Linear Algebra

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أساتذة المادة

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Linear Transformations and Matrices.

Def Let V and W be V.spaces. A function $L: V \to W$ is called Linear transformation of V in to W if

ⓐ $L(u+v) = L(u) + L(v)$ for u and v in V

ⓑ $L(cu) = cL(u)$ for u in V and c is a real number

if $V = W$, the linear trans. $L: V \to V$ is also called linear operator.

Examples and Remarks

1) $L: V \to W$ is a linear trans $\Leftrightarrow L(au+bv) = aL(u) + bL(v)$

For any vectors $u, v \in V$ and for any scalars a, b.

② Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Then L is a linear trans.

③ Let $L: P_1 \to P_2$ defined by

$$L(p(t)) = t\,p(t)$$

Then L is a linear trans.

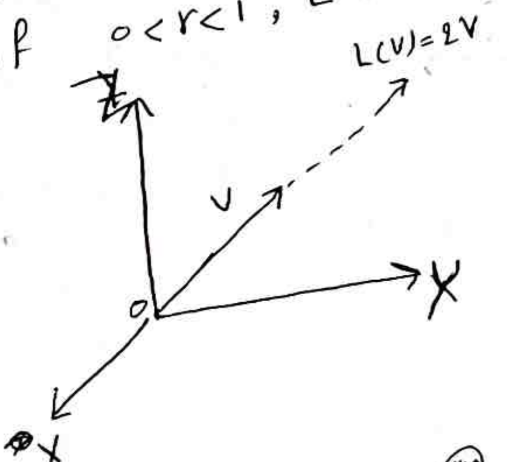
④ Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

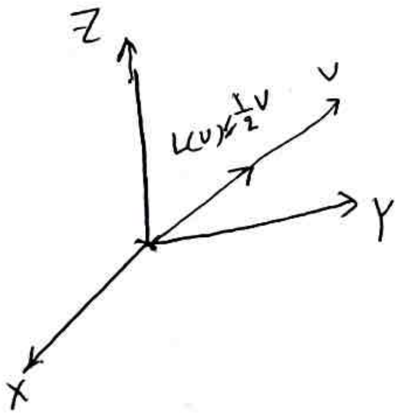
$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = r\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ where } r \in \mathbb{R}.$$

Then L is a linear operator.

If $r > 1$, L is called a dilation

If $0 < r < 1$, L " " a contraction





(5) Let $V = C[a,b]$ be the v.sp of all real valued functions that are integrable over the interval $[a,b]$

Let $L : V \longrightarrow \mathbb{R}$ defined by

$$L(f) = \int_a^b f(x)\,dx$$

Then L is a linear trans.

(6) Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by, A be a fixed 2×3 real matrix.

Let $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underset{2\times 3}{A} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Then L is a lin. trans

proof Let $X, Y \in \mathbb{R}^3$, Let $a, b \in \mathbb{R}$

$$\begin{aligned} L(aX + bY) &= A(aX + bY) \\ &= A(aX) + A(bY) \\ &= a(AX) + b(AY) \\ &= a\,L(X) + b\,L(Y) \end{aligned}$$

(7) Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 2a_1 + 1 \\ 2a_2 \\ a_3 \end{pmatrix}$$

show that L is not a lin-tran.

(8) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$L(a_1\ a_2) = (a_1^2\ 2a_2)$$

show that L is not a lin-trans.

ملاحظة. المتجهات في $\mathbb{R}^n$ يمكن ان تكتب على شكل مصفوفات صفية من النمط $1\times n$ او مصفوفات عمودية من النمط $n\times 1$ او على شكل $(a_1, a_2, \ldots, a_n)$

Def A Linear transformation $L : V \to W$ is called 1-1 if It is 1-1 function

Ex Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix}$$

Show that L is 1-1

Def Let $L : V \to W$ be a lin-trans of a V.sp V in to a V.sp W. The Kernel of L (denoted by Ker L) is the subset $\{v \in V : L(v) = 0_W\}$.

is the subset $\{v \in V : L(v) = 0_W\}$ is a subspace of V (check)

Note that Ker L is a subspace of V

Hence Ker $L \neq \varphi$.

EX Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ s.t $L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Find Ker L

Sol
$$\begin{aligned} \text{Ker } L &= \{v \in \mathbb{R}^3 : L(v) = 0_{\mathbb{R}^2}\} \\ &= \left\{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\ &= \{ \quad '' \quad a_1 = a_2 = 0 \} \\ &= \left\{\begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix} : a_3 \in \mathbb{R}\right\} \end{aligned}$$

Th Let $L : V \to W$ be a lin. trans. of a V.sp V into a V.sp W. Then

ⓐ Ker L is a subspace of V

ⓑ L is 1-1 $\Leftrightarrow$ Ker $L = \{0_V\}$

proof EXC.

EX Let $L : P_2 \to R$ be a lin. trans. defined by (4)

$$L(at^2+bt+c) = \int_0^1 (at^2+bt+c)\,dt$$

Find Ker L, find dim Ker L, IS L one to one.

Sol $\text{Ker} L = \{ v \in P_2 : L(v) = 0_R \}$

let $v = at^2 + bt + c \in \text{Ker} L$

$$L(v) = \int_0^1 (at^2+bt+c)\,dt$$

$$= \frac{at^3}{3} + \frac{bt^2}{2} + ct \Big]_0^1 = \frac{a}{3} + \frac{b}{2} + c$$

$L(v)=0 \Rightarrow \frac{a}{3} + \frac{b}{2} + c = 0$

$\Rightarrow c = -\frac{a}{3} - \frac{b}{2}$

$\Rightarrow v = at^2 + bt + \left(-\frac{a}{3} - \frac{b}{2}\right)$

To find dim Ker T ?

$\text{Ker} T = \{ at^2 + bt + (-\frac{a}{3} - \frac{b}{2}) : a, b \in \mathbb{R} \}$

$= \{ a(t^2 - \frac{1}{3}) + b(t - \frac{1}{2}) : a, b \in \mathbb{R} \}$

$S = \{ t^2 - \frac{1}{3}, t - \frac{1}{2} \}$ generates Ker T.

To show S is Lin-indep.

Assume $\alpha_1 (t^2 - \frac{1}{3}) + \alpha_2 (t - \frac{1}{2}) = 0$

$\alpha_1 t^2 + \alpha_2 t - \frac{\alpha_1}{3} - \alpha_2 \frac{1}{2} = 0$

$\Rightarrow \alpha_1 = \alpha_2$, So S is Lin-indep.

او نقول S هي lin-indep لان كل من المتجهين في S ليس مضاعفا الى الاخر

$\therefore$ S is basis for Ker L, hence dim Ker = 2

Now L is not 1-1 since $\text{Ker} L \neq \{0\}$

Def if $L: V \to W$ be a Lin trans of a V.sp V in to a V.sp W, then the range of L (or image of V under L) denoted by range L, is defined by

$$\text{rang}\, L = \{w \in W ;\ \exists v \in V \text{ s.t } L(v) = w\}.$$

Note that L is onto if rang $L = W$.

Theorem if $L: V \to W$ be a linear trans. of V.sp V into a V.sp W, then range of L is a subspace of W.

proof Let $w_1, w_2 \in$ rang L. Then $w_1 = L(v_1)$, $w_2 = L(v_2)$ for some $v_1, v_2 \in V$

Hence $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$

$\therefore w_1 + w_2 \in$ range L

Also $aw = a\,L(v)$ For any Real no. a

$= L(av)$

$\therefore aw \in$ rang L.

EX Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Is L onto ? Find dim Range L.

Sol Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$. It clear that for $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$ (where a_3 any number in $\mathbb{R}$), $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

To find dim range L ?

Since L is onto, range $L = \mathbb{R}^2$ $\therefore$ dim range $= 2$

EX Let $L: P_2 \to \mathbb{R}$ defined by $L(at^2+bt+c) = \int_0^1 (at^2+bt+c)\,dt$

Is L - onto. Find dim range L.

Is $L(v) = r$?

Sol Let $r \in \mathbb{R}$. we can find v s.t $L(v) = r$

$v = at^2 + bt + c$, so $L(v) = \frac{at^3}{3} + \frac{bt^2}{2} + c$

Let $a = 0$, $b = 0$, $c = r$. Hence $L(r) = r$

$\therefore L$ is onto, so range $L = \mathbb{R}$ and dim range $L = 1$

EX Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

a) is L onto ?
b) Find a basis for range L ?.
c) Find Ker L
d) is L 1-1 ?

Sol ⓐ Let $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, where $a, b, c \in \mathbb{R}$. so we seek

To find V s.t $L(V) = w$,

a solution of the lin-sys

$$AV = w$$

ie $$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right)$$

By transform this mat to r.r.e.f , we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right)$$

The solution exists only when $c-b-a = 0$

$\therefore L$ is not onto .

ⓑ range $L = \{L(V) : V \in \mathbb{R}^3\}$

$$= \left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3\right\}.$$

$$= \left\{\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + 2a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} : \text{〃 〃 〃}\right\}$$

Now $\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + 2a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\therefore S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ spans range L

Notice that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore S' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ spans range L

Also S' is Lin-indep, since any vector is not multiple of the other.

$\therefore S'$ is a basis of range L

$\therefore$ dim range L $= 2$.

(c) $\text{Ker } L = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \quad '' \quad : \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\therefore \left. \begin{array}{l} a_1 + a_3 = 0 \\ a_1 + a_2 + 2a_3 = 0 \\ 2a_1 + a_2 + 3a_3 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} a_1 = -a_3 \\ a_2 = -a_3 \end{array}$

$\therefore \text{Ker } L = \left\{ \begin{pmatrix} -a \\ -a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$

$= \left\{ a \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

So $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for Ker L

$\therefore$ dim Ker L $= 1$

Thus L is not 1-1

Theorem Let $L: V \to W$ be a lin. trans of n-dim. V.sp V into V.sp W. Then

$$\dim \operatorname{Ker} L + \dim \operatorname{range} L = \dim V$$

ملحوظة المبرهنة تعطى بدون برهان

Example Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a lin-trans defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix}$$

verify the previous th.

Sol $\operatorname{Ker} L = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

But $\left.\begin{array}{l} a_1 + a_3 = 0 \\ a_1 + a_2 = 0 \\ a_2 - a_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = -a_3 \\ a_2 = a_3 \end{array}$

$\therefore \operatorname{Ker} L = \left\{ \begin{pmatrix} -a \\ a \\ a \end{pmatrix} : a \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

$\therefore \dim \operatorname{Ker} L = 1$

Now, any vector in Range L is of the form $\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix}$

$\Rightarrow a_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\therefore S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ generate Range L

But $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, so $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ basis for rang L

$\therefore$ dime range $L = 2$

$\dim \mathbb{R}^3 = 3 = \dim \ker L + \dim \operatorname{rang} L$

$= 1 + 2$

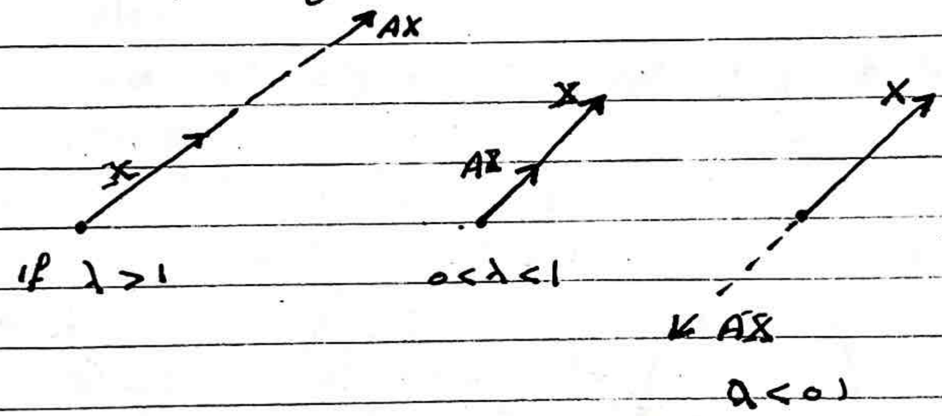
① Eigenvalues and Eigen vectors

Definition Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector X in $\mathbb{R}^n$ such that

$$AX = \lambda X$$

The vector x is called ~~ano~~ an eigenvector corresponding to λ.

geometricaly, the vector AX is the same direction as X depending on λ.



Computation of eiganvectors and Eigenvalues.

Let A be $n \times n$ matrix with eigenvalue λ and corresponding eigen vector X. Thus $AX = \lambda X$

Thus this equation Can be written as

$$AX - \lambda X = 0$$

$$(A - \lambda I) X = 0$$

This matrix equation is a homogenous linear system

Note that this system has a nonzero solution if $|A - \lambda I| = 0$

Hence, solving the equation $|A-\lambda I_n|=0$ for λ (leads to all eigenvalues of A).

On expanding $|A-\lambda I|=0$, we get a poly. in λ. This polynomial is called the characteristic poly. of A. The equation $|A-\lambda I_n|=0$ is called the characteristic equation of A.

The eigenvalues are then substituted back in the eq. $(A-\lambda I_n)X=0$ to find the corresponding eigenvectors.

Examples

① Find all eigenvalues and eigenvectors of the matrix

$$A=\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}$$

Sol.

$$A-\lambda I_2=\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{pmatrix}$$

The ch. poly. $|A-\lambda I|=(-4-\lambda)(6-\lambda)-(-6)3$

$$=\lambda^2-\lambda-2$$

Now, to solve the ch. equation $\lambda^2-\lambda-2=0$

$\therefore\ (\lambda-2)(\lambda+1)=0$

$\Rightarrow \lambda=2$ or $\lambda=-1$ (the eigenvalues of A)

The corresponding eigenvectors are found by using these values of λ in the equation

$$(A-\lambda I_2)X = 0$$

So, if $\lambda = 2$, we solve the eq. $(A-2I_2)X=0$

Hence $\left[\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} -6 & -6 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -6x_1 - 6x_2 = 0$$
$$3x_1 + 3x_2 = 0$$

Hence $x_1 = -x_2$, so the solution of this system eq. are $x_1 = -r$ $(x_2 = r)$, where r is a scalar

Thus the eigen vectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$r\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

when $\lambda = -1$, we solve the eq. $(A+1I_2)X=0$

i.e. $\left[\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad -3x_1 - 6x_2 = 0$$
$$3x_1 + 6x_2 = 0$$

Thus $x_1 = -2x_2$, so the solutions are $x_1 = -2s$ and $x_2 = s$, where s is a sclar.

Hence the eigenvectors of A corresponding to $\lambda = -1$ are nonzero vectors of the form

$$s\begin{pmatrix}-2\\1\end{pmatrix}$$

<u>Note</u> observe that the set of eigenvectors of $\lambda = 2$ together with zero vector, ie

$S = \left\{ r\begin{pmatrix}-1\\1\end{pmatrix} : r \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$ with dimension 1.

Also the set of eigenvectors corresponding to $\lambda = -1$ with the zero vector; ie

$S = \left\{ s\begin{pmatrix}-2\\1\end{pmatrix} : s \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$ with $\dim(S) = 1$.

<u>Theorem</u> Let A be an $n \times n$ matrix and λ an eigenvalue of A. The set of all eigenvectors corresponding to λ, together with zero vector is a subspace of $\mathbb{R}$. (This space is called the eigenspace of λ.

<u>Proof</u>. Let X_1, X_2 be two vectors corresponding to λ, so $AX_1 = \lambda X_1$, $AX_2 = \lambda X_2$

let c, d be any scalars.

T.P $cX_1 + dX_2$ is a vector in the eigen sp. of λ

$$A(cX_1 + dX_2) = AcX_1 + AdX_2$$
$$= cAX_1 + dAX_2$$
$$= c(\lambda X_1) + d(\lambda X_2)$$
$$= \lambda(cX_1) + \lambda(dX_2)$$
$$= \lambda(cX_1 + dX_2).$$

Thus $cX_1 + dX_2$ is a vector in the eigenspace of λ.
So eigenspace of λ is a subspace of $\mathbb{R}^n$.

<u>Example</u> Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

<u>Solution</u> $A - \lambda I_3 = \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}$

The ch. poly.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \xlongequal{-R_2+R_1} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$\xlongequal{C_1+C_2} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(9-\lambda)(2-\lambda) - 8] = (1-\lambda)(\lambda^2 - 11\lambda + 10)$$

بالنسبة ل R_1

5-

$$= (1-\lambda)(\lambda-10)(\lambda-1) = -(\lambda-10)(\lambda-1)^2$$

To solve the ch. eq. of A :

$$-(\lambda-10)(\lambda-1)^2 = 0$$

$\Longrightarrow \lambda = 10$ or $\lambda = 1$

$\therefore$ eigenvalues of A are 10 , 1 .

To find eigenvectors corresponding to $\lambda = 10$ we slove the eq: $(A - 10 I_3) X = 0$

$$\therefore \begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x_1 + 4x_2 + 2x_3 = 0$$
$$4x_1 - 5x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 - 8x_3 = 0$$

Hence $x_1 = 2r$, $x_2 = 2r$ and $x_3 = r$, where r is any scalar.

$\therefore$ Thus eigenvectors of $\lambda = 10$; $r\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $r \in \mathbb{R} - \{0\}$ & eigem space of $\lambda = 10$ is $\left\{ r\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} : r \in \mathbb{R} \right\}$ and It is One dimensional subspace of $\mathbb{R}^3$.

If $\lambda = 1$, then $(A - I)X = 0$, lead to :

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_2 = s$, $x_3 = 2t$, $x_1 = -s-t$ where s, t are scalar.

$\therefore$ Thus eigen space of $\lambda = 1$ is the space
$\left\{ \begin{pmatrix} -s-t \\ s \\ 2t \end{pmatrix} : s, t \in \mathbb{R} \right\}$ and its dim. is 2.
since $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis for this subsp.

Exercises