Linear Algebra

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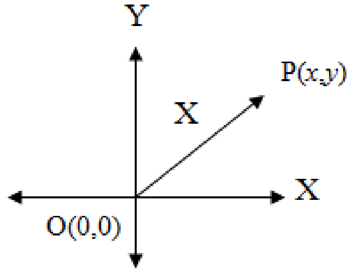
Chapter four
Vectors and Vector Spaces

Vectors: Consider the matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$ of degree (2×1) paired with X the line segment has a tail O(0,0) Its vertex is P(x,y) and vice versa with the directed line segment $\overrightarrow{OP}$.

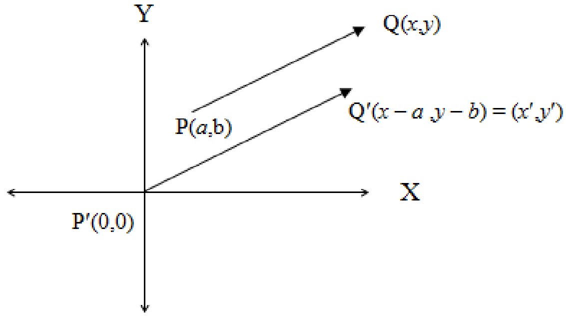


The vector is in the plane: It is the matrix $\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ of degree (2×1), where x and y are real numbers called components of $\vec{X}$.

Equal vectors: The vectors $\vec{X} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{Y} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are equal if and only if the corresponding elements are equal, that is $x_1 = x_2$ and $y_1 = y_2$.

Example: The vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are not equal since the corresponding elements in the second row are not equal

Remark: The beginning of the vector may not be the origin point, so its beginning may be the point (a,b). The line vector $\overrightarrow{PQ}$ beginning from P(a,b) (not the origin point) and ending with the point Q(x,y), so this vector can be represented by the vector P'O'(x',y') whose beginning is at O and its vertex is the point ($x - a$,$y - b$).

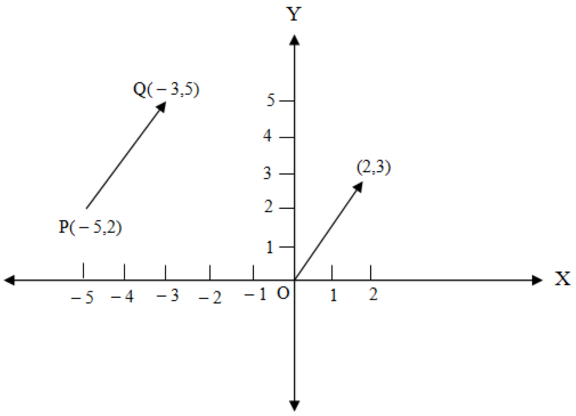


Examples:

(1) If Q(s,t) is the vertex of the vector $\overrightarrow{PQ}$, where $\overrightarrow{P'Q'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ whose beginning P(–5,2), we can find the values of s and t as follows:

$x - a = x' \;\Rightarrow\; s - (-5) = 2 \;\Rightarrow\; s = -3$

$y - b = y' \;\Rightarrow\; t - 2 = 3 \;\Rightarrow\; t = 5$

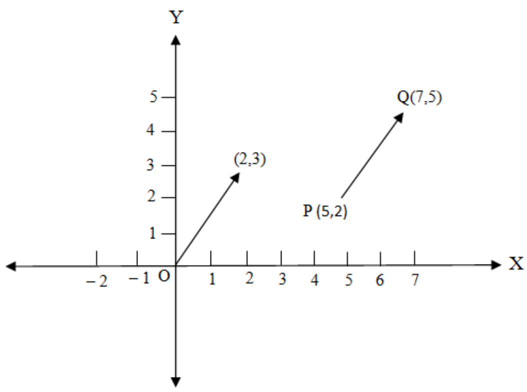


(2) If P(a,b) is the beginning of the vector $\overrightarrow{PQ}$, where $\overrightarrow{P'Q'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and the vertex is Q(7,5), find the value of each a, b ?

Solution:

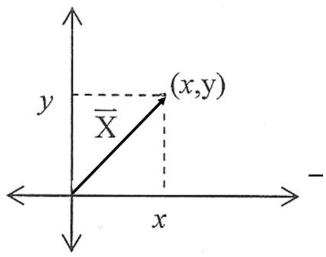
$7 - a = 2 \;\Rightarrow\; a = 5$ and

$5 - b = 3 \;\Rightarrow\; b = 2$



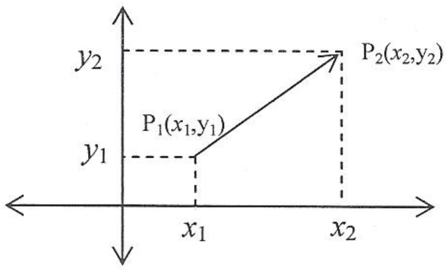
Definitions:

(1) The length of the vector $\overrightarrow{X}(x,y)$ is $\left\|\overrightarrow{X}\right\| = \sqrt{x^2 + y^2}$.



(2) The length of the straight line segment $\overrightarrow{P_1P_2}$ it is the distance between the two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ which is equal to

$$\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Examples:

(1) Find the length of the vector $\overrightarrow{X} = (6,-8)$?

Solution:

$$\left\|\overrightarrow{X}\right\| = \sqrt{x^2 + y^2} = \sqrt{(6)^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

(2) Find the distance between the two points P(2,3) and Q(5,–1) (The length of the straight line segment $\overrightarrow{PQ}$)?

Solution:

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{(5-2)^2 + (-1-3)^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Remark: The two vectors $\overrightarrow{X_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\overrightarrow{X_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are parallel if $x_1\, y_2 = x_2\, y_1$, that is, if and only if they are located on vertical or straight lines with the same slope,

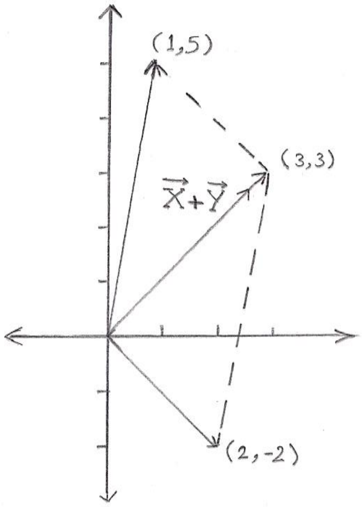
If $m_1 = \dfrac{y_1}{x_1},\ m_2 = \dfrac{y_2}{x_2} \Rightarrow m_1 = m_2 \Rightarrow \dfrac{y_1}{x_1} = \dfrac{y_2}{x_2} \Rightarrow x_1 y_2 = x_2 y_1.$

Operations on vectors:

Definition: Let both $\vec{X}=(x_1,y_1)$ and $\vec{Y}=(x_2,y_2)$ be vectors in the plane, so their sum is $\vec{X}+\vec{Y}=(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$.

Example: Let $\vec{X}=(1,5)$ and $\vec{Y}=(2,-2)$, then

$\vec{X}+\vec{Y}=(1,5)+(2,-2)=(3,3)$.

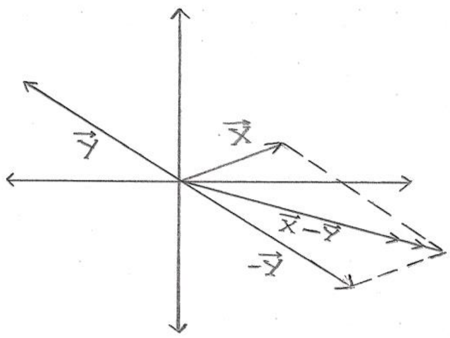


Definition: Let $\vec{X}=(x,y)$ and k is any real number, then $k\vec{X}=k(x,y)=(kx,ky)$

If k > 0, then $k\vec{X}$ has the same direction of $\vec{X}$.

If k < 0, then $k\vec{X}$ has opposite direction of $\vec{X}$.

Remark: The vector $\vec{O}=(0,0)$ is called the zero vector and $\vec{X}+\vec{O}=\vec{X}$. Also $\vec{X}+(-1)\vec{X}=\vec{O}$ and writes $(-1)\vec{X}$ as the form $-\vec{X}$ and called minus $\vec{X}$, and $\vec{X}-\vec{Y}=\vec{X}+(-\vec{Y})$ called the difference between $\vec{X}$ and $\vec{Y}$.



Note that adding two vectors represents one of the diagonals of a parallelogram and subtracting two vectors representing the other.

The angle between two vectors: the angle between two non-zero vectors $\vec{X}=(x_1,y_1)$ and $\vec{Y}=(x_2,y_2)$ is the angle θ and $0 \le \theta \le 180^\circ$

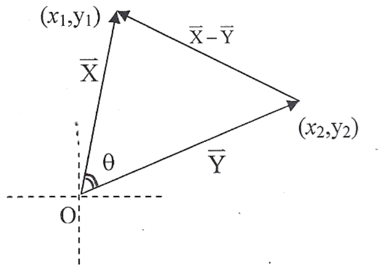
$$\left\|\vec{X}-\vec{Y}\right\|^2 = \left\|\vec{X}\right\|^2 + \left\|\vec{Y}\right\|^2 - 2\left\|\vec{X}\right\|\left\|\vec{Y}\right\|\cos\theta \qquad \text{(The law of the cosine)} \qquad \ldots(1)$$

$$\vec{X}-\vec{Y} = (x_1 - x_2) + (y_1 - y_2)$$

$$\left\|\vec{X}-\vec{Y}\right\|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= \underbrace{x_1^2 + y_1^2} + \underbrace{x_2^2 + y_2^2} - 2(x_1x_2 + y_1y_2)$$

$$= \left\|\vec{X}\right\|^2 + \left\|\vec{Y}\right\|^2 - 2(x_1x_2 + y_1y_2)$$



Substituting in (1) we get that

$$\boxed{\cos\theta = \frac{x_1x_2 + y_1y_2}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}} \quad \text{, where } \left\|\vec{X}\right\| \neq 0, \left\|\vec{Y}\right\| \neq 0$$

Inner Product

Definition: Let $\vec{X} = (x_1, y_1)$ and $\vec{Y} = (x_2, y_2)$ be two vectors, the inner product of the two vectors $\vec{X}$ and $\vec{Y}$ or dot product is defined as

$\vec{X}\cdot\vec{Y} = x_1x_2 + y_1y_2$

Accordingly, the previous law will be

$$\boxed{\cos\theta = \frac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}} \quad , 0 \leq \theta \leq \pi$$

Example: Let $\vec{X} = (2,4)$ and $\vec{Y} = (-1,2)$, then

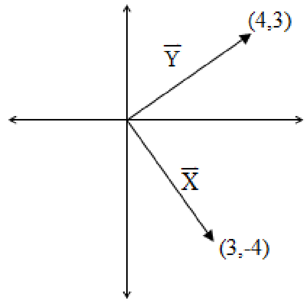
$\vec{X}\cdot\vec{Y} = (2)(-1) + (4)(2) = 6$, $\left\|\vec{X}\right\| = \sqrt{2^2 + 4^2} = \sqrt{20}$, $\left\|\vec{Y}\right\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$\cos\theta = \dfrac{6}{\sqrt{20}\sqrt{5}} = 0.6 \Rightarrow \theta = 53.2^\circ$ approximately

Remark: If $\vec{X}\cdot\vec{Y} = 0$, then $\cos\theta = 0$ and the vectors are orthogonal if and only if $\vec{X}\cdot\vec{Y} = 0$

Example: The two vectors $\vec{X} = (3,-4)$ and $\vec{Y} = (4,3)$ are orthogonal since

$\vec{X} \cdot \vec{Y}$ = (3)(4) +(– 4)(3) = 0.



Theorem: Let $\vec{X}$, $\vec{Y}$ and $\vec{Z}$ are vectors, k is any number, then

(1) $\vec{X} \cdot \vec{X} = \|\vec{X}\|^2 \geq 0$ satisfy the equality if and only if $\vec{X} = \vec{O}$.

(2) $\vec{X} \cdot \vec{Y} = \vec{Y} \cdot \vec{X}$ (Commutative property)

(3) $(\vec{X} + \vec{Y}) \cdot \vec{Z} = \vec{X} \cdot \vec{Z} + \vec{Y} \cdot \vec{Z}$ (The property of distributing multiplication on the addition)

(4) $(k\vec{X}) \cdot \vec{Y} = \vec{X} \cdot (k\vec{Y}) = k(\vec{X} \cdot \vec{Y})$.

Unit Vector: It is the vector whose length is equal to one unit.

If $\vec{X}$ a non-zero vector, the unit vector is the vector $\vec{U} = \dfrac{1}{\|\vec{X}\|}\vec{X}$

Example: Let $\vec{X} = (-4,3)$ be a vector, then

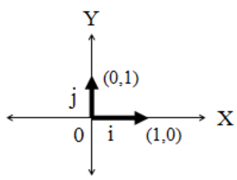
$$\|\vec{X}\| = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} + \sqrt{25} = 5$$

$\vec{U} = \dfrac{1}{5}(-4,3) = (\dfrac{-4}{5}, \dfrac{3}{5})$ it is the unit vector because

$$\|\vec{U}\| = \sqrt{(\frac{-4}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{16+9}{25}} = 1$$

Remark:

(1) i = (1,0) and j = (0,1) unit vector in $\mathbb{R}^2$ they are orthogonal to where i lies toward the positive X-axis and j toward the positive Y-axis.



(2) The vector $\vec{X}=(x,y)$ in $\mathbb{R}^2$ we writes in terms of i and j in $\mathbb{R}^2$ as follows $X = x\,i + y\,j$.

Example: Let $\vec{X}=(-3,7)$, then $X = -3\,i + 7j$.

Vector of type n

Definition: The matrix $\vec{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ is said to be a vector of type n and $x_1, x_2, \ldots, x_n$ called the components of the vector $\vec{X}$.

Remarks: The two vectors $\vec{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $\vec{Y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$ of the type n is equal if $(1 \le i \le n)$ and $(x_i = y_i)$.

Example: The two vectors $\vec{X}=\begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \end{bmatrix}$ and $\vec{Y}=\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ of type 4, $\vec{X}\neq\vec{Y}$ since the third component of them not equal $(4 \neq -2)$.

Remark: The vector can also be written in a row. For example, in the previous example the vectors can be written as $\vec{X}=[1,3,-2,5]$, $\vec{Y}=[1,3,4,5]$.

Operations on vectors:

Let $\vec{X}=[x_1, x_2, \ldots, x_n]$ and $\vec{Y}=[y_1, y_2, \ldots, y_n]$ be any vectors, k any number, then

(1) $k\overrightarrow{X} = k\,[x_1, x_2, \ldots, x_n] = [k\,x_1, k\,x_2, \ldots, k\,x_n]$

(2) $\overrightarrow{X} + \overrightarrow{Y} = [x_1, x_2, \ldots, x_n] + [y_1, y_2, \ldots, y_n] = [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$

(3) $\overrightarrow{X} - \overrightarrow{Y} = \overrightarrow{X} + (-\overrightarrow{Y}) = [x_1, x_2, \ldots, x_n] - [y_1, y_2, \ldots, y_n] = [x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n]$

Meaning multiplying a vector by any number, it is the same to the law of multiplying a matrix by a number, as well as addition and subtraction.

Example: Let $\overrightarrow{X_1} = [2,3,-4]$, $\overrightarrow{X_2} = [-1,2,6]$, $\overrightarrow{X_3} = \left[\frac{1}{2}, \frac{3}{4}, \frac{-5}{8}\right]$, find the value of

(1) $2\overrightarrow{X_1} + \overrightarrow{X_2} - 8\overrightarrow{X_3}$ **(2)** $\frac{1}{2}(\overrightarrow{X_1} - \overrightarrow{X_2})$

Solution:

(1) $2\overrightarrow{X_1} + \overrightarrow{X_2} - 8\overrightarrow{X_3} = 2\,[2,3,-4] + [-1,2,6] - 8\left[\frac{1}{2}, \frac{3}{4}, \frac{-5}{8}\right]$

$= [4,6,-8] + [-1,2,6] - [4,6,-5] = [-1,2,3]$

(2) $\frac{1}{2}(\overrightarrow{X_1} - \overrightarrow{X_2}) = \frac{1}{2}\,([2,3,-4] - [-1,2,6]) = \frac{1}{2}\,[3,1,-10]$

$= \left[\frac{3}{2}, \frac{1}{2}, -5\right]$

Exercises:

(1) Let $\overrightarrow{X_1} = [3,1,-4]$, $\overrightarrow{X_2} = [2,2,-3]$, $\overrightarrow{X_3} = [0,-4,1]$, $\overrightarrow{X_4} = [-4,-4,6]$, prove that

(a) $2\overrightarrow{X_1} - 5\overrightarrow{X_2} = [-4,-8,7]$

(b) $2\overrightarrow{X_2} + \overrightarrow{X_4} = \overrightarrow{O}$

(c) $2\overrightarrow{X_1} - 3\overrightarrow{X_2} - \overrightarrow{X_3} = \overrightarrow{O}$

(d) $2\overrightarrow{X_1} - \overrightarrow{X_2} - \overrightarrow{X_3} + \overrightarrow{X_4} = \overrightarrow{O}$

(2) Draw a diagram of a straight segment directed at $\mathbb{R}^2$ which represents the following

(a) $\overrightarrow{X_1} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **(b)** $\overrightarrow{X_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **(c)** $\overrightarrow{X_3} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ **(d)** $\overrightarrow{X_4} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

(3) Find the vertex for each of the following vectors and draw a diagram for it

(a) The vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the tail (3,2).

(b) The vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and the tail (1,2).

(4) Find $\overrightarrow{X} + \overrightarrow{Y}$, $\overrightarrow{X} - \overrightarrow{Y}$ and $3\overrightarrow{X} - 2\overrightarrow{X}$

(a) $\overrightarrow{X} = (2,3)$, $\overrightarrow{Y} = (-2,5)$.

(b) $\overrightarrow{X} = (0,3), \overrightarrow{Y} = (3,2)$.

(5) Let $\overrightarrow{X} = (1,2), \overrightarrow{Y} = (-3,4), \overrightarrow{Z} = (x,4), \overrightarrow{U} = (-2,y)$, find x and y such that

(a) $\overrightarrow{Z} = 2\overrightarrow{X}$ **(b)** $\frac{3}{2}\overrightarrow{U} = \overrightarrow{Y}$ **(c)** $\overrightarrow{Z} + \overrightarrow{U} = \overrightarrow{X}$

(6) Find the length for each of the following vectors

(a) (1,2) **(b)** (–3,–4) **(c)** (0,2)

(7) Find the distance for each pair of the following points

(a) (3,4),(2,3) **(b)** (3,4),(0,0) **(c)** (2,0),(0,3)

(8) Find the unit vector with the direction of $\overrightarrow{X}$

(a) $\overrightarrow{X} = (-3,4)$ **(b)** $\overrightarrow{X} = (-2,-3)$ **(c)** $\overrightarrow{X} = (5,0)$

(9) Find $\overrightarrow{X} \cdot \overrightarrow{Y}$ for each of the following vectors

(a) $\overrightarrow{X} = (1,2), \overrightarrow{Y} = (2,-3)$ **(b)** $\overrightarrow{X} = (-3,-4), \overrightarrow{Y} = (4,-3)$

(10) Prove that

(a) $i \cdot i = j \cdot j = 1$ **(b)** $i \cdot j = 0$

(11) Which of the following vectors $\overrightarrow{X_1} = (1,2)$, $\overrightarrow{X_2} = (0,1)$, $\overrightarrow{X_3} = (-2,-4)$, $\overrightarrow{X_4} = (-2,1)$, $\overrightarrow{X_5} = (2,5)$, $\overrightarrow{X_6} = (-6,3)$ are

(a) orthogonal **(b)** In the same direction

Theorem: Let $\vec{X}, \vec{Y}, \vec{Z}$ be a vectors in $\mathbb{R}^n$ and let c and d numbers, then

(a) $\vec{X} + \vec{Y}$ vector in $\mathbb{R}^n$ ($\mathbb{R}^n$ is closed under the addition of vectors operation)

(1) $\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$

(2) $\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{X} + \vec{Y}) + \vec{Z}$

(3) There exists unique vector $\vec{O}$ which is called zero vector in $\mathbb{R}^n$ such that $\vec{X} + \vec{O} = \vec{O} + \vec{X} = \vec{X}$.

(4) There exists unique vector $-\vec{X}$ in $\mathbb{R}^n$ such that $\vec{X} + (-\vec{X}) = (-\vec{X}) + \vec{X} = \vec{O}$

(b) $c\vec{X}$ vector in $\mathbb{R}^n$

(5) $c(\vec{X} + \vec{Y}) = c\vec{X} + c\vec{Y}$

(6) $(c+d)\vec{X} = c\vec{X} + d\vec{X}$

(7) $c(d\vec{X}) = (cd)\vec{X}$

(8) $1 \cdot \vec{X} = \vec{X}$

Definition: The length of the vector of the norm $\vec{X} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is

$$\boxed{\left\|\vec{X}\right\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}} \qquad \ldots(1)$$

Or it is a distance between the point $(x_1, x_2, \ldots, x_n)$ and the original point.

Example: Find the value of the numerical constant k to make the norm of the vector $\vec{A} = (5,3,k)$ equal to $\sqrt{50}$?

Solution:

$$\left\|\overrightarrow{A}\right\| = \sqrt{5^2 + 3^2 + k^2}$$

$$\sqrt{50} = \sqrt{25 + 9 + k^2}$$

$$\sqrt{50} = \sqrt{34 + k^2}$$

$50 = 34 + k^2$ Square both sides

$50 - 34 = k^2$

$16 = k^2 \Rightarrow k = \mp 4$

Definition: The distance between the points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is the length of the vector $\overrightarrow{X} - \overrightarrow{Y}$ where $\overrightarrow{X} = (x_1, x_2, \ldots, x_n)$ and $\overrightarrow{Y} = (y_1, y_2, \ldots, y_n)$.

$$\boxed{\left\|\overrightarrow{X} - \overrightarrow{Y}\right\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}} \quad \ldots(2)$$

Example: Let $\overrightarrow{X} = (2,3,2,-1), \overrightarrow{Y} = (4,2,1,3)$

$$\left\|\overrightarrow{X}\right\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2} = \sqrt{18}$$

$$\left\|\overrightarrow{Y}\right\| = \sqrt{4^2 + 2^2 + 1^2 + 3^2} = \sqrt{30}$$

$$\left\|\overrightarrow{X} - \overrightarrow{Y}\right\| = \sqrt{(2-4)^2 + (3-2)^2 + (2-1)^2 + (-1-3)^2} = \sqrt{22}$$

Example: Find the value of the numerical constant k to make the distance between the vectors $\overrightarrow{A} = (3,-1,6,3)$ and $\overrightarrow{B} = (2,k,1,-4)$ equal to 6 units? **(Home work)**

Remarks:

(1) The length of the vector represents the distance between the vector and the original point.

(2) The distance between two vectors in $\mathbb{R}^n$ represents the distance between the vertices points of the vectors.

(3) Prove that $\|\vec{X}-\vec{Y}\| = \|\vec{Y}-\vec{X}\|$.

Inner Product on $\mathbb{R}^n$

Definition: Let $\vec{X} = (x_1, x_2, \ldots, x_n)$ and $\vec{Y} = (y_1, y_2, \ldots, y_n)$ vectors in $\mathbb{R}^n$ then the inner product is defined as the form

$\vec{X}\cdot\vec{Y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

$$\boxed{\vec{X}\cdot\vec{Y} = \sum_{i=1}^{n} x_i y_i}$$

The inner product is also called point product.

Example: Let $\vec{X} = (2,3,2,-1)$ and $\vec{Y} = (4,2,1,3)$ two vectors, then

$\vec{X}\cdot\vec{Y} = (2)(4) + (3)(2) + (2)(1) + (-1)(3) = 8 + 6 + 2 - 3 = 13$

Cauchy-Schwarz Inequality

Theorem: Let $\vec{X},\vec{Y}$ be a vectors in $\mathbb{R}^n$, then $|\vec{X}\cdot\vec{Y}| \le \|\vec{X}\|\,\|\vec{Y}\|$.

Proof: If $\vec{Y} = \vec{O}$, then $\|\vec{Y}\| = 0$ and $\vec{X}\cdot\vec{Y} = 0$ and the theorem satisfy.

Let $\vec{X} \ne \vec{O}$, $\vec{Y} \ne \vec{O}$ and r arbitrary fixed, then

$(\vec{X} - r\vec{Y})\cdot(\vec{X} - r\vec{Y}) \ge 0$ (previous theorem)

$\vec{X}\cdot\vec{X} - 2r\vec{X}\cdot\vec{Y} + r^2\vec{Y}\cdot\vec{Y} \ge 0$

$\|\vec{X}\|^2 - 2r\vec{X}\cdot\vec{Y} + r^2\|\vec{Y}\|^2 \ge 0$

Because r represent any constant, the inequality above is true when $r = \dfrac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}$

$$\|\vec{X}\|^2 - 2\frac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}\,\vec{X}\cdot\vec{Y} + \left(\frac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}\right)^2 \|\vec{Y}\|^2 \ge 0$$

$$\|\vec{X}\|^2 - 2\frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} + \left(\frac{\left(\vec{X}\cdot\vec{Y}\right)}{\left\|\vec{Y}\right\|^2}\right)^2 \|\vec{Y}\|^2 \geq 0$$

$$\|\vec{X}\|^2 - 2\frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} + \frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} \geq 0$$

$\|\vec{X}\|^2\|\vec{Y}\|^2 - 2(\vec{X}\cdot\vec{Y})^2 + (\vec{X}\cdot\vec{Y})^2 \geq 0$ multiply by $\|\vec{Y}\|^2$

$\|\vec{X}\|^2\|\vec{Y}\|^2 - (\vec{X}\cdot\vec{Y})^2 \geq 0$

$\|\vec{X}\|^2\|\vec{Y}\|^2 \geq (\vec{X}\cdot\vec{Y})^2$

$|\vec{X}\cdot\vec{Y}| \leq \|\vec{X}\|\|\vec{Y}\|$ $\left(\sqrt{\vec{X}^2} = \left|\vec{X}\right|\right)$, taking the square root of both sides

Example: Let $\vec{X}$ = (2,3) and $\vec{Y}$ = (1,0), then
$\vec{X}\cdot\vec{Y}$ = (2)(1) + (3)(0) = 2

$\left\|\vec{X}\right\| = \sqrt{2^2+3^2} = \sqrt{13}$, $\left\|\vec{Y}\right\| = \sqrt{1^2+0^2} = 1$, $\left\|\vec{X}\right\|\left\|\vec{Y}\right\| = \sqrt{13}\,(1) = \sqrt{13}$

$\left|\vec{X}\cdot\vec{Y}\right| = 2 < \sqrt{13} = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|$

Definition: The angle between the two non-zero vectors $\vec{X}$ and $\vec{Y}$ is the unique number θ and 0 ≤ θ ≤ π, where $\cos\theta = \dfrac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}$

From Cauchy-Schwarz inequality we note that $\left|\dfrac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}\right| \leq 1$ also we know that |cos θ| ≤ 1.

Remark: We note that $\left|\vec{X}\cdot\vec{Y}\right| = \left|\cos\theta\right|\left\|\vec{X}\right\|\left\|\vec{Y}\right\|$. That is, the Cauchy - Schwarz inequality becomes equal if we multiply the right-hand side by the absolute cosine of the angle between the two vectors

Example: Let $\vec{X} = (1,0,0,1)$ and $\vec{Y} = (0,1,0,1)$, then

$\|\vec{X}\| = \sqrt{2}$, $\|\vec{Y}\| = \sqrt{2}$, $\vec{X} \cdot \vec{Y} = 1$.

Therefore,

$$\cos\theta = \frac{\vec{X} \cdot \vec{Y}}{\|\vec{X}\|\,\|\vec{Y}\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \;\Rightarrow\; \theta = \cos^{-1}\frac{1}{2} \;\Rightarrow\; \theta = 60^\circ.$$

Exercise: Find the angle between each of the following vectors:

(1) $\vec{X} = (0,1)$ and $\vec{Y} = (1,0)$ **Solution:** 90°

(2) $\vec{X} = (0,0,1)$ with itself **Solution:** 0°

(3) $\vec{X} = (0,0,1)$ and $\vec{Y} = (1,0,1)$ **Solution:** 45°

(4) $\vec{X} = (2,3,4)$ with itself **Solution:** 0°

Definition: Let $\vec{X}, \vec{Y}$ be a vectors in $\mathbb{R}^n$ we say that

(1) $\vec{X}$ and $\vec{Y}$ are orthogonal if $\vec{X} \cdot \vec{Y} = 0$ **or** if one of the vectors is zero.

(2) $\vec{X}$ and $\vec{Y}$ are parallel if $|\vec{X} \cdot \vec{Y}| = \|\vec{X}\|\|\vec{Y}\|$.

(3) $\vec{X}$ and $\vec{Y}$ are in the same direction if $\vec{X} \cdot \vec{Y} = \|\vec{X}\|\|\vec{Y}\|$

Remark: The previous definition can be formulated as follows:

Let $\vec{X}, \vec{Y}$ be a vectors in $\mathbb{R}^n$ and θ is the angle between them, then

(1) $\vec{X}$ and $\vec{Y}$ are orthogonal if cos θ = 0.

(2) $\vec{X}$ and $\vec{Y}$ are parallel if cos θ = ± 1.

(3) $\vec{X}$ and $\vec{Y}$ are in the same direction if cos θ = 1.

Example: Let $\vec{X} = (1,0,0,1)$, $\vec{Y} = (0,1,1,0)$ and $\vec{Z} = (3,0,0,3)$.

$\vec{X}\cdot\vec{Y}=0$, $\vec{Y}\cdot\vec{Z}=0$, so $\vec{X}$ and $\vec{Y}$ are orthogonal, also $\vec{Y}$ and $\vec{Z}$ are orthogonal. $\vec{X}\cdot\vec{Z}=6$, $\|\vec{X}\|=\sqrt{2}$, $\|\vec{Z}\|=\sqrt{18}$ $\Rightarrow$ $\|\vec{X}\|\|\vec{Z}\|=\sqrt{2}\sqrt{18}=\sqrt{36}=6$ $\Rightarrow \vec{X}\cdot\vec{Z}=\|\vec{X}\|\|\vec{Z}\|$.

Thus $\vec{X}$ and $\vec{Z}$ are parallel and because $\vec{X}\cdot\vec{Z}$ positive then $\vec{X}$ and $\vec{Z}$ are in the same direction.

Exercise: Any pair of the following vectors is parallel and which is orthogonal $\vec{X}_1$ = (–2,3,–1,–1), $\vec{X}_2$ = (–2,–1,–3,4), $\vec{X}_3$ = (1,2,3,–4), if there is a parallel pair are they in the same direction?

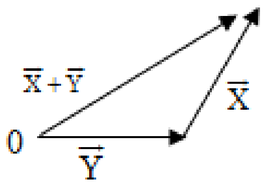
The following theorem is a result of the Cauchy-Schwarz inequality and it is called the triangle inequality

Theorem: Let $\vec{X},\vec{Y}$ be a vectors in $\mathbb{R}^n$, then $\|\vec{X}+\vec{Y}\|\le\|\vec{X}\|+\|\vec{Y}\|$.

Proof:

$\|\vec{X}+\vec{Y}\|^2=(\vec{X}+\vec{Y})\cdot(\vec{X}+\vec{Y})$ ($\vec{X}\cdot\vec{X}=\|\vec{X}\|^2\ge 0$ by previous theorem)

$=\vec{X}\cdot\vec{X}+2(\vec{X}\cdot\vec{Y})+\vec{Y}\cdot\vec{Y}$



$=\|\vec{X}\|^2+2(\vec{X}\cdot\vec{Y})+\|\vec{Y}\|^2$

$\le\|\vec{X}\|^2+2\|\vec{X}\|\|\vec{Y}\|+\|\vec{Y}\|^2$ Cauchy-Schwarz inequality

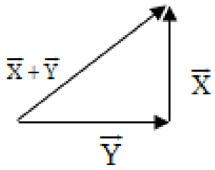
$=(\|\vec{X}\|+\|\vec{Y}\|)^2$

$\|\vec{X}+\vec{Y}\|\le\|\vec{X}\|+\|\vec{Y}\|$ taking the square root of both sides

The following theorem is also an important theorem which is called Pythagorean Theorem.

Theorem: Let $\vec{X}, \vec{Y}$ be a vectors in $\mathbb{R}^n$, then $\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$ if and only if $\vec{X}$ and $\vec{Y}$ are orthogonal.

Proof: $(\Rightarrow)$ By previous theorem $\vec{X}\cdot\vec{X} = \|\vec{X}\|^2 \geq 0$ we get



$\|\vec{X}+\vec{Y}\|^2 = (\vec{X}+\vec{Y})\cdot(\vec{X}+\vec{Y})$

$$= \|\vec{X}\|^2 + 2(\vec{X}\cdot\vec{Y}) + \|\vec{Y}\|^2 \qquad \ldots(1)$$

When $\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$ then $2(\vec{X}\cdot\vec{Y}) = 0$, then

$\vec{X}\cdot\vec{Y} = 0$ which is mean that $\vec{X}$ and $\vec{Y}$ are orthogonal.

$(\Leftarrow)$ $\vec{X}$ and $\vec{Y}$ are orthogonal, then $\vec{X}\cdot\vec{Y} = 0$ and so equation (1) become

$\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$

Example: Let $\vec{X}$ = (0,0,0,3) and $\vec{Y}$ = (0,– 4,3,0)

$\|\vec{X}\| = \sqrt{9} = 3$, $\|\vec{Y}\| = \sqrt{25} = 5$, $\|\vec{X}\| + \|\vec{Y}\| = 8$

$\vec{X}+\vec{Y}$ = (0,– 4,3,3) $\Rightarrow \|\vec{X}+\vec{Y}\| = \sqrt{34}$

$\|\vec{X}+\vec{Y}\|^2 = 34 = 3^2 + 5^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$

$\vec{X}\cdot\vec{Y} = 0$, so we get that $\vec{X}$ and $\vec{Y}$ are orthogonal

Definition: The unit vector $\vec{U}$ in $\mathbb{R}^n$ is a vector of length one unit.

If $\vec{X}$ is a non zero vector in $\mathbb{R}^n$ then the vector $\vec{U}$ defined as $\vec{U} = \dfrac{1}{\|\vec{X}\|}\vec{X}$ is a unit vector in the same direction of $\vec{X}$.

Example: Let $\vec{X}$ = (1,2,2) and $\vec{Y}$ = (2,0,0), then $\|\vec{X}\|$ = 3 and $\|\vec{Y}\|$ = 2.

The two vectors $\vec{U_1}$ and $\vec{U_2}$ defined as

$$\vec{U_1} = \frac{\vec{X}}{\|\vec{X}\|} = \frac{1}{3}(1,2,2) \qquad\qquad \vec{U_2} = \frac{\vec{Y}}{\|\vec{Y}\|} = \frac{1}{2}(2,0,0)$$

Are unit vectors in the same direction of $\overrightarrow{X}$ and $\overrightarrow{Y}$ respectively.

In the case $\mathbb{R}^3$ indicates unit vectors in the positive directions of the axes $\overrightarrow{X}$, $\overrightarrow{Y}$ and $\overrightarrow{Z}$ symbols i = (1,0,0), j = (0,1,0) and k = (0,0,1).

Also the vector $\overrightarrow{X} = (x,y,z)$ in $\mathbb{R}^3$ can be written by using the form of the unit vectors i, j and k as the form $\overrightarrow{X} = x\,i + y\,j + z\,k$.

As example the vector $\overrightarrow{X} = (2, -1, 3)$ writes as $\overrightarrow{X} = 2i - j + 3k$.

In general: in the case of $\mathbb{R}^n$ the unit vectors in the positive directions of the axes are $\overrightarrow{U_1} = (1,0,0,\ldots,0)$, $\overrightarrow{U_2} = (0,1,0,\ldots,0)$, …, $\overrightarrow{U_n} = (0,0,0,\ldots,1)$ which are mataually orthogonal.

If $\overrightarrow{X} = (x_1,x_2,\ldots,x_n)$ then $\overrightarrow{X} = x_1\overrightarrow{U_1} + x_2\overrightarrow{U_2} + \ldots + x_n\overrightarrow{U_n}$.

Exercises:

(1) Find the unit vector $\overrightarrow{U}$ in the same direction of $\overrightarrow{X}$ for each of the following

(a) $\overrightarrow{X} = (2,-1,3)$

(b) $\overrightarrow{X} = (1,2,3,4)$

(c) $\overrightarrow{X} = (0,1,-1)$

(d) $\overrightarrow{X} = (0,-1,2,-1)$

(2) Write $\overrightarrow{X}$ and $\overrightarrow{Y}$ In terms of unit vectors i, j and k, where $\overrightarrow{X} = (1,2,-3)$ and $\overrightarrow{Y} = (2,3,-1)$.

Vectors Space

Definition: We call V a real vector space (or V vector space over $\mathbb{R}$) if the set of elements has two operations:

$\oplus$: Binary operation on vector space, i.e. $\oplus : V \longrightarrow V$

$\odot$: The multiplication operation $\odot$ is an application from $\mathbb{R} \times V$ to V, and called the multiplication operation by a real number. Meaning $\odot: \mathbb{R} \times V \longrightarrow V$

Satisfy the following axioms:

First: Axioms of Addition

(1) Closure: The effect of vector addition, i.e. if $\vec{X}, \vec{Y} \in V$, then $(\vec{X} + \vec{Y}) \in V$.

(2) Associative: If $\vec{X}, \vec{Y}, \vec{Z} \in V$, then $(\vec{X} + \vec{Y}) + \vec{Z} = \vec{X} + (\vec{Y} + \vec{Z})$.

(3) Identity element: For addition there exists an element denoted by $(\vec{O})$ such that,
$\vec{X} + \vec{O} = \vec{O} + \vec{X} = \vec{X} \quad , \ \forall \ \vec{X} \in V$

(4) Addition Inverse: For all $\vec{X} \in V$ there exists $\vec{Y} \in V$ such that
$\vec{X} + \vec{Y} = \vec{Y} + \vec{X} = \vec{O}$, $\vec{Y}$ is called the inverse of $\vec{X}$ and denoted by $(-\vec{X})$.

(5) Commutative: For all $\vec{X}, \vec{Y} \in V$, $\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$.

Second: Axioms of Scalar (Real) Multiplication

(6) For any $\alpha \in \mathbb{R}$ and any $\vec{X} \in V$, then $\alpha \vec{X} \in V$.

(7) If $\alpha, \beta \in \mathbb{R}$ and $\vec{X} \in V$, then $(\alpha \ \beta)\vec{X} = \alpha(\beta \vec{X})$.

(8) For any $\vec{X} \in V$, then $1 \cdot \vec{X} = \vec{X}$.

Third: Axioms of Distributives

(9) If $\alpha, \beta \in \mathbb{R}$ and $\vec{X} \in V$, then $\vec{X} + \beta \vec{X} = \alpha \vec{X} \ (\alpha + \beta)$.

(10) If $\alpha \in \mathbb{R}$ and $\vec{X}, \vec{Y} \in V$, then $\alpha(\vec{X} + \vec{Y}) = \alpha \vec{X} + \alpha \vec{Y}$.

Remark: We call for any element of vector space by a vector.

Example: Show that $\mathbb{R}$ (set of real numbers) with addition and multiplication operations is a vector space

Solution:

First: axioms of addition:

(1) If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$ (closing by the effect of the addition operation)

(2) If $x, y, z \in \mathbb{R}$, then $(x + y) + z = x + (y + z) \in \mathbb{R}$ (the addition operation is associative)

(3) $0 \in \mathbb{R}$ (identity element), $\forall\, x \in \mathbb{R}$ then $x + 0 = 0 + x = x$.

(4) $x \in \mathbb{R}$ and $-x \in \mathbb{R}$, then $x + (-x) = (-x) + x = 0$

$(-x)$ is the additive inverse for x.

(5) $x, y \in \mathbb{R}$, then $x + y = y + x$ (the addition operation is commutative)

Second: Axioms of Scalar Multiplication

(6) If $x, y \in \mathbb{R}$, then $x\, y \in \mathbb{R}$ (closing by the effect of the multiplication operation)

(7) If $x, y, z \in \mathbb{R}$, then $(x\, y)\, z = x\, (y\, z) \in \mathbb{R}$ (the multiplication operation is associative)

(8) $1 \in \mathbb{R}$, $1 \cdot x = x \cdot 1$, $\forall\, x \in \mathbb{R}$.

Third: Axioms of Distributives

(9) If $x, y, z \in \mathbb{R}$, then $(x + y)\, z = x\, z + y\, z \in \mathbb{R}$.

(10) If $x, y, z \in \mathbb{R}$, then $z\, (x + y) = z\, x + z\, y \in \mathbb{R}$.

Thus $(\mathbb{R}, +, \cdot)$ is a vector space.

Remark: If V is a vector space on the set of the complex numbers $\mathbb{C}$ then it is called complex vector space.

Example: The set I (set of integer numbers) with addition and multiply be a real number not a vector space, i.e. $(\mathrm{I}, +, \cdot)$ is not a vector space.

Solution:

The operation of multiplication by a real number does not satisfy or what is called (closed in effect of the multiplication operation).

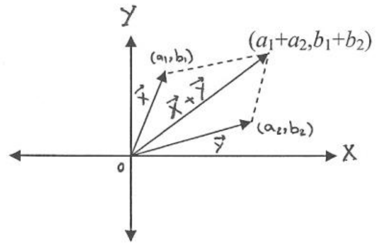
Take $x = 8 \in \mathrm{I}$, $\alpha = \dfrac{1}{3}$, then $\alpha x = \dfrac{1}{3}\cdot 8 = \dfrac{8}{3} \notin \mathrm{I}$.

So, $(\mathrm{I},+,\cdot)$ is not a vector space.

Example: Check in detail that $\mathbb{R}^2$ vector space, where $\mathbb{R}^2 = \{(a,b) : a,b \in \mathbb{R}\}$ if $(+)$ and $(\cdot)$ defined as the formulas

$(a_1,b_1)+(a_2,b_2)=(a_1+a_2,b_1+b_2)$

$k\,(a,b)=(ka,kb)$



Solution:

First: axioms of addition:

(1) Let $\vec{\mathrm{X}}=(a_1,b_1), \vec{\mathrm{Y}}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$

$$\vec{\mathrm{X}}+\vec{\mathrm{Y}}=(a_1,b_1)+(a_2,b_2)$$

$$=(\underbrace{a_1+a_2}_{\in\mathbb{R}},\underbrace{b_1+b_2}_{\in\mathbb{R}})\in\mathbb{R}^2 \qquad \text{(definition the addition operation of two vectors)}$$

$\therefore$ The closure operation with the effect of vectors addition satisfy

(2) Let $\vec{\mathrm{X}}=(a_1,b_1), \vec{\mathrm{Y}}=(a_2,b_2), \vec{\mathrm{Z}}=(a_3,b_3)\in\mathbb{R}^2, a_1,b_1,a_2,b_2,a_3,b_3\in\mathbb{R}$

$$(\vec{\mathrm{X}}+\vec{\mathrm{Y}})+\vec{\mathrm{Z}}=((a_1,b_1)+(a_2,b_2))+(a_3,b_3)$$

$$=(a_1+a_2,b_1+b_2)+(a_3,b_3)$$

$$=((a_1+a_2)+a_3,(b_1+b_2)+b_3) \qquad \text{(definition the addition operation of two vectors)}$$

$$=(a_1+(a_2+a_3),b_1+(b_2+b_3)) \qquad \text{(the addition of numbers is associative)}$$

$$(\vec{\mathrm{X}}+\vec{\mathrm{Y}})+\vec{\mathrm{Z}}=(a_1,b_1)+(a_2+a_3,b_2+b_3)$$

$$=(a_1,b_1)+((a_2,b_2)+(a_3,b_3))$$

$$=\vec{\mathrm{X}}+(\vec{\mathrm{Y}}+\vec{\mathrm{Z}})$$

$\therefore$ The associative operation is satisfy

(3) For all $\vec{\mathrm{X}}=(a,b)\in\mathbb{R}^2$ there exists $\vec{\mathrm{O}}=(0,0)\in\mathbb{R}^2$ such that

$$\vec{\mathrm{X}}+\vec{\mathrm{O}}=(a,b)+(0,0)$$

$$=(a+0,b+0) \qquad \text{(definition the addition operation of two vectors)}$$

$$=(a,b)=\vec{\mathrm{X}} \qquad \text{(Zero is the additive identity element of numbers)}$$

$\therefore$ The identity element exists

(4) For all $\vec{X}=(a,b)\in\mathbb{R}^2$ there exists $-\vec{X}=(-a,-b)\in\mathbb{R}^2$ such that

$$\begin{aligned}\vec{X}+(-\vec{X})&=(a,b)+(-a,-b)\\&=(a+(-a),b+(-b)) \quad \text{(definition the addition operation of two vectors)}\\&=(0,0)=\vec{O}\end{aligned}$$

$\therefore$ The addition Inverse exists

(5) Let $\vec{X}=(a_1,b_1),\vec{Y}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$

$$\begin{aligned}\vec{X}+\vec{Y}&=(a_1,b_1)+(a_2,b_2)\\&=(a_1+a_2,b_1+b_2)\\&=(a_2+a_1,b_2+b_1)\\&=(a_2,b_2)+(a_1,b_1) \quad \text{(the addition of numbers is commutative)}\\&=\vec{Y}+\vec{X}\end{aligned}$$

$\therefore$ The commutative property satisfy

Second: Axioms of Scalar Multiplication

(6) For any $\vec{X}=(a,b)\in\mathbb{R}^2$ and for any number $r\in\mathbb{R}$

$$r\vec{X}=r(a,b)=(\underset{\in\mathbb{R}}{ra},\underset{\in\mathbb{R}}{rb})\in\mathbb{R}^2$$

(7) For any $\vec{X}=(a,b)\in\mathbb{R}^2$ and for any numbers $r,t\in\mathbb{R}$

$$\begin{aligned}(rt)\vec{X}=(rt)(a,b)&=((rt)a,(rt)b) \quad \text{(definition the scalar multiplication)}\\&=(r(ta),r(tb)) \quad \text{(the multiplication of numbers is associative)}\\&=r(ta,tb) \quad \text{(definition the scalar multiplication)}\\&=r(t(a,b)) \quad \text{(definition the scalar multiplication)}\\&=r(t\vec{X})\end{aligned}$$

(8) For any $\vec{X}=(a,b)\in\mathbb{R}^2$ and for any number $1\in\mathbb{R}$

$$\begin{aligned}1\cdot\vec{X}&=1(a,b)\\&=(1a,1b) \quad \text{(definition the scalar multiplication)}\\&=(a,b)=\vec{X}\end{aligned}$$

Third: Axioms of Distributives

(9) For any $\overrightarrow{X}=(a,b)\in\mathbb{R}^2$ and for any numbers $\alpha,\beta\in\mathbb{R}$

$$(\alpha+\beta)\overrightarrow{X}=(\alpha+\beta)(a,b)=((\alpha+\beta)a,(\alpha+\beta)b)$$ (definition the scalar multiplication)

$$=((\alpha a+\beta a),(\alpha b+\beta b))$$
$$=((\alpha a,\alpha b)+(\beta a,\beta b))$$
$$=(\alpha(a,b)+\beta(a,b))$$
$$=\alpha\overrightarrow{X}+\beta\overrightarrow{X}$$

(10) For any $\overrightarrow{X}=(a_1,b_1),\overrightarrow{Y}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$ and any $\alpha\in\mathbb{R}$

$$\alpha(\overrightarrow{X}+\overrightarrow{Y})=\alpha((a_1,b_1)+(a_2,b_2))$$

$$=\alpha(a_1+a_2,b_1+b_2)$$ (definition the addition operation of two vectors)

$$=(\alpha(a_1+a_2),\alpha(b_1+b_2))$$ (definition the scalar multiplication)

$$=(\alpha a_1+\alpha a_2,\alpha b_1+\alpha b_2)$$ (the multiplication distribution over the addition of numbers)

$$=((\alpha a_1,\alpha b_1)+(\alpha a_2,\alpha b_2))$$ (definition the addition operation of two vectors)

$$=\alpha(a_1,b_1)+\alpha(a_2,b_2)$$ (definition the scalar multiplication)

$$=\alpha\overrightarrow{X}+\alpha\overrightarrow{Y}$$

$\therefore$ $\mathbb{R}^2$ is a vector space

Exercise: Prove that $\mathbb{R}^3$ is a vector space where $\mathbb{R}^3=\{(a,b,c):a,b,c\in\mathbb{R}\}$?

Definition: Suppose $\mathbb{R}^n=\{(x_1,x_2,...,x_n):x_i\in\mathbb{R}\}$ for $i=1,2,...,n$ and let $\overrightarrow{X}=(x_1,x_2,...,x_n),\overrightarrow{Y}=(y_1,y_2,...,y_n)\in\mathbb{R}^n$, define addition operation on $\mathbb{R}^n$ as

$$\overrightarrow{X}+\overrightarrow{Y}=(x_1,x_2,...,x_n)+(y_1,y_2,...,y_n)$$
$$=(x_1+y_1,x_2+y_2,...,x_n+y_n)$$

$\odot$ Multiplication by a scalar (standard) or real number on $\mathbb{R}^n$

$$r\overrightarrow{X}=r(x_1,x_2,...,x_n)=(rx_1,rx_2,...,rx_n)$$

Exercises:

(1) Show that $(\mathbb{R}^n, \oplus, \odot)$ is a vector space.

(2) Let $\mathrm{W} = \mathrm{M}_{2\times 2} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, a_i \in \mathbb{R}, i = 1,2,3,4 \right\}$ be a set of all matrices of degree 2×2 with the addition and multiplication by a real number on matrices. Prove that W is a vector space on $\mathbb{R}$.

Example: Let $\mathrm{V} = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}, x_2 = \frac{1}{2}x_1 + 1\}$. Is $(\mathrm{V},+,\cdot)$ vector space such that $(+)$ is the ordinary addition on $\mathbb{R}^2$ and $(\cdot)$ is the ordinary multiplication of an element in $\mathbb{R}^2$ by a real number k.

Solution:

Let $\overrightarrow{\mathrm{X}}, \overrightarrow{\mathrm{Y}} \in \mathrm{V}$ such that

$$\overrightarrow{\mathrm{X}} = (a_1, a_2), \overrightarrow{\mathrm{Y}} = (b_1, b_2) : a_1, a_2, b_1, b_2 \in \mathbb{R}, a_2 = \frac{1}{2}a_1 + 1, b_2 = \frac{1}{2}b_1 + 1$$

$$\overrightarrow{\mathrm{X}} + \overrightarrow{\mathrm{Y}} = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$a_2 + b_2 = (\frac{1}{2}a_1 + 1) + (\frac{1}{2}b_1 + 1)$$

$$= \frac{1}{2}a_1 + 1 + \frac{1}{2}b_1 + 1$$

$$= \frac{1}{2}(a_1 + b_1) + 2$$

$$\neq \frac{1}{2}(a_1 + b_1) + 1$$

$\therefore\ \overrightarrow{\mathrm{X}} + \overrightarrow{\mathrm{Y}} \notin \mathrm{V}$

For an example: $\overrightarrow{\mathrm{X}}, \overrightarrow{\mathrm{Y}} \in \mathrm{V}$ such that $\overrightarrow{\mathrm{X}} = (0,1)$, $1 = \frac{1}{2}(0) + 1$ and $\overrightarrow{\mathrm{Y}} = (2,2)$, $2 = \frac{1}{2}(2) + 1$

$\overrightarrow{\mathrm{X}} + \overrightarrow{\mathrm{Y}} = (0,1) + (2,2) = (2,3) \notin \mathrm{V}$ because $3 \neq \frac{1}{2}(2) + 1$.

So the addition operation not closed on V.

$\therefore$ $(\mathrm{V},+,\cdot)$ is not vector space.

Example: Let $V=\{(x_1,x_2,x_3):x_1,x_2,x_3 \in \mathbb{R}\}$. Is (V,+,·) vector space if (+) and (·) defined as the formulas $c\overrightarrow{X}=c(x_1,x_2,x_3)=(cx_1,x_2,x_3)$ and

$\overrightarrow{X}+\overrightarrow{Y}=(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$

Solution:

In this example all the conditions satisfy except the first condition of axioms of distributives in (Third).

Let $\overrightarrow{X}=(a,b,c)\in V,\ \alpha,\beta\in\mathbb{R}$

$$(\alpha+\beta)\overrightarrow{X}=(\alpha+\beta)(a,b,c)$$
$$=((\alpha+\beta)a,b,c)$$

$$\alpha\overrightarrow{X}+\beta\overrightarrow{X}=\alpha(a,b,c)+\beta(a,b,c)$$
$$=(\alpha a,b,c)+(\beta a,b,c)$$
$$=((\alpha+\beta)a,b+b,c+c)$$
$$=((\alpha+\beta)a,2b,2c)$$

$((\alpha+\beta)a,2b,2c)\neq((\alpha+\beta)a,b,c)$

$\therefore\ \alpha\overrightarrow{X}+\beta\overrightarrow{X}\neq(\alpha+\beta)\overrightarrow{X}$

$\therefore$ (V,+,·) is not vector space. (Make sure the rest of the conditions are satisfy)

Example: Is (V,⊕,⊙) vector space where ⊕ and ⊙ are defined as the formulas

$c\odot\overrightarrow{X}=c(x_1,x_2,x_3)=(cx_1,0,0)$

$\overrightarrow{X}\oplus\overrightarrow{Y}=(x_1,x_2,x_3)\oplus(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$

Solution:

(V,⊕,⊙) is not vector space, because if $\overrightarrow{X}=(x_1,x_2,x_3)\in V \quad x_1,x_2,x_3\in\mathbb{R}$

$1\odot\overrightarrow{X}=1(x_1,x_2,x_3)=(x_1,0,0)\neq(x_1,x_2,x_3)\Rightarrow 1\odot\overrightarrow{X}\neq\overrightarrow{X}$.

Example: Let V be the set of the solutions of the system of linear equations $A_{m\times n}\ X_{n\times 1}=B_{m\times 1}$ such that $B_{m\times 1}\neq O_{m\times 1}$. Is (V,+,·) vector space with the addition of matrix and multiply the matrix by a number.

Solution: Let $\overrightarrow{X},\overrightarrow{Y}\in V$

$A\overrightarrow{X}=B \quad ...(1) \qquad \& \qquad A\overrightarrow{Y}=B \quad ...(2)$

$$A\left(\overrightarrow{X}+\overrightarrow{Y}\right)=A\overrightarrow{X}+A\overrightarrow{Y}$$

$= B + B = 2B$ From (1) and (2)

$\neq B$

$\therefore \left(\overrightarrow{X}+\overrightarrow{Y}\right) \notin V$

$\therefore \left(\overrightarrow{X}+\overrightarrow{Y}\right)$ not necessary that the system of linear equations has solution.

$\therefore$ V not closed under the addition operation

$\therefore$ (V,+,·) is not vector space

Exercises:

(1) Let $V=\{(x,y): x,y \in \mathbb{R}, y=2x\}$. Is (V,+,·) vector space?

(2) Let V be a set of real functions defined on $\mathbb{R}$ then $(V,\oplus,\odot)$ is a vector space where $f,g \in V, c \in \mathbb{R}, (f \oplus g)(x) = f(x)+g(x), (c \odot g)(x) = c \cdot g(x)$

(3) Let $V=\{(x_1,x_2,0): x_1,x_2 \in \mathbb{R}\}$. Is $(V,\oplus,\odot)$ vector space where defined as the formulas $(x_1,x_2,0)\oplus(y_1,y_2,0)=(x_1+y_1,x_2+y_2,0)$ and $c \odot \overrightarrow{X} = c(x_1,x_2,x_3)=(cx_1,cx_2,0)$

(4) Let V be the set of all polynomial vectors of degree (2) or less, $p \in$ V, where for all $x \in \mathbb{R}$, $p(x) = a_0 + a_1x + a_2x^2$, $a_0, a_1, a_2 \in \mathbb{R}$, for $p, q \in$ V

$p(x) = a_0 + a_1x + a_2x^2$, $q(x) = b_0 + b_1x + b_2x^2$, $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$

$(p \oplus q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

$(c \odot p)(x) = ca_0 + (ca_1)x + (ca_2)x^2$; $c \in \mathbb{R}$

Prove that $(V,\oplus,\odot)$ is a vector space.

(5) Show that is the following sets with the operations defined below represent a vector space and if not, which conditions are not satisfy?

(a) The set of all ordered triads of real numbers (x,y,z) with the two defined operations $(x_1,y_1,z_1)\oplus(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$ and

$c \odot (x, y, z) = (x, 1, z)$.

(b) The set of all ordered triads of real numbers in the form (0,0,z) with the two defined operations $(0,0,z_1) \oplus (0,0,z_2) = (0,0,z_1 + z_2)$ and
$c \odot (0,0,z) = (0,0,cz)$.

(c) The set of all ordered pairs of real numbers (x,y) with the two defined operations $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c \odot (x, y) = (0,0)$.

(6) Let V = { f ; $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(0) = 1$}. Is (V,+,·) vector space?

(7) Which of the following is a vector space with ordinary addition and multiplication over $\mathbb{R}^2$

(i) W = {$(x,0)$; $x \in \mathbb{R}$ }   **(ii)** W = {$(x,1)$; $x \in \mathbb{R}$ }   **(iii)** W = {(x,x); $x \in \mathbb{R}$ }

(8) Which of the following is a vector space with ordinary addition and multiplication over $\mathbb{R}^3$?

(i) W = {$(a + 3b,a,b)$; a, $b \in \mathbb{R}$ }   **(ii)** W = {$(-a, -2a,a)$; $a \in \mathbb{R}$ }

(9) Which of the following is a vector space with addition of matrix and multiply the matrix by a number?

(i) W = {$[a_{ij}]_{2\times2}$; $a_{12} = 0$, $a_{ij} \in \mathbb{R}$ }   **(ii)** W = {$[a_{ij}]_{2\times2}$; $a_{11} = a_{22} = 0$}

(iii) $W = \left\{ \begin{bmatrix} a & c & 0 \\ b & d & 0 \end{bmatrix}_{2\times3}, a,b,c,d \in \mathbb{R} \right\}$

Subspaces

Definition: Let V be a vector space and W is a non empty subset of V, if W is a vector space for the two operations defined on V we say that W is a subspace of V.

Examples: Let W = {$(a,b,1)$; a, $b \in \mathbb{R}$ }, $W \subset \mathbb{R}^3$, is W a subspace of $\mathbb{R}^3$?

Solution: Let $\vec{X}, \vec{Y} \in W$, where

$\vec{X} = (a_1, b_1, 1), \vec{Y} = (a_2, b_2, 1)$

$$\overrightarrow{X} + \overrightarrow{Y} = (a_1, b_1, 1) + (a_2, b_2, 1)$$
$$= (a_1 + a_2, b_1 + b_2, 2) \neq (a_1 + a_2, b_1 + b_2, 1)$$
$$\therefore \left(\overrightarrow{X} + \overrightarrow{Y}\right) \notin W$$

Thus W is no a subspace of $\mathbb{R}^3$.

Theorem: Let V be a vector space with the two operations $\oplus$ and $\odot$, $\phi \neq W \subseteq V$. Then W is a subspace of V if and only if the two conditions satisfy

(1) If $\overrightarrow{X}, \overrightarrow{Y} \in W$, then $\overrightarrow{X} \oplus \overrightarrow{Y} \in W$

(2) If $t \in \mathbb{R}$ and $\overrightarrow{X} \in W$, then $t \odot \overrightarrow{X} \in W$

Proof: (⇒) Let $W \subseteq V$

$\therefore$ W is a subspace from the definition.

$\therefore$ W is closed for addition and closed for multiplication by a number i.e. properties (1) and (2) are satisfied.

(⇐) If W is closed for addition and closed for multiplication by a number, we must prove that W is a vector space.

First: axioms of addition:

(1) W closed for the addition from hypothesis.

(2) If $\overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z} \in W$, then $\overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z} \in V$ because $W \subseteq V$

But the addition is associative on V (since V is a vector space)

$(\overrightarrow{X} \oplus \overrightarrow{Y}) \oplus \overrightarrow{Z} = \overrightarrow{X} \oplus (\overrightarrow{Y} \oplus \overrightarrow{Z})$

$\therefore \oplus$ associative on W

(3) Let $\overrightarrow{X} \in W$, because W is closed for multiplication by a number, so $(-1)\overrightarrow{X} = -\overrightarrow{X} \in W$

$-\overrightarrow{X} \oplus \overrightarrow{X} = (-\overrightarrow{X}) \oplus \overrightarrow{X} = \overrightarrow{O}$ (W closed for the addition)

Thus $-\overrightarrow{X}$ the additive inverse for the vector $\overrightarrow{X}$.

(4) Because $(-\overrightarrow{X}) \in W$ and $\overrightarrow{X} \oplus (-\overrightarrow{X}) = (-\overrightarrow{X}) \oplus \overrightarrow{X} = \overrightarrow{O}$ (W closed for the addition)

Thus $\overrightarrow{O} \in W$. So $\overrightarrow{O}$ is the identity element for the addition operation.

(5) If $\overrightarrow{X}, \overrightarrow{Y} \in W$, then $\overrightarrow{X}, \overrightarrow{Y} \in V$ because $W \subseteq V$

But the addition is commutative on V (since V is a vector space)

$\therefore\ \vec{X}\oplus\vec{Y}=\vec{Y}\oplus\vec{X}$

$\therefore\ \oplus$ commutative on W

Second: Axioms of Scalar Multiplication

(6) W is closed for multiplication by a number by hypothesis.

(7) Let $\alpha,\ \beta\in\mathbb{R}$ and $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $(\alpha\beta)\vec{X}=\alpha(\beta\vec{X})\in W$.

(8) Let $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $1\odot\vec{X}=\vec{X}$.

Third: Axioms of Distributives

(9) Let $\alpha,\ \beta\in\mathbb{R}$ and $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $(\alpha+\beta)\vec{X}=\alpha\vec{X}\oplus\beta\vec{X}$.

(10) Let $\alpha\in\mathbb{R}$ and $\vec{X},\vec{Y}\in W$, then $\vec{X},\vec{Y}\in V$ because $W\subseteq V$

But V is a vector space, thus $\alpha(\vec{X}\oplus\vec{Y})=\alpha\vec{X}\oplus\alpha\vec{Y}$.

Therefore (W,⊕,⊙) is a vector space and also is a subspace of (V,⊕,⊙).

Example: Let W = {(a,b,0); a, $b\in\mathbb{R}$ }, $W\subset\mathbb{R}^3$, is W a subspace of $\mathbb{R}^3$?

Solution: Let $\vec{X}=(a_1,b_1,0),\vec{Y}=(a_2,b_2,0)\in W$

(1) $\vec{X}+\vec{Y}=(a_1,b_1,0)+(a_2,b_2,0)$

$=(a_1+a_2,b_1+b_2,0)$

Since the third component equal zero, so $\vec{X}+\vec{Y}\in W$

(2) Let $\vec{X}=(a_1,b_1,0)\in W,\alpha\in\mathbb{R}$, then $\alpha\vec{X}=\alpha(a_1,b_1,0)=(\alpha a_1,\alpha b_1,0)\in W$

$\therefore\alpha\vec{X}\in W$

The two properties are satisfy, thus W is a subspace of $\mathbb{R}^3$.

Exercise: Let W={(a,b); b = 2a,a, $b\in\mathbb{R}$ }, $W\subset\mathbb{R}^2$. Is W subspace of $\mathbb{R}^2$?

Remarks: If (V,⊕,⊙) is any vector space, then

(1) (V,⊕,⊙) is a subspace by itself because $V\subseteq V$ and V is a vector space.

(2) $W=\{\vec{O}\}$ is a subspace of V.

Therefore for any non zero vector space there exists at least two subspaces of it.

Proof (2): $\vec{O} \in W$, then $\vec{O} + \vec{O} = \vec{O} \in W$

$\therefore$ W is closed for addition operation

$\vec{O} \in W, \alpha \in \mathbb{R}$, then $\alpha \cdot \vec{O} = \vec{O} \in W$

$\therefore$ W is closed for multiplication by a number

$\therefore$ W is a subspace of V

Exercise: Let $V = \mathbb{R}^2$, W = $\{(x,y);\ a\,x + by = 0,\ x\,,y \in \mathbb{R}\}$ where a and b are real numbers. Prove that W is a subspace of $\mathbb{R}^2$.

Examples:

(1) Let $A\vec{X} = \vec{O}$ homogeneous system where A is a matrix of degree m×n and let $W \subseteq \mathbb{R}^n$ contains all the solutions for this homogeneous system, then W is a subspace of $\mathbb{R}^n$.

Solution: Let $\vec{Y}, \vec{Z} \in W \subseteq \mathbb{R}^n$, then $\vec{Y}$ and $\vec{Z}$ two solutions for the homogeneous system, i.e. $A\vec{Y} = \vec{O}$ and $A\vec{Z} = \vec{O}$

(a) $A(\vec{Y} + \vec{Z}) = A\vec{Y} + A\vec{Z} \ = \vec{O} + \vec{O}$

$\therefore\ A(\vec{Y} + \vec{Z}) = \vec{O}$

Thus $\vec{Y} + \vec{Z}$ solution for the homogeneous system, i.e. $\vec{Y} + \vec{Z} \in W$.

(b) Let $t \in \mathbb{R}, \vec{Y} \in W$

$A(t\vec{Y}) = t\,(A\vec{Y})$ (by previous theorem)

$= t\,(\vec{O})$

$A(t\vec{Y}) = \vec{O}$

$\therefore\ t\ \vec{Y} \in W$

Thus $t\ \vec{Y}$ is a solution for the homogeneous system, i.e. $t\ t\ \vec{Y} \in W$.

Therefore W is a subspace of $\mathbb{R}^n$.

(2) Let V be a vector space and let U and W two subspaces of V. Prove that G = U $\cap$ W is a subspace of V.

Proof: It is clear that G $\neq \phi$ because $\vec{O} \in$ U and $\vec{O} \in$ W, so $\vec{O} \in$ U $\cap$ W

(a) Let $\vec{X}, \vec{Y} \in G \ \Rightarrow\ \vec{X}, \vec{Y} \in U \ \Rightarrow\ \vec{X} + \vec{Y} \in U$ (U is a subspace)

Also, $\Rightarrow\ \vec{X}, \vec{Y} \in W \ \Rightarrow\ \vec{X} + \vec{Y} \in W$ (W is a subspace)

Thus $\vec{X} + \vec{Y} \in G$

(b) Let $\alpha \in \mathbb{R}$, $\alpha\overrightarrow{X} \in U$ and $\alpha\overrightarrow{X} \in W$, thus $\alpha\overrightarrow{X} \in G$

$\therefore$ G is a subspace of V

Exercises:

(1) Let W = {(x,y,z); $a\,x + by + cz = 0$}, a, b, $c \in \mathbb{R}$ not all zero. Show that W is a subspace of $\mathbb{R}^3$.

(2) Let W = {A: A is a square matrix of degree 2×2 and invertible matrix}. Is W subspace of V = $M_{2\times2}(\mathbb{R})$}.

(3) Let W be a set of the diagonal matrices of degree n×n. Is W subspace of V = $M_{n\times n}(\mathbb{R})$}.

(4) Let W = {A: A is a square matrix of degree 2×2, $A^2 = I$ }. Is W subspace of V = $M_{2\times2}(\mathbb{R})$}.

(5) Which of the following sets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, set of all vectors of the form:

(a) $(a,b,2)$

(b) (a,b,c), where $a + b = c$

(c) (a,b,c), where $c > 0$

(d) (a,b,c), where $a = c = 0$

(e) (a,b,c), where $1+2a = b$

(f) (a,b,c), where $a = -c$

(6) Show that the set of the solutions for the system $A\overrightarrow{X} = B$, where A is a matrix of degree m×n not a subspace of $\mathbb{R}^n$ if $B \neq O$.

(7) Let V be a vector space, U and W are two subspaces of V. Prove that

$G = U + W = \{ \vec{u} + \vec{w}, \vec{u} \in U \; and \; \vec{w} \in W \}$ is a subspace.

Linear Combination

Definition: We say that the vector $\overrightarrow{X}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}$ if we can write it as the form $\overrightarrow{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_n\overrightarrow{X_n}$, where $c_1, c_2, \ldots, c_n$ are numbers

Examples:

(1) Consider the vectors $\overrightarrow{X_1}=(1,2,1,-1)$, $\overrightarrow{X_2}=(1,0,2,-3)$, $\overrightarrow{X_3}=(1,1,0,-2)$ in $\mathbb{R}^4$, show that the vector $\overrightarrow{X}=(2,1,5,-5)$ is a linear combination of $\overrightarrow{X_1},\overrightarrow{X_2},\overrightarrow{X_3}$.

Solution: Suppose that we have c_1, c_2 and c_3 where

$c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+c_3\overrightarrow{X_3}=\overrightarrow{X}$

$c_1(1,2,1,-1) + c_2(1,0,2,-3) + c_3(1,1,0,-2) = (2,1,5,-5)$

$(c_1,2c_1,c_1,-c_1) + (c_2,0,2c_2,-3c_2) + (c_3,c_3,0,-2c_3) = (2,1,5,-5)$

$(c_1 + c_2 + c_3,\ 2c_1 + c_3, c_1 + 2c_2, - c_1 - 3c_2 - 2c_3) = (2,1,5,-5)$

$c_1 + c_2 + c_3 = 2 \qquad \dots(1)$

$2c_1 \quad + c_3 = 1 \qquad \dots(2)$

$c_1 + 2c_2 \quad = 5 \qquad \dots(3)$

$- c_1-3c_2-2c_3 =-5 \qquad \dots(4)$

We use Gauss-Jordan reduction method to find the values of c_1, c_2 and c_3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ -1 & -3 & -2 & -5 \end{array}\right] \xrightarrow[\substack{R_3=r_3-r_1 \\ R_4=r_4+r_1}]{R_2=r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & -1 & -3 \end{array}\right]$$

$$\xrightarrow[\substack{R_2=r_2+2r_3 \\ R_4=r_4+2r_3}]{R_1=r_1-r_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & -3 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$c_1 + 2c_3 = -1$

$- 3c_3 = 3 \Rightarrow c_3 = -1$

$c_2 - c_3 = 3$

$- 3c_3 = 3 \Rightarrow c_3 = -1$

$$\begin{cases} c_1 - 2 = -1 \Rightarrow c_1 = 1 \\ c_2+1=3 \Rightarrow c_2 = 2 \\ c_3 =-1 \end{cases}$$

Therefore

$1(1,2,1,-1) + 2(1,0,2,-3) - 1(1,1,0,-2) = (2,1,5,-5)$

$\therefore$ $\overrightarrow{X}$ is a linear combination of $\overrightarrow{X_1},\overrightarrow{X_2}$ and $\overrightarrow{X_3}$.

(2) Let $\overrightarrow{X_1}=(1,2,-1)$ and $\overrightarrow{X_2}=(1,0,-1)$ two vectors in $\mathbb{R}^3$. Is the vector $\vec{X}=(1,0,2)$ a linear combination of $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$.

Solution: The vector $\vec{X}$ to be a linear combination of $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ we must find two numbers c_1 and c_2 such that

$c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}=\vec{X}$

$c_1(1,2,-1) + c_2(1,0,-1) = (1,0,2)$

$(c_1,2c_1,-c_1) + (c_2,0,-c_2) = (1,0,2)$

$(c_1 + c_2, 2c_1,-c_1 - c_2) = (1,0,2)$

$c_1 + c_2 = 1 \qquad \ldots(1)$

$2c_1 \quad = 0 \qquad \ldots(2)$

$-c_1 - c_2 = 2 \qquad \ldots(3)$

We use Gauss-Jordan reduction method to find the values of c_1 and c_2

$$\left[\begin{array}{cc:c} 1 & 1 & 1 \\ 2 & 0 & 0 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow[R_3=r_3+r_1]{R_2=r_2-2r_1} \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 3 \end{array}\right]$$

The third row means $0\,c_1 + 0\,c_2 = 3 \Rightarrow 0 = 3$ which is impossible.
Thus the system has no solution.
$\therefore\ \vec{X}$ is not a linear combination of $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$.

Exercises:

(1) Which of the following vectors is a linear combination of the vectors $\overrightarrow{X_1}=(2,1,-2)$, $\overrightarrow{X_2}=(-2,-1,0)$, $\overrightarrow{X_3}=(4,2,-3)$

(a) $\vec{X}=(1,0,0)$ (b) $\vec{X}=(0,0,1)$ **(c)** $\vec{X}=(1,1,1)$ **(d)** $\vec{X}=(4,2,-6)$

(2) If possible express the vector (1,1,1) as a linear combination of the vectors in $\mathbb{R}^3$ $\overrightarrow{X_1}=(2,1,-3)$, $\overrightarrow{X_2}=(1,2,-2)$, $\overrightarrow{X_3}=(2,-5,-1)$.

(3) If possible express the vector $\vec{X}=(4,5,1)$ as a linear combination of the vectors in $\mathbb{R}^3$ $\overrightarrow{X_1}=(1,2,1)$, $\overrightarrow{X_2}=(2,3,1)$, $\overrightarrow{X_3}=(1,1,0)$. (the solution $c_1 = 2$, $c_2 = -1$, $c_3 = 4$)

Theorem: Let V be a vector space and $\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n} \in V$ and let W be a set of all linear combination of the vectors $\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}$, i.e.

$W=\{c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+...+c_n\overrightarrow{X_n}, c_1,c_2,...,c_n \in \mathbb{R}\}$, then W is a subspace of V.

Proof: because $\overrightarrow{O}=0\overrightarrow{X_1}+0\overrightarrow{X_2}+...+0\overrightarrow{X_n}$

$\therefore\ \overrightarrow{O}\in W \Rightarrow W\neq\phi$

$\therefore W\subseteq V$

(1) Let $\overrightarrow{X},\overrightarrow{Y}\in W$ so there exists $c_1,c_2,...,c_n\in\mathbb{R}$ such that $\overrightarrow{X}=c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+...+c_n\overrightarrow{X_n}$

and there exists $d_1,d_2,...,d_n\in\mathbb{R}$ such that $\overrightarrow{Y}=d_1\overrightarrow{Y_1}+d_2\overrightarrow{Y_2}+...+d_n\overrightarrow{Y_n}$

$$\overrightarrow{X}\oplus\overrightarrow{Y}=(c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+...+c_n\overrightarrow{X_n})\oplus(d_1\overrightarrow{Y_1}+d_2\overrightarrow{Y_2}+...+d_n\overrightarrow{Y_n})$$

$$=(c_1+d_1)\overrightarrow{X_1}+(c_2+d_2)\overrightarrow{X_2}+...+(c_n+d_n)\overrightarrow{X_n}$$

$\overrightarrow{X}\oplus\overrightarrow{Y}=e_1\overrightarrow{X_1}+e_2\overrightarrow{X_2}+...+e_n\overrightarrow{X_n}$, where $c_1+d_1=e_1, c_2+d_2=e_2,...,c_n+d_n=e_n$

$\therefore\ \overrightarrow{X}\oplus\overrightarrow{Y}\in W$ (W closed for the addition)

(2) Let $\overrightarrow{X}=c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+...+c_n\overrightarrow{X_n}\in W, t\in\mathbb{R}$

$t\overrightarrow{X}=t(c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+...+c_n\overrightarrow{X_n})$

$=(tc_1)\overrightarrow{X_1}+(tc_2)\overrightarrow{X_2}+...+(tc_n)\overrightarrow{X_n}$ Axioms of Scalar Multiplication: $(\alpha\beta)\overrightarrow{X}=\alpha(\beta\overrightarrow{X})$

$=k_1\overrightarrow{X_1}+k_2\overrightarrow{X_2}+...+k_n\overrightarrow{X_n}$ $k_i=tc_i$, $i=1,2,...,n$

$\therefore\ t\overrightarrow{X}\in W$ (W is closed for multiplication by a number)

Therefore W is a subspace of V.

Span (Generating) Set

Definition: Let $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ be a set of vectors in a vector space V. We say that the set S spans V or V generated by S if any vector in V is a linear combination of the vectors of S.

Examples:

(1) Show that the set S = {(1,0), (0,1)} spans $\mathbb{R}^2$.

Solution: We must prove that every vector in $\mathbb{R}^2$ can be written as a linear combination of the vectors (1,0), (0,1).

Let $(a,b) \in \mathbb{R}^2$, to find the values of the numbers c_1 and c_2 such that

$(a,b) = c_1(1,0) + c_2(0,1)$
$\quad\quad = (c_1,0) + (0,c_2)$
$\quad\quad = (c_1,c_2) \Rightarrow c_1 = a \;,\; c_2 = b$
$\therefore (a,b) = a(1,0) + b(0,1)$
$\therefore$ S spans $\mathbb{R}^2$.

(2) Let $\overrightarrow{X_1} = (1,2), \overrightarrow{X_2} = (-1,1)$ show that the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is $\mathbb{R}^2$.

Solution: Let $\overrightarrow{X} = (a,b) \in \mathbb{R}^2, a,b \in \mathbb{R}$, to find the values of the numbers c_1 and c_2 such that

$\overrightarrow{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2}$
$(a,b) = c_1(1,2) + c_2(-1,1)$
$\quad\quad = (c_1,2c_1) + (-c_2,c_2)$
$\quad\quad = (c_1 - c_2, 2c_1 + c_2)$

$c_1 - c_2 = a \quad\quad \ldots(1)$
$2c_1 + c_2 = b \quad\quad \ldots(2)$
-------------------------------------- by addition

$3c_1 = a + b \Rightarrow c_1 = \frac{1}{3}(a + b)$ and $c_2 = \frac{1}{3}(b - 2a)$

$\overrightarrow{X} = \frac{1}{3}(a+b)\overrightarrow{X_1} + \frac{1}{3}(b-2a)\overrightarrow{X_2}$

$\therefore \{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ spans $\mathbb{R}^2$.

(3) Find the space spans by the set $\{(-1,2,1)\}$.

Solution:

$W = \{c\,(-1,2,1);\, c \in \mathbb{R}\} = \{(-c,2c,c);\, c \in \mathbb{R}\}$
$\quad\quad = \{c\,(-1,2,1);\, c \in \mathbb{R}\}$

This is equation of a straight line passing through the two points $(-1,2,1)$ and $(0,0,0)$.

(4) Find the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ where $\overrightarrow{X_1} = (-1,2,3), \overrightarrow{X_2} = (-2,4,6)$.

Solution: Let W is the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$, i.e.

$W = \{c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2}; c_1, c_2 \in \mathbb{R}\}$

$= \{c_1(-1,2,3) + c_2(-2,4,6)\ ; c_1, c_2 \in \mathbb{R}\}$

$= \{(-c_1, 2c_1, 3c_1) + (-2c_2, 4c_2, 6c_2)\ ; c_1, c_2 \in \mathbb{R}\}$

$= \{(-c_1 - 2c_2, 2c_1 + 4c_2, 3c_1 + 6c_2)\ ; c_1, c_2 \in \mathbb{R}\}$

$= \{(-1\,(c_1 + 2c_2), 2(c_1 + 2c_2), 3(c_1 + 2c_2))\ ; c_1, c_2 \in \mathbb{R}\}$

$= \{(c_1 + 2c_2)(-1,2,3)\ ; c_1, c_2 \in \mathbb{R}\}$

$= \{c_3\,(-1,2,3)\ ; c_3 \in \mathbb{R}\}$, suppose $c_1 + 2c_2 = c_3$

This is equation of a straight line passing through the two points (0,0,0) and (–1,2,3).

Note that $\overrightarrow{X_2} = 2\overrightarrow{X_1}$

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} = c_1\overrightarrow{X_1} + c_2(2\overrightarrow{X_1})$$
$$= (c_1 + 2c_2)\overrightarrow{X_1}$$
$$= c_3\overrightarrow{X_1}\ , \qquad \text{suppose } c_1 + 2c_2 = c_3$$

$\therefore$ The space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is the same space spans by the set $\{\overrightarrow{X_1}\}$

or **Note that** $\overrightarrow{X_1} = \frac{1}{2}\overrightarrow{X_2}$

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} = c_1(\tfrac{1}{2}\overrightarrow{X_2}) + c_2\overrightarrow{X_2}$$
$$= (\tfrac{1}{2}c_1 + c_2)\overrightarrow{X_2}$$
$$= c_4\overrightarrow{X_2}\ , \qquad \text{suppose } \tfrac{1}{2}c_1 + c_2 = c_4$$

$\therefore$ The space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is the same space spans by the set $\{\overrightarrow{X_2}\}$

Remark: Every line passing through the origin point is a subspace of $\mathbb{R}^2$.

(5) Express the zero vector as a linear combination of the two vectors $\overrightarrow{X_1} = (2,3), \overrightarrow{X_2} = (-3,1)$ in $\mathbb{R}^2$

Solution: To find the values of the numbers c_1 and c_2 such that

$\overrightarrow{O} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2}$

$(0,0) = c_1(2,3) + c_2(-3,1)$

$\quad = (2c_1, 3c_1) + (-3c_2, c_2) = (2c_1 - 3c_2,\ 3c_1 + c_2)$

$2c_1 - 3c_2 = 0$

$3c_1 + c_2 = 0 \ \Rightarrow\ c_2 = -3c_1$

$\therefore 2c_1 + 9c_1 = 0 \Rightarrow 11c_1 = 0 \Rightarrow c_1 = 0$

$\Rightarrow c_2 = -3(0) = 0$

$(0,0) = 0\cdot\overrightarrow{X_1} + 0\cdot\overrightarrow{X_2} = 0\cdot(2,3) + 0\cdot(-3,1)$

(6) Let W = $\{(x,y,z): x - 2y + z = 0\}$

(a) Prove that W is a subspace of $\mathbb{R}^3$. **(Home work)**

(b) Find the two vectors $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ such that W is generated by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$

Solution (b):

$W = \{(x,y,z): x - 2y + z = 0\}$

$= \{(x,y,2y - x): x, y \in \mathbb{R}\}$

$= \{(0,y,2y) + (x,0,-x); x, y \in \mathbb{R}\}$

$= \{y(0,1,2) + x(1,0,-1); x, y \in \mathbb{R}\}$

This is a set of linear combination of the two vectors (0,1,2) and (1,0,–1).

Let $\overrightarrow{X_1} = (1,0,-1)$ and $\overrightarrow{X_2} = (0,1,2)$

$\therefore$ W is generated by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$, i.e. $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ spans W.

Exercises:

(1) Let $\overrightarrow{X_1} = (1,0)$ and $\overrightarrow{X_2} = (1,1)$ show that the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is $\mathbb{R}^2$.

(2) Find the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ in $\mathbb{R}^3$, where $\overrightarrow{X_1} = (1,-1,2)$ and $\overrightarrow{X_2} = (0,1,1)$.

(3) Which of the following sets spans $\mathbb{R}^2$

(a) (–1,1), (1,2) **(b)** (1,0), (0,1) **(c)** (0,0), (1,1), (–2,–2) **(d)** (2,4), (–1,2)

(4) Which of the following sets spans $\mathbb{R}^3$

(a) (1,0,0), (0,1,0), (0,0,1) **(b)** (1,–1,2), (0,1,1)

(c) (1,1,0), (1,0,1), (0,1,1) **(d)** (2,2,3), (–1,–2,1), (0,1,0)

Linear Independence

Definition: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_k}\}$ be a set of distinct vectors in a vector space V. We say the set S is non linearly independent (dependent) if there exists a constants $(c_1, c_2, \ldots, c_k)$ not all zero, such that

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_k\overrightarrow{X_k} = \overrightarrow{O} \qquad \ldots(1)$$

Conversely, the set S is said to be linearly independent. That is, S is linearly independent if it satisfy (1) only when $c_1 = c_2 = \dots = c_k = 0$.

Remark: If the set $S=\{\overrightarrow{X_1},\overrightarrow{X_2},\dots,\overrightarrow{X_k}\}$, then S is non linearly independent if there exists a linear combination of vectors of the set S equal to zero, i.e.
$c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+\dots+c_k\overrightarrow{X_k}=\overrightarrow{O}$, such that at least one of $c_i \neq 0$, $1 \le i \le k$.

Example: Let S = {(1,0),(0,1)}. Show that S is linearly independent set.
Solution: Let c_1, c_2 be a constants
$c_1 (1,0) + c_2 (0,1) = (0,0)$
$(c_1,0) + (0,c_2) = (0,0)$
$(c_1,c_2) = (0,0) \Rightarrow c_1 = 0, c_2 = 0$
$\therefore$ S is linearly independent set

Exercise: Let S = {(2,2),(0,1)}. Is the set S linearly independent or dependent?

Example: Let S ={(1,1,–1),(2,3,–4),(4,3,–2)}. Show that S is non linearly independent (dependent) set.
Solution: Let c_1 and c_2 and c_3 be a constants
$c_1 (1,1,-1) + c_2 (2,3,-4) + c_3 (4,3,-2) = (0,0,0)$
$c_1 + 2c_2 + 4c_3 = 0$
$c_1 + 3c_2 + 3c_3 = 0$
$-c_1 - 4c_2 - 2c_3 = 0$

The coefficients matrix $A=\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -2 \end{bmatrix}$, because $|A| = 0$

$\therefore$ The linear system or the set has no trivial solution.
Thus S is non linearly independent (dependent) set.

Remark: If the system of equations is homogeneous and the determinant of the coefficients matrix equal to zero, then this means that the system has a solution other than the zero solution, but if the determinant not equal to zero, this means that the system has a unique solution (the zero solution).

To make sure of this, we will find the values of c_1, c_2 and c_3 in the previous example

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -2 \end{bmatrix} \xrightarrow[R_3=r_3+r_1]{R_2=r_2-r_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow[R_3=r_3+2r_2]{R_1=r_1-2r_2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 6c_3 = 0 \Rightarrow c_1 = -6c_3$
$c_2 - c_3 = 0 \Rightarrow c_2 = c_3$
Let $c_3 = 1$, then $c_1 = -6$ and $c_2 = 1$.
$-6(1,1,-1) + 1(2,3,-4) + 1(4,3,-2) = (0,0,0)$
Thus S is non linearly independent (dependent) set.

Exercise: Show that S = {(0,1,1),(2,1,2),(1,2,1)} is linearly independent set by using the previous remark.

Examples:

(1) If $S=\{\overrightarrow{X_1},\overrightarrow{X_2}\}$ is a linearly independent set in the vector space V. Prove that the set $\{\overrightarrow{X_1}+\overrightarrow{X_2},\overrightarrow{X_1}-\overrightarrow{X_2}\}$ is also linearly independent.

Solution: Let c_1 and c_2 be a constants

$c_1(\overrightarrow{X_1}+\overrightarrow{X_2})+c_2(\overrightarrow{X_1}-\overrightarrow{X_2})=\overrightarrow{O}$

$(c_1+c_2)\overrightarrow{X_1}+(c_1-c_2)\overrightarrow{X_2}=\overrightarrow{O}$

Because $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ are linearly independent, so

$c_1 + c_2 = 0$
$c_1 - c_2 = 0$
-------------------- by addition
$2c_1 = 0 \Rightarrow c_1 = 0$, $c_2 = 0$

$\therefore$ The set $\{\overrightarrow{X_1}+\overrightarrow{X_2},\overrightarrow{X_1}-\overrightarrow{X_2}\}$ is also linearly independent

(2) If $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ are two vectors in the vector space $\mathbb{R}^2$ such that $\overrightarrow{E_1}=(1,0)$ and $\overrightarrow{E_2}=(0,1)$. Prove that $S=\{\overrightarrow{E_1},\overrightarrow{E_2}\}$ is a linearly independent set. Also prove that the set $\{\overrightarrow{E_1},\overrightarrow{E_2},...,\overrightarrow{E_n}\}$ in $\mathbb{R}^n$ is linearly independent, where

$$\overrightarrow{E_1} = (1,0,0,...,0), \overrightarrow{E_2} = (0,1,0,0,...,0),..., \overrightarrow{E_n} = (0,0,...,0,1)$$

Solution: Let c_1 and c_2 be a constants

$c_1\overrightarrow{E_1} + c_2\overrightarrow{E_2} = \overrightarrow{O}$

$c_1(1,0) + c_2(0,1) = (0,0)$

$(c_1,0) + (0,c_2) = (0,0)$

$(c_1,c_2) = (0,0) \Rightarrow c_1 = 0, c_2 = 0$

$\therefore$ S is a linearly independent set

In the same way (or more generally) the set $\{\overrightarrow{E_1}, \overrightarrow{E_2},..., \overrightarrow{E_n}\}$ is a linearly independent set.

(3) Let $P_1(t) = t^2 + t + 2$, $P_2(t) = 2t^2 + t$, $P_3(t) = 3t^2 + 2t + 2$. Is the set $S = \{P_1(t), P_2(t), P_3(t)\}$ linearly independent or not?

Solution: Let c_1 and c_2 and c_3 be a constants

$c_1P_1(t) + c_2P_2(t) + c_3P_3(t) = 0$

$c_1(t^2 + t + 2) + c_2(2t^2 + t) + c_3(3t^2 + 2t + 2) = 0$

$(c_1 + 2c_2 + 3c_3)\, t^2 + (c_1 + c_2 + 2c_3)\, t + (2c_1 + 2c_3) = 0\, t^2 + 0\, t + 0$

$c_1 + 2c_2 + 3c_3 = 0$

$c_1 + c_2 + 2c_3 = 0$

$2c_1 \quad + 2c_3 = 0$

This homogeneous system has infinity of solutions **(check it)**

One of these solutions is $c_1 = 1$, $c_2 = 1$, $c_3 = -1$. So that $P_1(t) + P_2(t) - P_3(t) = \overrightarrow{O}$.

$\therefore$ The set S is non linearly independent (dependent).

Exercises:

(1) Show whether the following sets are linearly independent in V, where V is the vector space of all polynomials of second degree or less

(a) $S_1=\{x, 2x - 1, 1\}$ **(b)** $S_2=\{2x^2, 3x^2\}$ **(c)** $S_3=\{1, 2x + 3, x^2 + 2x + 1, 3x^2 - 2x\}$

(2) Show whether the following sets are linearly independent in the vector space V, where V is the vector space of 2×2 matrices

(a) $S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

(b) $S_2 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -7 & 3 \end{bmatrix} \right\}$

Theorem: S is a set of nonzero vectors in a vector space V. S is non linearly independent if and only if one of these vectors is a linear combination of all other vectors in S.

Proof: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, \ldots, \overrightarrow{X_k}\}$

Because S is non linearly independent

$\therefore$ There exists a constants $c_1, c_2, \ldots, c_i, \ldots, c_k$ not all zero ($c_i \neq 0$)

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + c_i\overrightarrow{X_i} + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k} = \overrightarrow{O}$$

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + \ldots + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k} = -c_i\overrightarrow{X_i}$$

$$\left(\frac{-c_1}{c_i}\right)\overrightarrow{X_1} + \left(\frac{-c_2}{c_i}\right)\overrightarrow{X_2} + \ldots + \left(\frac{-c_{i-1}}{c_i}\right)\overrightarrow{X_{i-1}} + \ldots + \left(\frac{-c_{i+1}}{c_i}\right)\overrightarrow{X_{i+1}} + \ldots + \left(\frac{-c_k}{c_i}\right)\overrightarrow{X_k} = \overrightarrow{X_i}$$

$\therefore$ $\overrightarrow{X_i}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_{i+1}}, \ldots, \overrightarrow{X_k}$.

Conversely, $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, \ldots, \overrightarrow{X_k}\}$

If $\overrightarrow{X_i}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_{i+1}}, \ldots, \overrightarrow{X_k}$

$\therefore$ There exists a constants $c_1, c_2, \ldots, c_{i-1}, \ldots, c_{i+1}, \ldots, c_k$ such that

$$\overrightarrow{X_i} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + \ldots + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k}$$

$$\overrightarrow{O} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + (-1)\overrightarrow{X_i} + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k}$$

$\therefore$ The set $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, \ldots, \overrightarrow{X_k}\}$ is non linearly independent

Examples:

(1) The set S = {(2,6),(3,9),(1,3)} is non linearly independent (dependent) since

$1(3,9) = 1(2,6) + 1(1,3)$

That is the vector (3,9) is a linear combination of the other vectors.

or solve using the definition

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + c_3\overrightarrow{X_3} = \overrightarrow{O} \quad \Rightarrow \quad 1(2,6) + 1(1,3) - 1(3,9) = (0,0)$

(2) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$ and $j(x) = (x+1)^2$, for all $x \in \mathbb{R}$. Show that the set $S = \{f, g, h, j\}$ is non linearly independent in the vector spaces for all real functions

Solution:

$j(x) = (x+1)^2 = x^2 + 2x + 1$

$= 1\cdot f(x) + 2\cdot g(x) + 1\cdot h(x)$

$j = 1\cdot f + 2\cdot g + 1\cdot h$

$\therefore$ $S = \{f, g, h, j\}$ is non linearly independent.

Theorem: In $\mathbb{R}^n$ any set containing more than n elements is a non linearly independent set.

Proof: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ set in $\mathbb{R}^n$ such that m > n, if

$\overrightarrow{X_1} = (a_{11}, a_{12}, \ldots a_{1n})$, $\overrightarrow{X_2} = (a_{21}, a_{22}, \ldots a_{2n})$, …, $\overrightarrow{X_m} = (a_{m1}, a_{m2}, \ldots a_{mn})$

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_m\overrightarrow{X_m} = \overrightarrow{O}$,

$c_1(a_{11}, a_{12}, \ldots a_{1n}) + c_2(a_{21}, a_{22}, \ldots a_{2n}) + \ldots + c_m(a_{m1}, a_{m2}, \ldots a_{mn}) = (0,0,\ldots,0)$

$c_1a_{11} + c_2a_{21} + \ldots + c_ma_{m1} = 0 \qquad \ldots(1)$

$c_1a_{12} + c_2a_{22} + \ldots + c_ma_{m2} = 0 \qquad \ldots(2)$

$\vdots \qquad \vdots \qquad \vdots$

$c_1a_{1n} + c_2a_{2n} + \ldots + c_ma_{mn} = 0 \qquad \ldots(n)$

The set of homogeneous equations (homogeneous system) contains (n) of equations and (m) of unknowns i.e.

The number of unknowns (m) > The number of equations (n).

$\therefore$ The homogeneous system has a non-trivial solution.

$\therefore$ S is non linearly independent

Example: Is the set S = {(1,1,1), (1,2,1), (1,3,5), (0,1,–1)} in $\mathbb{R}^3$ linearly independent or dependent set?

Solution: Because 4 > 3 (m > n), so S is non linearly independent (linearly dependent).

Basis and Dimension

Definition of the basis: A set of the vectors $S=\{\overrightarrow{X_1},\overrightarrow{X_2},\ldots,\overrightarrow{X_n}\}$ in a vector space V is said to be the basis of the vector space V if **(1)** S spans V
(2) S is linearly independent

Example: Let $S=\{\overrightarrow{E_1},\overrightarrow{E_2}\}$ where $\overrightarrow{E_1}$ = (1,0) and $\overrightarrow{E_2}$ = (0,1), then S is a basis for $\mathbb{R}^2$.

Solution:

(1) Let c_1 and c_2 be a constants

$$c_1\overrightarrow{E_1}+c_2\overrightarrow{E_2}=\overrightarrow{O}$$
$$c_1(1,0)+c_2(0,1)=(0,0)$$
$$(c_1,0)+(0,c_2)=(0,0)$$
$$(c_1,c_2)=(0,0)\Rightarrow c_1=0,\ c_2=0$$

$\therefore$ S is linearly independent set.

(2) Let $(x,y)\in\mathbb{R}^2$ and k_1, k_2 constants

$$(x,y)=k_1\overrightarrow{E_1}+k_2\overrightarrow{E_2}$$
$$=k_1(1,0)+k_2(0,1)$$
$$(x,y)=(k_1,k_2)\Rightarrow k_1=x,\ k_2=y$$

$\therefore$ $(x,y) = x\,(1,0) + y(0,1)$

$\therefore$ S spans $\mathbb{R}^2$

Thus S is a basis for $\mathbb{R}^2$.

Remark: S in this case called the normal basis for $\mathbb{R}^2$.

Exercise: The set $S=\{\overrightarrow{E_1},\overrightarrow{E_2},\overrightarrow{E_3}\}$ where $\overrightarrow{E_1}$ = (1,0,0) $\overrightarrow{E_2}$ = (0,1,0) and $\overrightarrow{E_3}$ = (0,0,1) is a basis for $\mathbb{R}^3$.

In general:

The set S = {(1,0,…,0),(0,1,…,0),…,(0,0,…1)} is a basis for $\mathbb{R}^n$.

Example: Let $S_1 = \{(1,0),(1,1)\}$ and $S_2 = \{(-2,1),(2,3)\}$

(a) Show that S_1 is a basis for $\mathbb{R}^2$.

(b) Show that S_2 is a basis for $\mathbb{R}^2$.

Solution (a):

(1) Let c_1 and c_2 be a constants

$c_1(1,0) + c_2(1,1) = (0,0)$

$(c_1,0) + (c_2,c_2) = (0,0)$

$(c_1 + c_2, c_2) = (0,0) \Rightarrow c_2 = 0$

$c_1 + 0 = 0 \Rightarrow c_1 = 0$

$\therefore$ S_1 is linearly independent set.

(2) Let $(a,b) \in \mathbb{R}^2$ and k_1, k_2 constants

$(a,b) = k_1(1,0) + k_2(1,1)$

$= (k_1,0) + (k_2,k_2)$

$(a,b) = (k_1 + k_2, k_2) \Rightarrow k_2 = b$

$k_1 + k_2 = a \Rightarrow k_1 + b = a \Rightarrow k_1 = a - b$

$\therefore$ $(a,b) = (a-b)(1,0) + b(1,1)$

$\therefore$ S_1 spans $\mathbb{R}^2$

Thus S_1 is a basis for $\mathbb{R}^2$.

Solution (b):

(1) Let c_1 and c_2 be a constants

$c_1(-2,1) + c_2(2,3) = (0,0)$

$(-2c_1, c_1) + (2c_2, 3c_2) = (0,0)$

$(-2c_1 + 2c_2, c_1 + 3c_2) = (0,0)$

$-2c_1 + 2c_2 = 0 \Rightarrow -2c_1 = -2c_2 \Rightarrow c_1 = c_2$

$c_1 + 3c_2 = 0 \Rightarrow c_1 = -3c_2 \Rightarrow -3c_2 = c_2 \Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0$

$c_2 = 0, c_1 = c_2 \Rightarrow c_1 = 0$

$\therefore$ S_2 is linearly independent set.

(2) Let $(a,b) \in \mathbb{R}^2$ and k_1, k_2 constants

$$(a,b)=k_1(-2,1)+k_2(2,3)$$

$$=(-2k_1+2k_2,k_1+3k_2)\Rightarrow \left.\begin{array}{l}-2k_1+2k_2=a\\ k_1+3k_2=b\end{array}\right\}\Rightarrow \begin{array}{l}-k_1+k_2=\frac{1}{2}a\\ k_1+3k_2=b\end{array}$$

$$-k_1+k_2=\frac{1}{2}a$$
$$k_1+3k_2=b$$

---------------------- by addition

$$4k_2=\frac{1}{2}a+b \quad\Rightarrow\quad k_2=\frac{1}{4}(\frac{1}{2}a+b) \quad\Rightarrow\quad k_1=\frac{1}{4}(-\frac{3}{2}a+b)$$

$\therefore$ (a,b) = $\frac{1}{4}(-\frac{3}{2}a+b)(-2,1)$ + $\frac{1}{4}(\frac{1}{2}a+b)(2,3)$

$\therefore$ S_2 spans $\mathbb{R}^2$

Thus S_2 is a basis for $\mathbb{R}^2$.

Example: Find the basis for the space of the solutions of system of homogeneous equations.

$x_1 + x_2 - x_3 = 0$ …(1)

$x_1 + 2x_2 + x_3 + x_4 = 0$ …(2)

$3x_1 + 5x_2 + x_3 + 3x_4 = 0$ …(3)

$2x_1 + x_2 - 4x_3 - x_4 = 0$ …(4)

Solution:

Using the Causs elimination method or the Causs Jordan elimination method, we get that $x_1 = 3x_3 + x_4$, $x_2 = -2x_3 - x_4$. So the solution space is

W = $\{3x_3 + x_4, - 2x_3 - x_4, x_3,x_4;\ x_3,x_4 \in \mathbb{R}\}$

= $\{(3x_3,- 2x_3,x_3,0) + (x_4,- x_4,0,x_4);\ x_3,x_4 \in \mathbb{R}\}$

= $\{x_3\,(3,- 2,1,0) + x_4\,(1,- 1,0, 1);\ x_3,x_4 \in \mathbb{R}\}$

$\therefore$ Every vector in W is a linear combination of the vectors (1,– 1,0, 1) and (3,– 2,1,0)

$\therefore$ B = {(3,– 2,1,0), (1,– 1,0, 1)} spans W.

Also, B = {(3,– 2,1,0), (1,– 1,0, 1)} is linearly independent set

$\therefore$ The set B is a basis for W.

Exercises:

(1) Show that

(a) $B_1 = \{(1,1,0),(0,1,1),(0,0,1)\}$ basis for $\mathbb{R}^3$.

(b) $B_2 = \{(1,2,-1),(2,2,1),(1,1,3)\}$ basis for $\mathbb{R}^3$.

(c) $B_3 = [(1,0,1,0),(0,1,-1,2),(1,0,0,1),(0,2,2,1)\}$ basis for $\mathbb{R}^4$.

(2) Let the set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, is B a basis for $M_{2\times 2}(\mathbb{R})$?

(3) Let the set $B = \{1,x,x^2\}$. Show that B is a basis for the vector space V where V is the vector space of all polynomials of second degree or less

(4) Let the set $B = \{1 - x, 1 + x + x^2, 1 - x - x^2, 1 + 2x + x^2\}$. Is B a basis for the vector space V where V is the vector space of all polynomials of second degree or less?

Theorem: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}\}$ be a basis for the vector space V, then every vector in V can be written in only one form as a linear combination of S vectors.

Proof:

Every vector $\overrightarrow{X}$ in V can be written as a linear combination of S vectors because S spans V. Let

$$\overrightarrow{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_n\overrightarrow{X_n} \qquad \ldots(1)$$

$$\overrightarrow{X} = d_1\overrightarrow{X_1} + d_2\overrightarrow{X_2} + \ldots + d_n\overrightarrow{X_n} \qquad \ldots(2)$$

By subtracting equation (2) from equation (1) we get that

$$\overrightarrow{O} = (c_1 - d_1)\overrightarrow{X_1} + (c_2 - d_2)\overrightarrow{X_2} + \ldots + (c_n - d_n)\overrightarrow{X_n}$$

Because S is linearly independent we have

$c_1 - d_1 = 0 \Rightarrow c_1 = d_1$,

$c_2 - d_2 = 0 \Rightarrow c_2 = d_2, \ldots$

$c_n - d_n = 0 \Rightarrow c_n = d_n$

i.e. $c_i = d_i \quad (1 \le i \le n)$.

Theorem: (without proof)

If $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ is a basis for the vector space V and $T=\{\overrightarrow{Y_1},\overrightarrow{Y_2},...,\overrightarrow{Y_r}\}$ is linearly independent set of the vectors of V then $n \geq r$.

Corollary: If $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ and $T=\{\overrightarrow{Y_1},\overrightarrow{Y_2},...,\overrightarrow{Y_m}\}$ are two basis for the vector space V then $m = n$.

Proof:

Because S is a basis for V and T is linearly independent set (since T is a basis for the vector space V)

$\Rightarrow$ $n \geq m$ (by previous theorem) …(1)

Because T is a basis for V and S is linearly independent set (since S is a basis for the vector space V)

$\Rightarrow$ $m \geq n$ (by previous theorem) …(2)

From (1) and (2) we get that $m = n$.

Examples:

(1) The set S = {(1,0),(0,1)} is a basis for $\mathbb{R}^2$, so S contains only two vectors

$\Rightarrow$ any basis for $\mathbb{R}^2$ must contains two vectors.

(2) The set S = {(1,0,0,…,0),(0,1,0,…,0),…,(0,0,0,…,1)} is a basis for $\mathbb{R}^n$, so S contains n vectors

$\Rightarrow$ any basis for $\mathbb{R}^n$ must contains n vectors (by corollary)

Theorem: (without proof)

Let $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_m}\}$ be a set of nonzero vectors generating subspace W of vector space V, then a subset of S is a basis for W.

Example: Let W be a subspace of $\mathbb{R}^4$ generator by the vectors

$$\overrightarrow{X_1}=(1,2,-2,1),\ \overrightarrow{X_2}=(-3,0,4,3),\ \overrightarrow{X_3}=(2,1,1,-1),\ \overrightarrow{X_4}=(-3,3,-9,6)$$

Find the set B such that $B\subseteq\{\overrightarrow{X_1},\overrightarrow{X_2},\overrightarrow{X_3},\overrightarrow{X_4}\}=S$ and B basis for W.

Solution: We test whether S is linearly independent or dependent set

In the case S is linearly independent set, then S is the basis we need.

If

$$c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+c_3\overrightarrow{X_3}+c_4\overrightarrow{X_4}=\overrightarrow{O} \qquad \ldots(1)$$

$$c_1(1,2,-2,1)+c_2(-3,0,4,3)+c_3(2,1,1,-1)+c_4(-3,3,-9,6)=(0,0,0,0)$$

By solving this system of equations (Using the Causs elimination method or the Causs Jordan elimination method), we get $c_1 = -c_2 - 9c_4$ ، $c_3 = 2c_2 + 3c_4$

$\therefore$ The system has infinite number of solutions.

$\therefore$ S is non linearly independent set.

Take $c_2 = 1$ and $c_4 = 0$, then $c_1 = -1$ and $c_3 = 2$

$-\overrightarrow{X_1}+\overrightarrow{X_2}+2\overrightarrow{X_3}+0\overrightarrow{X_4}=\overrightarrow{O}$ by compensation in equation (1)

$\overrightarrow{X_2}=\overrightarrow{X_1}-2\overrightarrow{X_3}-0\overrightarrow{X_4}$

This means that $\overrightarrow{X_2}$ is linear combination for the vectors $\overrightarrow{X_1}$, $\overrightarrow{X_3}$ and $\overrightarrow{X_4}$.

Thus $B_1=\{\overrightarrow{X_1},\overrightarrow{X_3},\overrightarrow{X_4}\}$ spans W

We test whether B_1 is an independent or dependent set.

In the case that B_1 is independent set, then B_1 is the required basis.

If $d_1\overrightarrow{X_1}+d_2\overrightarrow{X_3}+d_3\overrightarrow{X_4}=\overrightarrow{O} \qquad \ldots(2)$

$$d_1(1,2,-2,1)+d_2(2,1,1,-1)+d_3(-3,3,-9,6)=(0,0,0,0)$$

By solving this system of equations we get that $d_1 = -3d_3$ ، $d_2 = 3d_3$

$\therefore$ The system has infinite number of solutions.

$\therefore$ B_1 is non linearly independent set.

Take $d_3 = 1$, then $d_1 = -3$ and $d_2 = 3$

$-3\overrightarrow{X_1}+3\overrightarrow{X_3}+\overrightarrow{X_4}=\overrightarrow{O}$ by compensation in equation (2)

$\overrightarrow{X_4}=3\overrightarrow{X_1}-3\overrightarrow{X_3}$

This means that $\overrightarrow{X_4}$ is linear combination for the vectors $\overrightarrow{X_1}$ and $\overrightarrow{X_3}$.

Thus $B_2=\{\overrightarrow{X_1},\overrightarrow{X_3}\}$ spans W.

We test whether B_2 is an independent or dependent set.

In the case that B_2 is independent set, then B_2 is the required basis.

Note that B_2 is independent set **(Prove that)**

$\therefore$ $B_2=B=\{\overrightarrow{X_1},\overrightarrow{X_3}\}$ is a basis for W.

Dimension of Space

Definition of the dimension: Let V be a no zero vector space, V is called a space with a finite dimension if it has a base the number of its elements is a natural number.

The number of the vectors in the basis is called the dimension and denoted by dim (V) i.e. dim (V) = the number of the vectors in the basis of the vector space V.

If $V=\{\vec{O}\}$, then dim (V) = 0.

If $S=\{\vec{X}_1,\vec{X}_2,\ldots,\vec{X}_n\}$ basis for the vector space V, then dim (V) = n

Remark: The spaces that we will discuss are vector spaces with finite dimensions (that is, their distance = a natural number). There are vector spaces of infinite dimensions.

Examples:

(1) $\dim(\mathbb{R}^2)=2$, $\dim(\mathbb{R}^3)=3$, $\dim(\mathbb{R}^n)=n$.

(2) dim (P_2) = 3, where P_2 is a quadratic polynomial.

(3) dim (P_3) = 4, where P_3 is a polynomial of the third degree.

Generally: $\boxed{\dim(P_n) = n + 1}$, where P_n is a polynomial of degree (n).

(4) Find the dimension for the vector space $M_{2\times3}(\mathbb{R})$.

Solution: $M_{2\times3}(\mathbb{R})=\left\{\begin{bmatrix} a & b & c \\ d & e & f\end{bmatrix}; a,b,c,d,e,f\in\mathbb{R}\right\}$

$$S=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$$

S is a basis for $M_{2\times3}(\mathbb{R})$ because

(a) $$c_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+c_2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}+c_3\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+c_4\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}+c_5\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}+$$
$$c_6\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\quad\Rightarrow\quad c_1 = c_2= c_3= c_4= c_5 = c_6 = 0$$

$\therefore$ S is linearly independent set.

(b) $$\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}=k_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+k_2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}+k_3\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+k_4\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}+$$

$$k_5\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}+k_6\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}=\begin{bmatrix}k_1&k_2&k_3\\k_4&k_5&k_6\end{bmatrix} \quad \Rightarrow k_1 = a,\ k_2 = b,\ k_3 = c, k_4 = d, k_5 = e, k_6 = f$$

$\therefore$ S spans $M_{2\times3}(\mathbb{R})$.

Thus S is a basis for $M_{2\times3}(\mathbb{R})$.

$\therefore\ \dim(M_{2\times3}(\mathbb{R}))=6$ (the number of the vectors in S is 6)

Generally: $\boxed{\dim(M_{m\times n}(\mathbb{R}))=m\times n}$

(5) Let V be the vector space of all polynomials of degree 2 or less. Find the dimension for this vector space?

Solution:

Let $p\in V$ ، $p(x) = a_0 + a_1x + a_2x^2$, $\forall\ a_0,a_1,a_3\in\mathbb{R}, x\in\mathbb{R}$ and $S = \{1,x,x^2\}$

$\therefore$ S is a basis for the vector space V because

(a) $c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2 = \vec{O} = 0 + 0\cdot x + 0\cdot x^2 \Rightarrow c_1 = c_2 = c_3 = 0$

$\therefore$ S is linearly independent set.

(b) $k_1\cdot 1 + k_2\cdot x + k_3\cdot x^2 = a_0 + a_1x + a_2x^2 \Rightarrow k_1 = a_0$, $k_2 = a_1$, $k_3 = a_2$

$\therefore$ $S = \{1,x,x^2\}$ spans V

Thus S is a basis for V.

$\therefore$ dim (V) = 3 (the number of the vectors in S is 3)

(6) Find the dimension for the subspace W generated by the vectors {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)} in $\mathbb{R}^4$?

Solution: To find the dimension for subspace W generated by the vectors {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)} we must first find a basis for W.

Let S = {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)}

Therefore, we must make sure that S is linearly independent or linearly dependent set

So, if

$c_1(1,0,-1,5) + c_2(3,2,1,0) + c_3(0,-1,0,1) + c_4(-1,-5,-3,13) = (0,0,0,0)$, we get

$$\begin{aligned}
c_1 + 3c_2 \qquad - c_4 &= 0\\
2c_2 - c_3 - 5c_4 &= 0\\
-c_1 + c_2 \qquad - 3c_4 &= 0\\
5c_1 \qquad + c_3 + 13c_4 &= 0
\end{aligned}$$

From this it follows that this system has a non-trivial solution (because the determinant of the coefficients matrix = zero) **(check it)**.
And by solving this system by the Causs-Jordan elimination method, we get that:
$c_1 = -2c_4,\ c_2 = c_4,\ c_3 = -3c_4$
Take $c_4 = 1 \Rightarrow c_1 = -2,\ c_2 = 1,\ c_3 = -3$
$\therefore$ S is a non linearly independent set (linearly dependent) because the constants are not zeros.
$-2(1,0,-1,5) + 1(3,2,1,0) - 3(0,-1,0,1) + 1(-1,-5,-3,13) = (0,0,0,0)$
We take the vector of one factor for convenience
$$\begin{aligned}(-1,-5,-3,13) &= 2(1,0,-1,5) - 1(3,2,1,0) + 3(0,-1,0,1)\\ &= (2,0,-2,10) - (3,2,1,0) + (0,-3,0,3)\end{aligned}$$
Thus W generated by the set B = {(1,0,–1,5), (3,2,1,0), (0,–1,0,1)}
Now we must check whether B is linearly independent set or linearly dependent set.
So if:
$k_1(1,0,-1,5) + k_2(3,2,1,0) + k_3(0,-1,0,1) = (0,0,0,0)$
$$\begin{aligned} k_1 + 3k_2 \qquad &= 0\\ 2k_2 - k_3 &= 0\\ -k_1 + k_2 \qquad &= 0\\ 5k_1 \qquad + k_3 &= 0\end{aligned}$$
By solving this system, we get $k_1 = k_2 = k_3 = 0$.
$\therefore$ B is a linearly independent set.
But the set B spans the subspace W, so B is a basis for W and from that we obtain dim (W) = 3 (the number of the vectors in B is 3)

Exercises:

(1) Find the dimension for the subspace of $\mathbb{R}^2$ span by the vectors (2,4), (4,2), (0,0)?

(2) Find the dimension for the subspace of $\mathbb{R}^3$ span by the vectors
(a) (1,0,0), (–1,2,1), (3,2,2) **(b)** (2,3,4) ,(1,1,–1)

(3) Find the dimension for the vector space span by $\{1 + x,\ 1 + x + x^2,\ 1 - x - x^2,\ 1 + 2x + x^2\}$ for polynomials of second degree or less.

Theorem: (without proof)
If S is a set of linearly independent vectors in the finite dimension vector space V, there exists a basis T for the vector space V contains S.

Example: Find the basis for $\mathbb{R}^3$ contain the vector $\overrightarrow{X_1} = (1,0,2)$?

Solution: Let $S = \{\overrightarrow{X_1}\}, S_1 = \{\overrightarrow{X_1}, \overrightarrow{E_1}, \overrightarrow{E_2}, \overrightarrow{E_3}\}$ where $\overrightarrow{E_1} = (1,0,0), \overrightarrow{E_2} = (0,1,0), \overrightarrow{E_3} = (0,0,1)$

S_1 spans $\mathbb{R}^3$. We find the set T such that $T \subseteq S_1$, $\overrightarrow{X_1} \in T$ and T is a basis for $\mathbb{R}^3$.

S_1 is non linearly independent set (it contains four vectors from $\mathbb{R}^3$ and $\dim((\mathbb{R}^3)) = 3$)

Moreover if $c_1\overrightarrow{X_1} + c_2\overrightarrow{E_1} + c_3\overrightarrow{E_2} + c_4\overrightarrow{E_3} = \overrightarrow{O}$ we get that $c_2 = -c_1$, $c_4 = -2c_1$, $c_3 = 0$

Take $c_1 = 1 \Rightarrow c_2 = -1$, $c_3 = 0$, $c_4 = -2$

$\overrightarrow{X_1} - \overrightarrow{E_1} + 0\overrightarrow{E_2} - 2\overrightarrow{E_3} = \overrightarrow{O} \qquad \Rightarrow \qquad \overrightarrow{X_1} - 0\overrightarrow{E_2} - 2\overrightarrow{E_3} = \overrightarrow{E_1}$

$\therefore \; T = \{\overrightarrow{X_1}, \overrightarrow{E_2}, \overrightarrow{E_3}\}$ spans $\mathbb{R}^3$. But T is linearly independent set of vectors in $\mathbb{R}^3$.

Therefore T is a basis for $\mathbb{R}^3$ and contains the set $S = \{\overrightarrow{X_1}\}$.

Exercises:

(1) Find a basis for $\mathbb{R}^3$ contains the two vectors $\overrightarrow{X_1} = (1,0,2)$ and $\overrightarrow{X_2} = (0,1,2)$.

(2) Find a basis for $\mathbb{R}^4$ contains the two vectors $\overrightarrow{X_1} = (1,0,1,0)$ and $\overrightarrow{X_2} = (-1,1,-1,0)$.

(3) Find a basis for $\mathbb{R}^4$ contains the two vectors $\overrightarrow{X_1} = (1,0,1,0)$ and $\overrightarrow{X_2} = (0,1,-1,0)$.

Rank of Matrix

Definition: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ matrix of degree m×n and

$\overrightarrow{X_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \overrightarrow{X_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \overrightarrow{X_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$, the subspace W span by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}\}$ is called matrix column space A or the space span by the column of the matrix A. The dimension of the matrix column space A is called the column rank for the matrix A.

Example: Let $A=\begin{bmatrix}1 & -2 & 0\\ 3 & 2 & 8\\ 2 & 3 & 7\\ -1 & 2 & 0\end{bmatrix}$. Find the column rank for the matrix A?

Solution: The subspace of the columns of the matrix A is

$$W=\left\{c_1\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}+c_3\begin{bmatrix}0\\8\\7\\0\end{bmatrix}, c_1,c_2,c_3\in\mathbb{R}\right\}.$$

Let $S=\left\{\begin{bmatrix}1\\3\\2\\-1\end{bmatrix},\begin{bmatrix}-2\\2\\3\\2\end{bmatrix},\begin{bmatrix}0\\8\\7\\0\end{bmatrix}\right\}$ the spans set for W.

If

$$c_1\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}+c_3\begin{bmatrix}0\\8\\7\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$. Then we get

$c_1 - 2c_2 = 0$
$3c_1 + 2c_2 + 8c_3 = 0$
$2c_1 + 3c_2 + 7c_3 = 0$
$-c_1 + 2c_2 = 0$
$\Rightarrow c_1 = 2c_2,\ c_3 = -c_2$
Let $c_2 = 1 \Rightarrow c_1 = 2$ and $c_3 = -1$
$\therefore$ The set S is non linearly independent.

$$2\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}-\begin{bmatrix}0\\8\\7\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}0\\8\\7\\0\end{bmatrix}=2\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}$$

Let $S' = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$ be the set spans W. It is a linearly independent set.

$\therefore$ S' is a basis for W. Thus dim (W) = 2.

$\therefore$ The dimension of the column space of the matrix A = 2 (the column rank of A).

Example: Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & -1 \\ -1 & 2 & -2 & 3 \\ 4 & 1 & 5 & 2 \end{bmatrix}$. Find the column rank for the matrix A?

Solution: The column subspace of the matrix A is

$$W = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}, c_1, c_2, c_3, c_4 \in \mathbb{R} \right\}$$ and let

$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right\}$ be the set spans W, if we have

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + c_2 + c_3 = 0$$

$$2c_1 + 3c_2 + 4c_3 - c_4 = 0$$

$$-c_1 + 2c_2 - 2c_3 + 3c_4 = 0$$

$$4c_1 + c_2 + 5c_3 + 2c_4 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & -1 \\ -1 & 2 & -2 & 3 \\ 4 & 1 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -1 \\ 2 & -2 & 3 \\ 1 & 5 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 4 & -1 \\ -1 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & 3 \\ 4 & 1 & 2 \end{vmatrix} + 0 \neq 0$$

$\therefore$ This homogeneous linear system has a trivial solution, i.e. $c_1 = c_2 = c_3 = c_4 = 0$.
$\therefore$ S is a linearly independent set.
$\therefore$ S is a basis for W. Thus dim (W) = 4.
$\therefore$ The dimension of the column space of the matrix A = 4 (the column rank of A).

Definition: Let A be a matrix of degree m×n, if A is row equivalent to a matrix B (where B is a reduced echelon form matrix (r.e.f)) then the number of non-zero rows of the matrix B is called the row rank of A.

Examples:

(1) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -1 & 3 \end{bmatrix}$. Find the row rank of A?

Solution: $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -1 & 3 \end{bmatrix} \xrightarrow[R_3=r_3+r_1]{R_2=r_2-2r_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1=r_1-2r_3} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = B$

$\therefore$ The matrix B is a reduced echelon form matrix (r.e.f). Thus the row rank of A = 2.

(2) Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 8 \\ 2 & 3 & 7 \\ -1 & 2 & 0 \end{bmatrix}$. Find the row rank of A?

Solution:

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 8 \\ 2 & 3 & 7 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[\substack{R_3=r_3-2r_1 \\ R_4=r_4+r_1}]{R_2=r_2-3r_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 8 & 8 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3=\frac{1}{7}r_3]{R_2=\frac{1}{8}r_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3=r_3-r_2]{R_1=r_1+2r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

$\therefore$ The matrix B is a reduced echelon form matrix (r.e.f). Thus the row rank of A = 2.

Remarks:

(1) The rank of the zero matrix of degree $m\times n = 0$.

(2) The rank of the identity matrix of degree $n\times n = n$. (because it is a reduced echelon form matrix (r.e.f) and all rows are linearly independent).

(3) If the degree of the matrix is $m\times n$, then its rank is not greater than the smallest of the two numbers m and n, i.e. $r(A) \le \min\{m,n\}$.

Another definition of matrix rank: The rank of a matrix is the largest number of linearly independent rows (columns) in the matrix.

Theorem: (without proof)

Let A be a matrix of degree $(m\times n)$ the row rank and the column rank of a matrix A are equal.

Remark: The row rank of A = The column rank = The rank of a matrix A.

Example: Find the rank of a matrix A if $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 3 \\ 3 & -5 & 1 \\ 1 & -4 & 5 \end{bmatrix} \square\ B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution: Because the matrix A is equal to the matrix B which has reduced echelon form.

$\therefore$ The rank of a matrix A = 2 (the number of non zero rows) **(check that A ~ B)**.

Exercise: Find the rank of the matrix A where $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & -1 \\ -1 & 2 & -2 & 3 \\ 4 & 1 & 5 & 2 \end{bmatrix}$? **(Ans. 4)**

Remark: To find the rank of the matrix do the following

(1) Transform the matrix A to reduced echelon form and let the resulting matrix is B.

(2) The rank of the matrix A = the number of the non zero rows in the matrix B.

Theorem: Let A and B be two matrices of degree (m×n) row equivalent then the two rows spaces are identical.

Proof: Because A and B are two row equivalent matrices, then B can be obtained from A rows after performing a finite number of elementary transformations on it.

That is, each row in B is a linear combination of A rows.

This means that the rows of B are a subset of the row space of A.

That is, the row space of B is contained in the row space of A ... (1)

In the same way we get that

The row space of A is contained in the row space of B ... (2)

From (1) and (2) we get that the row space of A is equal to the row space of B.

Theorem: (without proof)

Let A be a matrix of degree (m×n) then the non zero rows in a matrix B which is the matrix A after transform it to the reduced echelon form which is the basis for the row space of A.

Remark: We can use the above theorem to find the basis for the vector space V spans by the set of vectors $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}\}$ in $\mathbb{R}^n$, i.e. spans S= V ⊆ $\mathbb{R}^n$ as follows:

(1) Form the matrix A defined by the shape $A = \begin{bmatrix} \overrightarrow{X}_1 \\ \overrightarrow{X}_2 \\ \vdots \\ \overrightarrow{X}_n \end{bmatrix}$ whose rows represent the S vectors

(2) Transform the matrix A to the reduced echelon forms and let the resulting matrix is B.

(3) The non zero rows in the matrix B are the basis for the matrix A.

Example: Let S = {(1,–2,0,3,–4),(3,2,8,1,4),(2,3,7,2,3),(–1,2,0,4,–3)} and V be a subspace of $\mathbb{R}^5$. Find the basis for V?

Solution: Form the matrix A where the rows of it are the S vectors and V is the row space for it

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$$

And by using the elementary transformations on the rows, we convert the matrix A to the reduced echelon forms, and we get the matrix

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows in the matrix B represent the basis for the row space of the matrix A which is represent the basis for V. That is, $\{(0,0,0,1,-1),(0,1,1,0,1),(1,0,2,0,1)\}$ is the basis for V.

Remark: From the above theorem and example we note the following

(1) The result basis is not subset of the given vectors.

(2) The way to represent any vector as a linear combination of these basis elements is done in the same way that the vector is represented as a linear combination of the elements of the natural base.

That is: in the previous example when we want to represent the vector $\overrightarrow{X} = (5,4,14,6,3)$ we note that the base vectors contain the axis element 1 in the first, second, and fourth positions in the first, second and fourth vectors, respectively. Thus, we will use the base vectors with the first, second and fourth projections of the vector $\overrightarrow{X}$, as follows:

$$\overrightarrow{X} = (\underline{\underline{5}},\underline{\underline{4}},14,\underline{\underline{6}},3) = 5(1,0,2,0,1) + 4(0,1,1,0,1) + 6(0,0,0,1,-1)$$

$$= (5,0,10,0,5) + (0,4,4,0,4) + (0,0,0,6,-6)$$

Theorem: Let A be a matrix of degree (n×n), then

A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n.

Proof: $\Rightarrow$ Let A be a square invertible matrix

$\therefore$ A is a row equivalent to the identity matrix I_n (A square matrix of degree n×n has an inverse if it is a row equivalent to the identity matrix)

$\therefore$ The rank of the matrix A = n.

$\Leftarrow$ Let The rank of the matrix A = n

Since A is a row equivalent to the identity matrix I_n and $|I_n| \neq 0$

$\therefore$ A is invertible matrix.

Corollary (1): Let A be a matrix of degree (n×n), then

The rank of the matrix A = n $\Leftrightarrow$ $|A| \neq 0$

Proof: $\Rightarrow$ The rank of the matrix A = n

A is invertible matrix (previous theorem: let A be a matrix of degree (n×n), then A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n)

Thus $|A| \neq 0$.

$\Leftarrow$ Let $|A| \neq 0$, then A is invertible matrix

The rank of the matrix A = n (previous theorem: let A be a matrix of degree (n×n), then A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n)

Corollary (2): Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}\}$ be a set of n vectors in $\mathbb{R}^n$ and A columns (rows) are $\overrightarrow{X_1}$, $\overrightarrow{X_2}$, …, $\overrightarrow{X_n}$ then S is linearly independent set $\Leftrightarrow$ $|A| \neq 0$.

Proof: $\Rightarrow$ Suppose that S is linearly independent set

From definition of linearly independent

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_n\overrightarrow{X_n} = \overrightarrow{O}$ such that $c_1 = c_2 = ... = c_n = 0$

We can write this relation as

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1a_{11} + c_2a_{12} + \ldots + c_na_{1n} \\ c_1a_{21} + c_2a_{22} + \ldots + c_na_{2n} \\ \vdots \\ c_1a_{n1} + c_2a_{n2} + \ldots + c_na_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \quad A\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This system is homogeneous this lead to $|A| \neq 0$ (because the column of the unknowns = 0) this is possible when $|A| \neq 0$.

$\Leftarrow$ Suppose that $|A| \neq 0$.

From Corollary (1), the rank of the matrix A = n $\Leftrightarrow$ $|A| \neq 0$.

From Theorem, A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n.

$\therefore$ S is linearly independent set when the columns (rows) of the matrix are the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}$ invertible matrix this means $|A| \neq 0$.

Corollary (3): The homogeneous system $A\overrightarrow{X} = \overrightarrow{O}$ former of n linear equation and n unknowns has non zero solution $\Leftrightarrow$ the rank of the matrix A < n.

That is : $A\overrightarrow{X} = \overrightarrow{O}$ has non zero solution $\Leftrightarrow$ the rank of the matrix A < n.

Proof: From Corollary (1), the rank of the matrix A < n $\Leftrightarrow$ $|A| = 0$.

This is: the rank of the matrix A < n $\Leftrightarrow$ A is a non invertible matrix.

$\therefore$ The homogeneous system $A\overrightarrow{X} = \overrightarrow{O}$ has non zero solution $\Leftrightarrow$ A is a non invertible matrix.

Example: Consider the homogeneous system

$2x_1 \qquad + x_3 = 0$

$3x_1 + 3x_2 + x_3 = 0$

$x_1 - 3x_2 + x_3 = 0$

Solution: The coefficient matrix is $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \\ 1 & -3 & 1 \end{bmatrix}$.

$$A=\begin{bmatrix}2&0&1\\3&3&1\\1&-3&1\end{bmatrix}\xrightarrow[R_2=r_2-3r_3]{R_1=r_1-r_3}\begin{bmatrix}1&3&0\\0&12&-2\\1&-3&1\end{bmatrix}\xrightarrow[R_3=r_3-r_1]{R_2=\frac{1}{2}r_2}\begin{bmatrix}1&3&0\\0&6&-1\\0&-6&1\end{bmatrix}$$

$$\xrightarrow{R_3=r_3+r_2}\begin{bmatrix}1&3&0\\0&6&-1\\0&0&0\end{bmatrix}\xrightarrow{R_2=\frac{1}{6}r_2}\begin{bmatrix}1&3&0\\0&1&-\frac{1}{6}\\0&0&0\end{bmatrix}\xrightarrow{R_1=r_1-3r_2}\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-\frac{1}{6}\\0&0&0\end{bmatrix}$$

This matrix is in a reduced echelon forms. Therefore the rank of the matrix $A < 3$.

Theorem: Let A be a matrix of degree $n\times n$, then

The homogeneous system $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ A is non invertible matrix.

Proof: $\Rightarrow$ Suppose that A is invertible matrix that is mean A^{-1} exists.

$A^{-1}(A\vec{X})=A^{-1}\vec{O}$ By multiplying both sides of the homogeneous system equation by A^{-1}

$(A^{-1}A)\vec{X}=\vec{O}$

$I_n\vec{X}=\vec{O} \Rightarrow \vec{X}=\vec{O}$

This means that the unique solution for this homogeneous system is the zero solution $\vec{X}=\vec{O}$ which is a contradiction since the homogeneous system has non zero solution.

$\therefore$ A is non invertible matrix.

$\Leftarrow$ **(Home work)**

Example: Find the rank of the matrix $A=\begin{bmatrix}1&2&0\\0&1&3\\2&1&3\end{bmatrix}$. And from it conclude whether it is invertible matrix or not and if the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution or not?

Solution: We transform the matrix A in the reduced echelon forms.

$$A=\begin{bmatrix}1&2&0\\0&1&3\\2&1&3\end{bmatrix}\xrightarrow{R_3=r_3-2r_1}\begin{bmatrix}1&2&0\\0&1&3\\0&-3&3\end{bmatrix}\xrightarrow[R_3=r_3+3r_2]{R_1=r_1-2r_2}\begin{bmatrix}1&0&-6\\0&1&3\\0&0&12\end{bmatrix}\xrightarrow{R_3=\frac{1}{12}r_3}\begin{bmatrix}1&0&-6\\0&1&3\\0&0&1\end{bmatrix}$$

$$\xrightarrow[R_2=r_2-3r_3]{R_1=r_1+6r_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3$$

Thus the rank of the matrix $A = 3$.

$\therefore$ A is invertible matrix (By previous theorem: Let A be a matrix of degree $(n\times n)$, then A is invertible matrix $\Leftrightarrow$ the rank of the matrix $A = n$)

or A square matrix of degree n×n has an inverse if it is row equivalent to the identity matrix

So by the previous theorem (Let A be a matrix of degree n×n, then the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ A is non invertible matrix).
The homogeneous system has no solution only the zero solution.

Exercise: Find the rank of the matrix $B=\begin{bmatrix}1 & 0 & -6\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}$ And from it conclude whether it is invertible matrix or not and if the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution or not?

Corollary: (without prove)

The linear system has solution $A\vec{X}=\vec{B}$ $\Leftrightarrow$ The rank of the matrix A = the rank of [A:B].

Example: Consider the linear system $\begin{bmatrix}2 & 1 & 3\\ 1 & 2 & 2\\ 0 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$.

Since the rank of the matrix A = the rank of [A:B], so the linear system has solution

Example: Consider the linear system $\begin{bmatrix}1 & 2 & 3\\ 1 & -3 & 4\\ 2 & -1 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$

The rank of the matrix A = 2 and the rank of [A:B] = 3.
$\therefore$ the rank of the matrix A $\neq$ the rank of [A:B]
$\therefore$ The system has no solution.

Exercises:

(1) Let $S=\{\overrightarrow{X_1},\overrightarrow{X_2},\overrightarrow{X_3},\overrightarrow{X_4},\overrightarrow{X_5}\}$, where $\overrightarrow{X_1}=(1,2,3)$, $\overrightarrow{X_2}=(2,1,4)$, $\overrightarrow{X_3}=(-1,-1,2)$, $\overrightarrow{X_4}=(0,1,2)$ and $\overrightarrow{X_5}=(1,1,1)$ find the basis for the subspace?

(2) Find the row and column rank for the matrix $A=\begin{bmatrix}1 & 2 & 3 & 2 & 1\\ 3 & 1 & -5 & -2 & 1\\ 7 & 8 & -1 & 2 & 5\end{bmatrix}$?

(3) Is the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix}$ invertible or not invertible matrix?

(4) Does the following system have a solution or not

$x_1 - 2x_2 - 3x_3 + 4x_4 = 1$

$4x_1 - x_2 - 5x_3 + 6x_4 = 2$

$2x_1 + 3x_2 + x_3 - 2x_4 = 2$

(5) Let $S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$, is it is linearly independent in $\mathbb{R}^3$?