Linear Algebra

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Chapter two
Determinants

Determinant:

A function whose domain is the set of square matrices and their range (codomain) the set F.

Where: (F is the set of real or complex numbers)

The value of the square matrix function is called "determinant" for that matrix and is written as follows:

$f\,(A) = \det\,(A) = |A|$

$f\,([a]) = |a| = a$

Examples:

(1) $f\,([-8]) = |-8| = -8$

(2) $f\left(\left[\frac{2}{3}\right]\right) = \left|\frac{2}{3}\right| = \frac{2}{3}$

Determinant of the matrix of degree 2×2

It is the product of the elements of the main diagonal minus the product of the elements of the secondary diagonal. That is:

$$\det\left(\left[a_{ij}\right]_{2\times 2}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Examples:

(1) $\det\left(\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}_{2\times 2}\right) = \begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} = (4)(1) - (2)(-3) = 4 + 6 = 10$

(2) $\det\left(\begin{bmatrix} 7 & -2 \\ -1 & 3 \end{bmatrix}_{2\times 2}\right) = \begin{vmatrix} 7 & -2 \\ -1 & 3 \end{vmatrix} = (7)(3) - (-2)(-1) = 21 - 2 = 19$

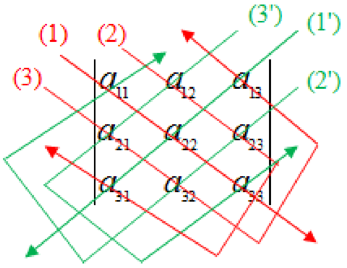
(3) $\det\left(\begin{bmatrix} -8 & 5 \\ 7 & 0 \end{bmatrix}_{2\times 2}\right) = \begin{vmatrix} -8 & 5 \\ 7 & 0 \end{vmatrix} = (-8)\,(0) - (5)(7) = 0 - 35 = -35$

(4) $\det\left(\begin{bmatrix} -\frac{2}{7} & -3 \\ 2 & 6 \end{bmatrix}_{2\times 2}\right) = \begin{vmatrix} -\frac{2}{7} & -3 \\ 2 & 6 \end{vmatrix} = (-\frac{2}{7})(6) - (-3)(2) = -\frac{12}{7} - (-6) = \frac{30}{7}$

Determinant of the matrix of degree 3×3

$$\det\left(A = \left[a_{ij}\right]_{3\times 3}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}\right)$$

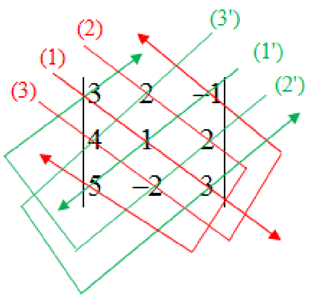
The calculation of the determinant is as follows:



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$$
$$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

Example: Find the determinant of the matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 5 & -2 & 3 \end{bmatrix}$

Solution:



$$= (3)(1)(3) + (2)(2)(5) + (4)(-2)(-1)$$
$$-(-1)(1)(5) - (2)(-2)(3) - (2)(4)(3)$$
$$= (9 + 20 + 8) + (5 + 12 - 24) = 54 - 24 = 30$$

Second method: which is to write the first and second columns after the third column, then we find the multiplication products as follows:

$$\det\left(A = \left[a_{ij}\right]_{3\times 3}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}\right) =$$

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

(1) (2) (3) (1') (2') (3')

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

or

$$\begin{array}{c} + \quad + \quad + \quad - \quad - \quad - \\ |A| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right. \end{array}$$

Examples: Find the determinant of the following matrices

(1) $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 5 & -2 & 3 \end{bmatrix}$

Solution:

(1) (2) (3) (1') (2') (3')

$$|A| = \left|\begin{array}{ccc|cc} 3 & 2 & -1 & 3 & 2 \\ 4 & 1 & 2 & 4 & 1 \\ 5 & -2 & 3 & 5 & -2 \end{array}\right.$$

$= (3)(1)(3) + (2)(2)(5) + (-1)\,(4)(-2) - (-1)(1)(5) - (3)(2)(-2) - (2)(4)(3)$

$= (9 + 20 + 8) + (5 + 12 - 24) = 54 - 24 = 30$

(2) $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 0 & -4 & 0 \end{bmatrix}$

Solution:

$|A| = (1)\,(2)\,(0) + (0)\,(-1)\,(0) + (3)\,(1)\,(-4)$
$\quad - (3)\,(2)\,(0) - (1)\,(-1)\,(-4) - (0)\,(1)\,(0)$
$\quad = 0 + 0 + (-12) - 0 - 4 - 0 = -16$

$$\begin{array}{c} + \quad + \quad + \quad - \quad - \quad - \\ |A| = \left|\begin{array}{ccc|cc} 1 & 0 & 3 & 1 & 0 \\ 1 & 2 & -1 & 1 & 2 \\ 0 & -4 & 0 & 0 & -4 \end{array}\right. \end{array}$$

(3) $A = \begin{bmatrix} \frac{1}{4} & -\frac{1}{6} & \frac{1}{4} \\ 0 & 0 & -2 \\ -2 & 8 & 0 \end{bmatrix}$

Solution:

$$|A| = (\frac{1}{4})(0)\ (0) + (-\frac{1}{6})(-2)(-2) + (\frac{1}{2})(0)\ (8)$$
$$-\ (\frac{1}{2})\ (0)(-2) - (\frac{1}{4})(-2)\ (8) - (-\frac{1}{6})\ (0)\ (0)$$
$$= \ 0 + \left(-\frac{2}{3}\right) + 0 - 0 + 4 - 0 = \frac{10}{3}$$

+ + + – – –

$$|A| = \left|\begin{array}{ccc|cc} 1/4 & -1/6 & 1/2 & 1/4 & -1/6 \\ 0 & 0 & -2 & 0 & 0 \\ -2 & 8 & 0 & -2 & 8 \end{array}\right.$$

Finding the determinant of the matrix using cofactor method

Definition: C_{ij} the cofactor of the element a_{ij} of the elements of the square matrix $A = [a_{ij}]_{n\times n}$ "is a product of $(-1)^{i+j}$ by the determinant of the matrix A after remove the i-th row and j-th column. This determinant is called the miner determinant for a_{ij} and denoted by M_{ij}"

Examples:

(1) The cofactor of the element a_{22} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$C_{22} = (-1)^{2+2}M_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

(2) The cofactor of the element a_{23} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$C_{23} = (-1)^{2+3}M_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

(3) The cofactor of the element (– 4) in the position (1,1) in the matrix $B = \begin{bmatrix} -4 & 3 & 1 \\ 2 & -1 & -4 \\ 3 & 1 & 5 \end{bmatrix}$ is

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^{1+1}\begin{vmatrix} -1 & -4 \\ 1 & 5 \end{vmatrix} = (-1)(5) - (-4)\ (1) = -5 + 4 = -1$$

Theorem: Matrix determinant: is the sum of the product of the elements of a row (column) by their cofactors.

Let C_{ij} is the cofactor of the element a_{ij} in the matrix $A = [a_{ij}]_{n\times n}$, so the determinant of the matrix is as follows:

(1) If chose the row i, then $|A| = \det A = \sum_{j=1}^{n} a_{i\,j}C_{i\,j} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

(2) If chose the column j, then $|A| = \det A = \sum_{i=1}^{n} a_{i\,j}C_{i\,j} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

Remark: When using this method, we choose the row or column that contains the largest number of zeros.

Examples:

(1) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Find det A?

Solution: chose the row 1

$$|A| = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12}$$

$$= a_{11}(-1)^{1+1}|a_{22}| + a_{12}(-1)^{1+2}|a_{21}| = a_{11}\,a_{22} - a_{12}\,a_{21}$$

(2) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find det A?

Solution: chose the row 1

$$|A| = \det A = A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}(a_{21}\,a_{32} - a_{22}\,a_{31})$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}a_{21}\,a_{32} - a_{13}a_{22}\,a_{31}$$

$$= a_{11}a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} - a_{13}a_{22}\,a_{31}$$

(3) Find $|A| = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 5 & -2 & 3 \end{vmatrix}$

Solution: chose the row i = 2

$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= 4(-1)^{2+1}\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix}$

$= -4(6-2) + 1(9+5) - 2(-6-10) = -4(4) + 1(14) - 2(-16) = -16 + 14 + 32 = 30$

(3) Find $|B| = \begin{vmatrix} 3 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 4 & -2 & 0 & 1 \\ -1 & 3 & 0 & 2 \end{vmatrix}$

Solution: We can chose the row i = 2 or the column j = 3

Now we chose the row i = 2

$|B| = b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23} + b_{24}C_{24}$

$|B| = 0(-1)^{2+1}\begin{vmatrix} 2 & 1 & 4 \\ -2 & 0 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 3 & 2 & 4 \\ 4 & -2 & 1 \\ -1 & 3 & 2 \end{vmatrix} +$

$0(-1)^{2+4}\begin{vmatrix} 3 & 2 & 1 \\ 4 & -2 & 0 \\ -1 & 3 & 0 \end{vmatrix}$

$|B| = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix}$

To find the value of this determinant we can find it directly or by using the cofactor by chose the second column

$|B| = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} + 0(-1)^{2+2}\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix}$

$$= -\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} = -(8+1) = -9$$

Exercise: Resolve the previous example using the third column

Example: Prove that $|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos 2\theta$.

Proof:

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

Example: Prove that $|B| = \begin{vmatrix} \cos\theta & 1 & 1 \\ 1 & \cos\theta & 1 \\ \sin\theta & 1 & 1 \end{vmatrix} = (\cos\theta - \sin\theta)(\cos\theta - 1)$.

Proof:

$$|B| = \begin{vmatrix} \cos\theta & 1 & 1 \\ 1 & \cos\theta & 1 \\ \sin\theta & 1 & 1 \end{vmatrix} = \cos^2\theta + 1 + \sin\theta - \sin\theta\cos\theta - \cos\theta - 1$$

$$= \cos^2\theta - \sin\theta\cos\theta - \cos\theta + \sin\theta$$

$$= \cos\theta(\cos\theta - \sin\theta) - 1(\cos\theta - \sin\theta)$$

$$= (\cos\theta - \sin\theta)(\cos\theta - 1)$$

Some Properties of Determinants:

Property (1):

If all elements of row or column of any matrix are zero, then their determinant equal to zero.

Proof: Let the row i all elements of it are zero.

$a_{i1} = a_{i2} = \ldots = a_{in} = 0 \Rightarrow (0,0,\ldots,0)$ the row i

We open the determinant around the row i

$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

$= 0\,C_{i1} + 0\,C_{i2} + \ldots + 0\,C_{in}$

$= 0 + 0 + \ldots + 0 = 0$

The same method remains if all elements of one of the columns are zeros. **(Home work)**

Examples:

(1) $\begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0$ since all elements of the first row are zero

(2) $\begin{vmatrix} -4 & 0 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 0$ since all elements of the second column are zero

Property (2):

The determinant of the square matrix A equal to the determinant of its transpose, i.e. $|A| = |A^t|$

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$, so $A^t = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$ (by opening the determinant about 1^{st} row)

$|A^t| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$ (by opening the determinant about 1^{st} column)

Thus $|A| = |A^t|$.

Examples:

(1) $|A| = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 12 + 2 = 14$ and $|A^t| = \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} = 12 + 2 = 14$

$\therefore |A| = |A^t|$

(2) Show the correctness of $|A| = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 4 & 3 & -1 & 3 \\ 0 & 2 & 0 & 0 \\ -1 & 2 & 3 & 4 \end{vmatrix} = 80$ and $|A^t| = 80$?

Solution: To find the value of the determinant to the matrix A, we choose the third row (why ?)

$|A| = a_{31}C_{31} + a_{32}\,C_{32} + + a_{33}\,C_{33} + a_{34}\,C_{34}$

$$= 0(-1)^{3+1}\begin{vmatrix} -2 & 2 & 1 \\ 3 & -1 & 3 \\ 2 & 3 & 4 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & -1 & 3 \\ -1 & 3 & 4 \end{vmatrix} + 0(-1)^{3+3}\begin{vmatrix} 1 & -2 & 1 \\ 4 & 3 & 3 \\ -1 & 2 & 4 \end{vmatrix} +$$

$$0(-1)^{3+4}\begin{vmatrix} 1 & -2 & 2 \\ 4 & 3 & -1 \\ -1 & 2 & 3 \end{vmatrix}$$

$$|A| = -2\begin{vmatrix} 1 & 2 & 1 \\ 4 & -1 & 3 \\ -1 & 3 & 4 \end{vmatrix} = -2[(-4) + (-6) + (12) - (1) - (9) - (32)]$$

$$= -2\,(2 - 42) = -2\,(-40) = 80$$

To find the value of the determinant to the transpose of the matrix A, we choose the third column.

$|A^t| = a_{13}\,C_{13} + a_{23}\,C_{23} + a_{33}\,C_{33} + a_{43}\,C_{43}$

$$|A^t| = \begin{vmatrix} 1 & 4 & 0 & -1 \\ -2 & 3 & 2 & 2 \\ 2 & -1 & 0 & 3 \\ 1 & 3 & 0 & 4 \end{vmatrix} = 0(-1)^{1+3}\begin{vmatrix} -2 & 3 & 2 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix} +$$

$$0(-1)^{3+3}\begin{vmatrix} 1 & 4 & -1 \\ -2 & 3 & 2 \\ 1 & 3 & 4 \end{vmatrix} + 0(-1)^{4+3}\begin{vmatrix} 1 & 4 & -1 \\ -2 & 3 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$|A^t| = -2\begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= -2[(-4) + (-6) + (12) - (1) - (9) - (32)]$$

$$= -2\,(2 - 42) = -2\,(-40) = 80$$

$$\therefore |A| = |A^t|$$

(3) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 7 & 9 \\ 4 & 5 & 8 \end{bmatrix}$

$|A| = 168 + 0 + 0 - 0 - 135 - 0$
$= 168 - 135$
$= 33$

$$|A| = \left|\begin{array}{ccc|cc} 3 & 0 & 0 & 3 & 0 \\ 1 & 7 & 9 & 1 & 7 \\ 4 & 5 & 8 & 4 & 5 \end{array}\right| \qquad + \; + \; + \; - \; - \; -$$

And

$A^t = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 7 & 5 \\ 0 & 9 & 8 \end{bmatrix}$

$|A^t| = 168 + 0 + 0 - 0 - 135 - 0$
$= 168 - 135$
$= 33$

$$|A^T| = \left|\begin{array}{ccc|cc} 3 & 1 & 4 & 3 & 1 \\ 0 & 7 & 5 & 0 & 7 \\ 0 & 9 & 8 & 0 & 9 \end{array}\right| \qquad + \; + \; + \; - \; - \; -$$

Property (3):

If a matrix B results from interchanging row by row or column by column in matrix A, then det B = – det A

In other words: If replace a row with a row (column by column) in the matrix, then the signal determinant is changed.

Proof: when $n = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ so } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

If we replace the second row with the first row $R_1 \leftrightarrow R_2$

$$B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}, \text{ so } |B| = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$$

$$= -(a_{11}a_{22} - a_{21}a_{12})$$

$$= -|A|$$

when $n = 3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = \underbrace{a_{11}a_{22}\,a_{33} + a_{12}a_{23}\,a_{31} + a_{21}a_{32}a_{13}}_{\text{Multiply towards the main diagonal}} - \underbrace{a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}}_{\text{Multiply towards the secondary diagonal}}$$

If we replace the second row with the first row $R_1 \leftrightarrow R_2$

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|B| = a_{21}a_{12}\,a_{33} + a_{22}a_{13}\,a_{31} + a_{11}a_{32}a_{23} - a_{23}a_{12}a_{31} - a_{11}a_{22}a_{33} - a_{13}a_{32}a_{21}$$

$$= -\,(-\,a_{21}a_{12}\,a_{33} - a_{22}a_{13}\,a_{31} - a_{11}a_{32}a_{23} + a_{23}a_{12}a_{31} + a_{11}a_{22}a_{33} + a_{13}a_{32}a_{21})$$

$$= -\,(\underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}}_{\text{Multiply towards the main diagonal}} - \underbrace{a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{32}a_{23}a_{11}}_{\text{Multiply towards the secondary diagonal}})$$

$$= -\,|A|$$

As well as if we replace the first row with the third row $R_1 \leftrightarrow R_3$, we get $|B| = -\,|A|$
In the same way if the replace was on the columns.
And so in the same way we prove when n = k.
$\therefore$ The property is true for all values of n.

Examples:

(1) $A = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$ (replace the first row with the second row $R_1 \leftrightarrow R_2$)

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix} = 6 - (-3) = 9$$

$$|B| = \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} = -3 - 6 = -9$$

$$\therefore\ |B| = -|A|$$

(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ (replace the third row with the second row $R_2 \leftrightarrow R_3$)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = (1)(-1)(4) + (2)(1)(2) + (3)(1)(3) - (3)(-1)(2) - (1)(3)(1) - (2)(1)(4)$$

$$= (-4 + 4 + 9) + (6 - 3 - 8) = 9 - 5 = 4$$

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix} = (1)(3)(1) + (2)(4)(1) + (3)(2)(-1) - (3)(3)(1) - (1)(4)(-1) - (2)(2)(1)$$

$$= (3 + 8 - 6) - (9 + 4 - 4) = 5 - 9 = -4$$

$$\therefore |B| = -|A|$$

Let $C = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & -1 \\ 2 & 4 & 3 \end{bmatrix}$ (replace the third column with the second column $C_2 \leftrightarrow C_3$).

Show that $|C| = -|A|$ **(Home work)**

Let $D = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ (replace the third column with the first column $C_1 \leftrightarrow C_3$).

Show that $|D| = -|A|$ **(Home work)**

Property (4):

If the elements of two rows (two columns) are equal in a square matrix then its determinant is equal to zero.

Proof: Suppose in the matrix A the elements of the row (i) equal to the elements of the row (ℓ).

Suppose the matrix B results from replace the row (i) by the row (ℓ).

$|B| = -|A|$ (by Property (3))

But $B = A$, so $|B| = |A|$ (the row (i) = the row (ℓ))

$|A| = -|A|$ (replaced)

$2|A| = 0 \Rightarrow 2 \neq 0$

$\therefore |A| = 0$

Examples:

(1) $\begin{vmatrix} 4 & -2 & 3 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 0$ (the second and third rows are equal)

(2) $\begin{vmatrix} 3 & 1 & 2 & 2 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 1 & 1 \end{vmatrix} = 0$ (the third and fourth columns are equal)

Property (5):

Determinant of the product of two square matrices of the same degree = product of the determinants of those two matrices. i.e. $|AB| = |A| \cdot |B|$

Proof: We prove this property when n = 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$

After opening and arranging it, produce:

$|AB| = \boxed{a_{11}b_{11}a_{22}b_{22} - a_{11}a_{22}b_{12}b_{21} - b_{11}b_{22}a_{12}a_{21} + a_{12}a_{21}b_{12}b_{21}}$

$|A| = a_{11}a_{22} - a_{12}a_{21}$

$|B| = b_{11}b_{22} - b_{12}b_{21}$

$|A| \cdot |B| = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$

$= \boxed{a_{11}b_{11}a_{22}b_{22} - a_{11}a_{22}b_{12}b_{21} - b_{11}b_{22}a_{12}a_{21} + a_{12}a_{21}b_{12}b_{21}}$

$\therefore \ |AB| = |A| \cdot |B|$

As the same way when n = 3.

Thus this property can be proved when n = k.

Examples:

(1) Let $A=\begin{bmatrix}2 & -3\\ -1 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$, show that $|AB| = |A|\cdot|B|$?

Solution: $|AB| = |A|\cdot|B|$

$$\left|\begin{bmatrix}2 & -3\\ -1 & 5\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\right| = \begin{vmatrix}2 & -3\\ -1 & 5\end{vmatrix}\begin{vmatrix}1 & 4\\ 2 & 3\end{vmatrix}$$

$$\begin{vmatrix}-4 & -1\\ 9 & 11\end{vmatrix} = (10-3)\cdot(3-8)$$

$$-44+9 = (7)\,(-5)$$

$$-35 = -35$$

(2) Let $A=\begin{bmatrix}0 & 2 & 2\\ 3 & -1 & 0\\ 1 & 4 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 3 & 0\\ 0 & 2 & 1\\ 2 & -1 & 3\end{bmatrix}$, show that $|AB| = |A|\cdot|B|$?

Solution: $|AB| = |A|\cdot|B|$

$$\left|\begin{bmatrix}0 & 2 & 2\\ 3 & -1 & 0\\ 1 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 0\\ 0 & 2 & 1\\ 2 & -1 & 3\end{bmatrix}\right| = \begin{vmatrix}0 & 2 & 2\\ 3 & -1 & 0\\ 1 & 4 & 1\end{vmatrix}\cdot\begin{vmatrix}1 & 3 & 0\\ 0 & 2 & 1\\ 2 & -1 & 3\end{vmatrix}$$

$$\begin{vmatrix}4 & 2 & 8\\ 3 & 7 & -1\\ 3 & 10 & 7\end{vmatrix} = (20)\,(13)$$

$$260 = 260$$

(3) Let $A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 7 & 2\end{bmatrix}$, find $|A|$, $|B|$, $|AB|$, $|A^3B|$, $|B^2A|$? **(Home work)**

(4) If $|A| = 4$, find $|A^3|$?

Solution:

$|A^3| = |A\cdot A^2|$

$\quad = |A|\cdot|A^2|$ (by property (5))

$\quad = |A|\,|A\cdot A|$

$\quad = |A|\cdot|A|\cdot|A|$ (by property (5))

$|A^3| = |A|^3$

$\quad = 4^3$

$|A^3| = 64$

This example leads to the following corollary:

Corollary: If A is a square matrix, then $|A^k| = |A|^k$, where k is a positive integer number. **(The prove home work)**

Generalization: If $A_1, A_2, \ldots, A_n$ are square matrices of the same degree, then $|A_1 A_2 \ldots A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$. **(The prove home work)**

Remark: $|A + B| \neq |A| + |B|$, for example:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$$

$$|A+B| = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4, \quad |A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7, \quad |B| = \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} = 3$$

$$|A + B| \neq |A| + |B|$$
$$4 \neq 7 + 3$$
$$4 \neq 10$$

Property (6):

If A, B and D are three matrices of degree (n×n) equal in all rows except in the row (i) in the matrix D such that $d_{ij} = a_{ij} + b_{ij}$, $j = 1,2,\ldots, n$, then det D = det A + det B. The same way if it is a column.

Proof: Let

$$A = \begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ \boxed{a_{i1}} & a_{i2} & \cdots & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} \text{i-th row}, \quad B = \begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ b_{i1} & b_{i2} & \cdots & \cdots & \cdots & b_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} \text{i-th row}$$

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{i1} + b_{i1} & a_{i2} + b_{i2} & \cdots & \cdots & \cdots & a_{in} + b_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} \text{i-th row}$$

Where $d_{ij} = a_{ij} + b_{ij}$, j = 1,2,…,n

We open the determinant of the matrix D about the row i

$$|D| = \sum_{k=1}^{n}((a_{ik} + b_{ik})C_{ik})$$

$$= \sum_{k=1}^{n} a_{ik}\, C_{ik} + \sum_{k=1}^{n} b_{ik}\, C_{ik}$$

$$|D| = \quad |A| \quad + \quad |B|$$

By theorem:
Matrix determinant is the sum of the product of the elements of a row (column) by their cofactors.

In the same way if it is

The column j from D = The column j from A + The column j from B, i.e.

$$D = \begin{array}{c} \text{j-th column} \\ \begin{vmatrix} d_{11} & d_{12} & \cdots & a_{1j} + b_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & a_{2j} + b_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & a_{nj} + b_{nj} & \cdots & d_{nn} \end{vmatrix} \end{array}$$

$$= \begin{array}{c} \begin{vmatrix} d_{11} & d_{12} & \cdots & a_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & a_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & a_{nj} & \cdots & d_{nn} \end{vmatrix} \\ \text{j-th column} \end{array} + \begin{array}{c} \begin{vmatrix} d_{11} & d_{12} & \cdots & b_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & b_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & b_{nj} & \cdots & d_{nn} \end{vmatrix} \\ \text{j-th column} \end{array}$$

$$|D| = |A| + |B|$$

Examples:

(1) $$|C| = \begin{vmatrix} 1 & 0 & 2 \\ 4 & 6 & 8 \\ -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ -2 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ -2 & 1 & 0 \end{vmatrix}$$

24 = 9 + 15

(2) $$\begin{vmatrix} \boxed{3} & 1 & 0 \\ \boxed{6} & 2 & -1 \\ \boxed{9} & 1 & 4 \end{vmatrix} = \begin{vmatrix} \boxed{2} & 1 & 0 \\ \boxed{2} & 2 & -1 \\ \boxed{2} & 1 & 4 \end{vmatrix} + \begin{vmatrix} \boxed{1} & 1 & 0 \\ \boxed{4} & 2 & -1 \\ \boxed{7} & 1 & 4 \end{vmatrix}$$

$$24 + (-9) + (0) - (0) - (-3) - (24) = 8 + (-14)$$

$$-6 = -6$$

Property (7):

If all the elements of a row (column) in a square matrix A multiplying by a fixed ($k \neq 0$), then the determinant of it multiply by the fixed k. i.e.

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\,|A|$$

Proof: Let B is the matrix obtained by multiply the row (i) in the matrix A by the fixed ($k \neq 0$), when we open the determinant of the matrix B about the row i, we get

$$|B| = k\, a_{i1}C_{i1} + k\, a_{i2}C_{i2} + \ldots + k\, a_{in}C_{in}$$

$$= k\,(a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in})$$

$$|B| = k\,|A|$$

The same way when the column of the matrix is multiply.

Examples:

equal

(1) $$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 2 & 8 & 2 \end{vmatrix} = 3(0) = 0$$

(2) $$\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = \begin{vmatrix} (2)(1) & (2)(3) \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & (3)(1) \\ 1 & (3)(4) \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = (6)(3) = 18$$

(3) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, so $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} = (-15)$

Let B is the matrix obtained by multiply the first column of the matrix A by the fixed $k = 10$, then

$$|B| = \begin{vmatrix} 10 & -1 & 2 \\ 20 & 3 & 2 \\ 30 & 2 & 1 \end{vmatrix} = (-150) = 10(-15)$$

$|B| = 10\,|A|$

Corollary:

If $B = [b_{ij}]_{n\times n}$ is a matrix obtained from multiply each entry of $A = [a_{ij}]_{n\times n}$ by c, i.e. $b_{ij} = c\, a_{ij}$, $\forall$ i, j, $1 \le i, j \le n$. Then $|B| = c^n\,|A|$.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} c\,a_{11} & c\,a_{12} & \dots & c\,a_{1n} \\ c\,a_{21} & c\,a_{22} & \dots & c\,a_{2n} \\ \vdots & \vdots & & \vdots \\ c\,a_{i1} & c\,a_{i2} & \dots & c\,a_{in} \\ \vdots & \vdots & & \vdots \\ c\,a_{n1} & c\,a_{n2} & \dots & c\,a_{nn} \end{bmatrix}$.

Then

$$|B| = c\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ c\,a_{21} & c\,a_{22} & \dots & c\,a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c\,a_{n1} & c\,a_{n2} & \dots & c\,a_{nn} \end{vmatrix}$$ Extract the number c from the first row (R_1)

$$= c \cdot c\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c\,a_{n1} & c\,a_{n2} & \dots & c\,a_{nn} \end{vmatrix}$$ Extract the number c from the second row (R_2)

By repeating the process n-times, we get $|B| = c^n\,|A|$.

Example: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$, let c = 3. Show that $3^3\,|A| = |B|$, where B = 3A.

Property (8):

If the corresponding elements of two rows (two columns) in a square matrix are proportional then the determinant of that matrix = zero.

Proof: Let the row (i) proportional with the row (ℓ) in the matrix $A_{n\times n}$, then:

row (ℓ) = row (i) × (fixed k) , where $k \neq 0$ (by property (7))

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{i1} & a_{i1} & \dots & a_{in} \\ a_{\ell 1} & a_{\ell 2} & \dots & a_{\ell n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ \text{row (i)} \\ \text{row } (\ell) \\ \\ \end{matrix} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{i1} & a_{i1} & \dots & a_{in} \\ ka_{i1} & ka_{i2} & \dots & ka_{in} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ \text{row (i)} \\ \text{row } (\ell) \\ \\ \end{matrix}$$

replacing the row (ℓ)

$$|A| = k \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{i1} & a_{i2} & \dots & a_{in} \\ a_{i1} & a_{i2} & \dots & a_{in} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

equal

(by property (7))

$|A| = k\,(0) = 0$ (by property (4))

Examples:

(1) $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 0$ the first row proportional with the second row where the ratio between them $\frac{3}{1}$

(2) $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 2 & 4 & 6 \end{vmatrix} = 0$ the first row proportional with the third row where the ratio between them $\frac{1}{2}$

or the first column proportional with the third column where the ratio between them $\frac{1}{3}$

(3) $|B| = \begin{vmatrix} 4 & 0 & 6 \\ 2 & -1 & 3 \\ 10 & 3 & 15 \end{vmatrix} = 0$ the first column proportional with the third column where the ratio between them $\frac{2}{3}$

Property (9):

If add to the elements of a row (column) in a square matrix the multiplying of the corresponding elements in another row (column) by the constant k (k $\neq$ 0), then the value of its determinant does not change.

Proof: Let

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{row (i)} \\ \\ \text{row (s)} \\ \\ \\ \end{array}$$

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ ka_{i1} + a_{s1} & ka_{i2} + a_{s2} & \cdots & ka_{in} + a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ Adding the row (i) multiplying by k to the row (s)

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

(by property 6)

two rows are proportional

But the determinant of the second matrix = 0 by property (8), so

$|A| = |B|$.

Example: If $A = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$

(1) $B = \begin{bmatrix} -1 & 2+10(-1) \\ 3 & 5+10(3) \end{bmatrix}$ Adding to the second column ten times of the first column

$$|B| = \begin{vmatrix} -1 & -8 \\ 3 & 35 \end{vmatrix} = -35 + 24 = -11$$

$$|A| = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$$

$\therefore |A| = |B|$

(2) Adding to the first row double the second row (multiplying the second row by 2 and adding it to the first row)

$$B = \begin{bmatrix} -1+2(3) & 2+2(5) \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 3 & 5 \end{bmatrix}$$

Is $|A| = |B|$

$$\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 12 \\ 3 & 5 \end{vmatrix}$$

(–11) = (–11)

Property (10):

The determinant of a triangular matrix A equal to the product of the elements of the main diagonal, i.e. If $A = [a_{ij}]_{n\times n} \Rightarrow |A| = \det A = a_{11}a_{22} \ldots a_{nn}$.

Proof: Let A be an (n×n) upper triangular matrix, to prove $|A| = \det A = a_{11}a_{22} \ldots a_{nn}$. We use the method of mathematical induction.

(1) When n = 2, $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22}$

$\therefore$ The property true when n = 2.

(2) Suppose the property is true when k = n

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ 0 & a_{22} & a_{23} & \dots & a_{2k} \\ 0 & 0 & a_{33} & \dots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{kk} \end{vmatrix} = a_{11}a_{22}a_{33}\dots a_{kk}$$

(3) Is the property stay true when n = k + 1

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & a_{1(k+1)} \\ 0 & a_{22} & a_{23} & \dots & a_{2k} & a_{2(k+1)} \\ 0 & 0 & a_{33} & \dots & a_{3k} & a_{3(k+1)} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & a_{(k+1)k} & a_{(k+1)(k+1)} \end{vmatrix}$$

We open the determinant around the last row (k + 1).

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ 0 & a_{22} & a_{23} & \dots & a_{2k} \\ 0 & 0 & a_{33} & \dots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{kk} \end{vmatrix} a_{(k+1)(k+1)} = a_{11}a_{22}a_{33}\dots a_{kk}a_{(k+1)(k+1)}$$

Apply the same proof if the matrix is lower triangular matrix. **(Home work)**

Examples:

(1) $|A| = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ a & -1 & 5 \end{vmatrix} = (3)(1)(5) = 15$

(2) $|B| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 8 \end{vmatrix} = (1)(4)(8) = 32$

(3) $|A| = \begin{vmatrix} 4 & -7 & 8 \\ 0 & 5 & 4 \\ 0 & 0 & -1 \end{vmatrix} = (4)(5)(-1) = -20$

(4) $|B| = \begin{vmatrix} -7 & 0 & 0 \\ 12 & 8 & 0 \\ 15 & 6 & -2 \end{vmatrix} = (-7)(8)(-2) = 112$

Remark: The determinant of the unity matrix = 1, since it is upper and lower triangular matrix and the elements of its main diagonal $1 \times \ldots \times 1 \times 1$.

Property (11):

The sum of the products of multiplying the elements of a row (column) in the square matrix A by the cofactors for the corresponding elements in another row (column) of the matrix A equal to zero.

If $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ a_{s1} & a_{s2} & \ldots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{array}{l} \\ \\ \\ \text{row (i)} \\ \text{row (s)} \\ \\ \\ \end{array}$

And $C_{i1}, C_{i2}, \ldots, C_{in}$ the cofactors for the row (i), then

$a_{s1} C_{i1} + a_{s2} C_{i2} + \ldots + a_{sn} C_{in} = 0$

The same way if the cofactors for any column multiply by the corresponding elements in another column respectively.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and C_{11}, C_{12}, C_{13} the cofactors for the first row respectively, if we multiply the cofactors for the first row by the elements of the second row or the third row, then

$a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} = 0$

$a_{31}\, C_{11} + a_{32}\, C_{12} + a_{33}\, C_{13} = 0$

Example: If $|A| = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 4 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix} = 8$ (the cofactors for the element $a_{11} = 2$)

$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} = 3$ (the cofactors for the element $a_{12} = -1$)

$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix} = -18$ (the cofactors for the element $a_{13} = 2$)

$a_{21}\, C_{11} + a_{22}\, C_{12} + a_{23}\, C_{13} =$
$3(8) + 4(3) + 2(-18) = 24 + 12 - 36 = 0$
Also,
$a_{31}\, C_{11} + a_{32}\, C_{12} + a_{33}\, C_{13} =$
$3(8) + (-2)(3) + 1(-18) = 24 - 6 - 18 = 0$

Property (12):

If the elements of the matrix is complex number, then
The determinant of the conjugate matrix = conjugate determinant of the matrix, i.e. $|\bar{A}| = \overline{|A|}$

Example: Let $A = \begin{bmatrix} i & -2i \\ 2+i & 3i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} -i & 2i \\ 2-i & -3i \end{bmatrix}$

$|\bar{A}| = \begin{vmatrix} -i & 2i \\ 2-i & -3i \end{vmatrix} = 3i^2 - 2i(2-i)$

$= 3i^2 - 4i + 2i^2 = 5i^2 - 4i = \boxed{-5 - 4i}$

$|A| = \begin{vmatrix} i & -2i \\ 2+i & 3i \end{vmatrix} = 3i^2 - (-2i)(2+i)$

$= 3i^2 + 4i + 2i^2 = 5i^2 + 4i = -5 + 4i$

$\overline{|A|} = \overline{-5+4i} = \boxed{-5 - 4i}$

$\therefore\ |\bar{A}| = \overline{|A|}$

Examples:

(1) Find the value of the determinant of each matrix by using the properties of the determinant (without opening it mathematically)

(a) $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

$|A| = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -3 & 3 \\ 0 & 0 & 1 \end{vmatrix}$ Multiply the third column by k= –1 and added it to the second column (Property 9)

$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{vmatrix}$ Multiply the first row by k= 1 and added it to the second row (Property 9)

$= -2$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(b) $B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Solution:

$|B| = 2\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}$ Take out a common factor the number 2 from the third row (Property 7)

$|B| = 2\begin{vmatrix} 4 & 3 & 2 \\ 4 & 0 & 8 \\ 1 & 2 & 3 \end{vmatrix}$ Multiply the third row by k= 1 and added it to the second row (Property 9)

$|B| = (2)(4)\begin{vmatrix} 4 & 3 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}$ Take out a common factor the number 4 from the second row (Property 7)

$$|B| = -8\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$ Replacement the first column with the second column with change the signal (Property 3)

$$|B| = -8\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$ Multiply the second row by k= – 1 and added it to the third row (Property 9)

$$|B| = (-8)(3)\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$ Take out a common factor the number 3 from the first row (Property 7)

$$|B| = -24\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 0 & \frac{-8}{3} & \frac{-1}{3} \end{vmatrix}$$ Multiply the first row by k= – 2 and added it to the third row (Property 9)

$$|B| = -24\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$ Multiply the second row by k= $\frac{8}{3}$ and added it to the third row (Property 9)

$|B| = -24\,[(1)(1)(5)] = -120$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(c) $C = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$

Solution:

$$|C| = 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$ Take out a common factor the number 2 from the first row (Property 7)

$$|C| = 2\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}$$ Multiply the first row by k= – 3 and added it to the second row (Property 9)

$|C| = (2)(4) = 8$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(2) Prove the correctness of the following $\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = 0$ without opening it mathematically.

Proof:

$$\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = k \begin{vmatrix} 1 & -4 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$ Take out a common factor k from the third column (Property 7)

$= k(0) = 0$ The first column equal to the third column (Property 4)

Other proof:

Since the elements of the first column proportional with the elements of the third column and the ratio is k : 1, then

$$\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = 0$$ If two columns are proportional then the value of the determinant equal to zero (Property 8)

(3) Prove that $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = 0$ without opening it mathematically.

Proof:

$$\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = \begin{vmatrix} a & 1 & a+b+c \\ b & 1 & a+b+c \\ c & 1 & a+b+c \end{vmatrix}$$ Multiplying the first column by k= 1 and added it to the third column (Property 9)

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$ Take out a common factor $(a+b+c)$ from the third column (Property 7)

$= (a+b+c)\,(0) = 0$ If the elements of two rows (two columns) are equal in a square matrix then its determinant is equal to zero (Property 4)

(4) Prove that $\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ without opening it mathematically.

Proof: We take the left side

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ 2(a_1+b_1+c_1) & 2(a_2+b_2+c_2) & 2(a_3+b_3+c_3) \end{vmatrix}$$

Multiplying the first and second rows by k= 1 and added it to the third row (Property 9)

$$= 2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \end{vmatrix}$$

Take out a common factor the number 2 from the third row (Property 7)

$$= 2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Multiplying the second row by k= –1 and added it to the third row (Property 9)

$$= 2\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Multiplying the third row by k= –1 and added it to the first row (Property 9)

$$= 2\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Multiplying the first row by k= –1 and added it to the second row (Property 9)

$$= -2\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Replacement the first row with the third row with change the signal (Property 3)

$$= 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Replacement the second row with the third row with change the signal (Property 3)

(5) Without opening the determinant. Prove that

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = -(a_1-a_2)(a_2-a_3)(a_3-a_1)$$

Proof: We take the left side

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1^2 - a_2^2 & a_1 - a_2 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Multiplying the second row by k= –1 and added it to the first row (Property 9)

$$= (a_1 - a_2)\begin{vmatrix} a_1 + a_2 & 1 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Take out a common factor ($a_1 - a_2$) from the first row (Property 7)

$$= (a_1 - a_2)\begin{vmatrix} a_1 + a_2 & 1 & 0 \\ a_2^2 - a_3^2 & a_2 - a_3 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Multiplying the third row by k= –1 and added it to the second row (Property 9)

$$= (a_1 - a_2)(a_2 - a_3)\begin{vmatrix} a_1 + a_2 & 1 & 0 \\ a_2 + a_3 & 1 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Take out a common factor ($a_2 - a_3$) from the second row (Property 7)

$$= (a_1 - a_2)(a_2 - a_3)\begin{vmatrix} a_1 + a_2 & 1 & 0 \\ a_3 - a_1 & 0 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Multiplying the first row by k= –1 and added it to the second row (Property 9)

$$= (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\begin{vmatrix} a_1 + a_2 & 1 & 0 \\ 1 & 0 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Take out a common factor ($a_3 - a_1$) from the second row (Property 7)

$$= -(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\begin{vmatrix} 1 & 0 & 0 \\ a_1 + a_2 & 1 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Replacement the first row with the second row with change the signal (Property 3)

$$= -(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$$

The determinant of a triangular matrix equal to the product of the elements of the main diagonal = 1 (Property 10)

(6) Find the value of x without opening the determinant mathematically.

$$\begin{vmatrix} 2 & 2 - x \\ 1 + x & 0 \end{vmatrix} = 0$$

Proof:

$(1+x)\begin{vmatrix} 2 & 2-x \\ 1 & 0 \end{vmatrix} = 0$ Take out a common factor (1 + x) from the second row (Property 7)

$-(1+x)\begin{vmatrix} 1 & 0 \\ 2 & 2-x \end{vmatrix} = 0$ Replacement the first row with the second row with change the signal (Property 3)

$-(1+x)(2-x) = 0$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

$(1 + x)(2 - x) = 0$ Multiply each side by – 1

$1+x = 0 \Rightarrow x = -1$ or $2-x = 0 \Rightarrow x = 2$

(7) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 4 & 3 \end{bmatrix}$. Is $|A + B| = |A| + |B|$?

Solution: $A + B = \begin{bmatrix} 1 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow |A + B| = 7 - 49 = -42$

$|A| = 4 - 6 = -2$, $|B| = 0 - 20 = -20$.

So we get, $|A + B| \neq |A| + |B|$ since $-42 \neq -2 + -20$ (i.e. $-42 \neq -22$)

Exercises:

(1) Find the value of the determinant of each matrix by using the properties of the determinant or by reduction to triangular matrix (without opening it mathematically)

(a) $\begin{vmatrix} 1 & 0 & -3 \\ 2 & 2 & -1 \\ 4 & 1 & 4 \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix}$ **(c)** $\begin{vmatrix} 2 & 3 & 4 \\ 0 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}$ **(d)** $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ **(e)** $\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

(f) $\begin{vmatrix} 2 & 1 & 2 \\ -1 & 0 & 3 \\ x & y & z \end{vmatrix}$ **(g)** $\begin{vmatrix} 3 & 1 & -2 \\ 2a & 2b & 2c \\ 2 & 1 & 2 \end{vmatrix}$ **(h)** $\begin{vmatrix} 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 4 \end{vmatrix}$ **(j)** $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix}$

(2) Find the value of a if $\begin{vmatrix} a-1 & 2 \\ 3 & a-2 \end{vmatrix} = 0$ without opening it mathematically?

(3) If $|A| = -5$ and $|B| = 2$, find the value of $|A|^2$, $|A^4|$, $|A^2B^2|$.

(4) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 0$ without opening it mathematically.

(5) Prove that $\begin{vmatrix} 2a_1+b_1 & 2b_1+c_1 & 2c_1+a_1 \\ 2a_2+b_2 & 2b_2+c_2 & 2c_2+a_2 \\ 2a_3+b_3 & 2b_3+c_3 & 2c_3+a_3 \end{vmatrix} = 9\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ without opening it mathematically

(6) Prove that $\begin{vmatrix} yz & x^2 & x^2 \\ y^2 & xz & y^2 \\ z^2 & z^2 & xy \end{vmatrix} = \begin{vmatrix} yz & xy & xz \\ xy & xz & yz \\ xz & yz & xy \end{vmatrix}, xyz \neq 0$ without opening it mathematically.

(7) Without opening the determinant, show that the equation obtaining from the value of the following determinant is of degree two and has the roots a and b

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = 0, a \neq b$$

(8) If the points (x_1,y_1), (x_2,y_2), (x_3,y_3) on a straightened one, then prove that

$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ without opening it mathematically.

(9) Find the value of x for all the following without opening it mathematically

(a) $\begin{vmatrix} x-2 & 7 \\ 3 & x+2 \end{vmatrix} = 0$ **(b)** $\begin{vmatrix} x-1 & 0 & 1 \\ 0 & x-1 & 0 \\ 1 & 0 & x-1 \end{vmatrix} = 0$

(10) If $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix}$, show that

(a) $AB \neq BA$.

(b) $|AB| = |A| \cdot |B|$.

(c) $|AB| = |BA|$.

(11) Prove that $|AB| = |BA|$ for any matrices A and B?

(12) Let k be any real number and A is a matrix of degree $n \times n$, prove that $|kA| = k^n |A|$

(13) Without opening the determinant prove that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}, a \neq 0, b \neq 0, c \neq 0.$$

Note: Multiply and divided by *abc*.

(14) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -2$, find $\begin{vmatrix} a_1 & -\frac{1}{2}a_3 & a_2 & a_3 \\ b_1 & -\frac{1}{2}b_3 & b_2 & b_3 \\ c_1 & -\frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix}$?