frac{2x+8}{x} \Rightarrow x = \frac{2x+8}{x}$

(c) $x_{n+1} = \frac{x_n^2 - 8}{2}$ — g_3 ③ $x = \frac{x^2 - 8}{2}$

$x^2 = 2x + 8$

Numerical results for the corresponding iterations, starting with $x_0 = 5$, are given in Matlab code 2.11 with the Table.

$g_1(x) = (2x+8)^{1/2} \rightarrow g_1'(x) = \frac{1}{\cancel{2}}(2x+8)^{-1/2} \cdot \cancel{2} = (2x+8)^{-1/2} \rightarrow$ is $|g_1'(5)| < 1$?

$g_2(x) = \frac{2x+8}{x} = 2 + \frac{8}{x} \rightarrow g_2'(x) = -\frac{8}{x^2} \longrightarrow |g_2'(5)| < 1$?

$g_3(x) = \frac{x^2 - 8}{2} \rightarrow g_3'(x) = \frac{\cancel{2}x}{\cancel{2}} = x$ (diverge)

Solution:

k	Xa	Xb	Xc
1	4.24264069	3.60000000	8.50000000
2	4.06020706	4.22222222	32.12500000
3	4.01502355	3.89473684	512.0078125
4	4.00375413	4.05405405	131072.0000
5	4.00093842	3.97333333	8589934592.0
6	4.00023460	4.01342282	3.6893e+19

Consider that the sequence converges for (a) and (b), but diverges for (c).
This example highlights the need for a mathematical analysis of the method. Sufficient conditions for the convergence of the fixed point iteration are given in the following (without proof) theorem.
Theorem 1 : *If g' exists on an interval $I = [\alpha - A, \alpha + A]$ containing the starting value x_0 and fixed point α, then x_n converges to α provided*

$$|g'(x)| < 1 \quad on \quad I$$

We can now explain the results of Example 2.5

(a) If $g(x) = (2x+8)^{\frac{1}{2}}$ then $g'(x) = (2x+8)^{-1/2}$ Theorem 2.6 guarantees convergence to the positive root $\alpha = 4$, because $|g'(x)| < 1$ on the interval $I = [3,5] = [\alpha - 1, \alpha + 1]$ containing the starting value $x_0 = 5$. which is in agreement with the results of column Xa in the Table.

(b) If $g(x) = \frac{(2x+8)}{x}$ then $g'(x) = \frac{-8}{x^2}$ Theorem 1 guarantees convergence to the positive root $\alpha = 4$, because $|g'(x)| < 1$ as (a), which is in agreement with the results of column Xb in the Table.

(c) If $g(x) = \frac{(x^2-8)}{2}$ then $g'(x) = x$ Theorem1. cannot be used to guarantee convergence, which is in agreement with the results of column Xc in the Table.

Example 2 *Find the approximate solution for the equation*

$$f(x) = x^4 - x - 10 = 0$$

by fixed point iteration method starting with $x_0 = 1.5$ *with* $|x_n - x_{n-1}| < 0.009$

Solution

The function $f(x)$ has a root in the interval $(1, 2)$, **Why ?**, rearrange the equation as

حلها صفر في الفترة (1,2)

$x_0 = 1.5$

$$x_{n+1} = g(x_n) = \sqrt{x_n + 10}$$

1. $x = g(x)$

$\boxed{x = \sqrt[4]{x+10}}$

then

$$g'(x) = \frac{(x+10)^{\frac{-3}{4}}}{4}$$

Achieving the condition

$$|g'(x)| \leq 0.04139 \qquad on \quad (1, 2)$$

then we get the solution sequence $\{1.5, 1.8415, 1.85503, 1.8556, \cdots\}$. consider that $|1.85503 - 1.8556| = 0.00057 < 0.009$.

$|x_{n+1} - x_n| < \epsilon$

بتحقيق الشرط (شرط التوقف)

$$|g'(x)| = |g'(1.5)| = \left|\frac{1}{4}(1.5+10)^{-3/4}\right| = 0.04 < 1$$

باستعمال $x_{n+1} = g(x_n)$

x_n	1.5	1.8415	1.855033	1.8556
	1.8415	1.855033	1.8556	

2. $\boxed{x = x^4 + 10}$ diverge
3. $x^4 = x + 10 \longrightarrow x = \frac{x+10}{4x^3} \longrightarrow |g'(1.5)| > 1 \rightarrow$ diverge

FIXED POINT ITERATION METHOD

ALGORITHM

1. Consider f(x) =0 and transform it to the form x= φ (x)
2. Choose an arbitrary x_0
3. Do the iterations $x_{k+1}=\varphi(x_k)$; k=0,1,2,3.........

STOPING CRITERIA

Let "ϵ" be the tolerance value

1. $|x_k - x_{k-1}| \le \epsilon$
2. $|x_k - f(x_k)| \le \epsilon$
3. Maximum number of iterations reached.
4. Any combination of above.

CONVERGENCE CRITERIA

Let "x" be exact root such that r=f(x) out iteration is $x_{n+1} = f(x_n)$

Define the error $\epsilon_n = x_n - r$ Then

$$\epsilon_{n+1} = x_{n+1} - r = f(x_n) - r = f(x_n) - f(r) = f'(\S)(x_n - r)$$

(Where $\S \in (x_n, r)$; since f is continuous)

$$\epsilon_{n+1} = f'(\S)\epsilon_n \Rightarrow \epsilon_{n+1} \le |f'(\S)||\epsilon_n|$$

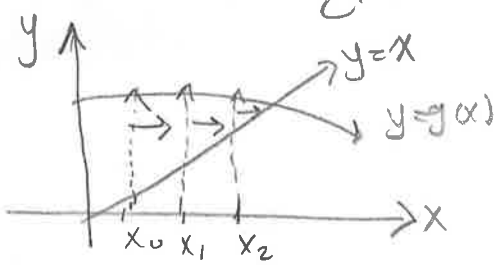
OBSERVATIONS

If $|f'(\S)| < 1$, error decreases, the iteration converges (linear convergence)

If $|f'(\S)| \ge 1$, error increases, the iteration diverges.

REMEMBER: If $|\varphi'(x)| < 1$ in questions then take that point as initial guess.

لايجاد الجذور للمعادلة f(x)=0 بالطريقة التكرارية نكتب المعادلة بالشكل x=g(x) فيكون الطرف الايسر خط مستقيم والطرف الايمن دالة مختلفة g(x) وتكون قيمة الجذر عند تقاطع هاتين الدالتين كما موضح



Example 3: Find the approximate solution for the equation $f(x): x^2 - x - 3 = 0$ by fixed point iteration method.

Solution :

يوجد جذراً للمعادلة في الفترة $\left(\overset{-}{2},\overset{+}{3}\right) \Rightarrow x_0 = 2.5$

نكتب المعادلة على شكل $x = g(x)$

هناك اكثر من طريقة واسلوب لكتابة المعادلة بالصيغة $x = g(x)$

بالقسمة على x: $x^2-x-3=0$, $x^2=x+3 \to \boxed{x=1+\frac{3}{x}}$

a) $x = 1 + \frac{3}{x} = g_1(x)$

b) $x = x^2 - 3 = g_2(x)$ $\longrightarrow x^2-x-3=0 \longrightarrow \boxed{x=x^2-3}$

c) $x = \frac{9x - x^2 + 3}{8} = g_3(x)$ $\longleftarrow x^2-9x+8x-3=0 \to \boxed{\frac{8x}{8}=\frac{9x-x^2+3}{8}} \to$

d) $x = \frac{x^2+3}{2x-1} = g_4(x)$ $\longleftarrow +2x^2 - x^2 - x - 3 = 0 \to 2x^2 - x = x^2+3 \to (2x-1)x = x^2+3 \to \boxed{x=\frac{x^2+3}{(2x-1)}}$

x	$x_{n+1} = g_1(x_n)$	$x_{n+1} = g_2(x_n)$	$x_{n+1} = g_3(x_n)$	$x_{n+1} = g_4(x_n)$
x_0	2.5	2.5	2.5	2.5
x_1	2.2	3.25	2.40625	2.3125
x_2	2.36364	7.5625	2.35828	2.302802
x_3	2.26923	54.1914	2.33288	2.302776
x_4	2.32203	2933.71	2.31920	2.302776
x_5	2.29197	8606642.63	2.31176	2.302776
x_6	2.30892	7.41×10^{13}	2.30770	2.302776

هنا التقارب ثابت للجذر غير متذبذب

كيف نختار g ؟ ان الشرط الكافي لتقارب الصيغة التكرارية $x_{n+1} = g(x_n)$ هو $|g'(x)| < 1$ بحسب المبرهنة التالية

Fixed Point Theorem:

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in [a, b]. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| < k \quad \forall x \in (a, b).$$

Then, for any number p_0 in [a, b], the sequence defined by $p_{n+1} = g(p_n)$ $n = 0,1,2,\ldots$ converges to the unique fixed point p in [a, b].

For the above example we note

$g_1{}'(x) = \frac{-3}{x^2} \Rightarrow |g_1{}'(2.5)| = 0.48 < 1$

$g_2{}'(x) = 2x \Rightarrow |g_2{}'(2.5)| = 5 > 1$ تباعد

$g_3{}'(x) = \frac{9}{8} - \frac{x}{4} \Rightarrow |g_3{}'(2.5)| = 0.5 < 1$

$g_4{}'(x) = \frac{(2x-1)2x - (x^2+3)2}{(2x-1)^2} \Rightarrow |g_4{}'(2.5)| = 0.0938 < 1$

افضل لانها اقل قيمة

EXAMPLE 4

Find the root of equation $2x = cosx + 3$ correct to three decimal points using fixed point iteration method.

SOLUTION

Given that $f(x) = 2x - cosx - 3 = 0$

X	0	1	2
F(X)	-4	-1.5403	1.4161

Root lies between "1" and "2"

Now $2x - cosx - 3 = 0 \Rightarrow x = \frac{cosx+3}{2} = \varphi(x)$

$\Rightarrow \varphi'(x) = \frac{1}{2}(-\sin x) \Rightarrow |\varphi'(x)| = |\frac{1}{2}(-\sin x)|$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{1}{2}(cos x_n + 3)$

Here we will take "x_0" as mid-point. So

If by putting 1 we get $|\varphi'(x)| < 1$ then take it as "x_0" if not then check for 2 rather take their mid-point

$x_0 = \frac{1+2}{2} = 1.5$

$x_1 = \frac{1}{2}(\cos x_0 + 3) = 1.9998$ $\qquad f(x_1) = 0.0002$

$x_2 = \frac{1}{2}(\cos x_1 + 3) = 1.9997$ $\qquad f(x_2) = 0.000008$

$x_3 = \frac{1}{2}(\cos x_2 + 3) = 1.9997$ $\qquad f(x_3) = 0.000008$

Hence the real root is 1.9997

هنا التقارب للجذر هو ثابت غير متذبذب

EXAMPLE 5

Find the root of equation $e^{-x} = 10x$ correct to four decimal points using fixed point iteration method.

SOLUTION

Given that

$f(x) = e^{-x} - 10x = 0$

X	0	1
F(X)	1	-9.6321

Root lies between "0" and "1"

Now $e^{-x} - 10x = 0 \Rightarrow x = \frac{e^{-x}}{10} = \varphi(x)$

$\Rightarrow \varphi'(x) = -\frac{e^{-x}}{10}$

Now since $|\varphi'(0)| = 0.1$ is less than "1" therefore $x_0 = 0$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{e^{-x_n}}{10}$

$X_1 = \frac{e^{-x_0}}{10} = \frac{e^{-0}}{10} = 0.1000$	$F(x_1) = -0.0952$
$X_2 = 0.0905$	$F(x_2) = 0.0085$
$X_3 = 0.0913$	$F(x_3) = -0.0003$
$X_4 = 0.0913$	$F(x_4) = -0.0003$

متساوي

Hence the real root is 0.0913