

Linear Algebra

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CHAPTER THREE

INVERSE OF MATRIX

Definition: The inverse of the square matrix A denoted by A^{-1} it is a matrix has the same degree of the matrix A such that $AA^{-1} = A^{-1}A = I$.

Or A is called **inverse matrix** or **invertible** if there exists a matrix B such that $AB = BA = I$, in this case the B is called the inverse of the matrix A and written $B = A^{-1}$.

Remark: The degree of matrix B (the inverse of the matrix A) must be equal to the degree of the matrix A and the rectangular (non-square) matrix has no inverse.

Examples:

(1) Find the inverse of the matrix $A = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$?

Solution: Let $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^{-1}$ is the inverse of the matrix A .

It must be prove that $AB = BA = I_2$.

$$\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2a+b & -2c+d \\ 3a-b & 3c-d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-2a+b = 1 \quad \dots(1)$$

$$-2c+d = 0 \quad \dots(3)$$

$$3a-b = 0 \quad \dots(2)$$

$$3c-d = 1 \quad \dots(4)$$

by addition by addition

$$a = 1$$

$$c = 1$$

$$\text{so, } b = 3$$

$$\text{so, } d = 2c \Rightarrow d = 2(1) = 2$$

$$\therefore B = A^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \text{ which is the inverse of the matrix } A.$$

Investigation:

$$\text{It must be } AB = I_2 \Rightarrow AB = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{It must be } BA = I_2 \Rightarrow BA = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\therefore B = A^{-1}$$

(2) Find the inverse of the matrix $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ if exists?

Solution: Let $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^{-1}$ is the inverse of the matrix A.

It must be prove that $AB = BA = I_2$.

$$AB = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a - 4b & 2c - 4d \\ a - 2b & c - 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2a - 4b = 1 \quad \dots(1) \qquad 2c - 4d = 0 \quad \dots(3)$$

$$a - 2b = 0 \quad \dots(2) \qquad c - 2d = 1 \quad \dots(4)$$

$$\begin{array}{rcl} \hline 2a - 4b = 1 & \dots(1) & 2c - 4d = 0 \end{array} \quad \text{by multiply the equations (2) and (4) by 2}$$

$$\begin{array}{rcl} 2a - 4b = 0 & \dots(2) & 2c - 4d = 2 \end{array}$$

$$\begin{array}{rcl} \hline 0 = 1 & \text{by subtraction} & 0 = -2 \end{array} \quad \text{by subtraction}$$

This is not possible. So there is no solution to the system.

$\therefore$ We hypothesized that there is an inverse of the matrix A is not true.

$\therefore$ There is no inverse to the matrix A.

Properties:

(1) The square matrix A is called not invertible if $\det(A) = |A| = 0$.

(2) The square matrix A is called invertible if $\det(A) = |A| \neq 0$ and $|A| = \frac{1}{|A^{-1}|}$.

Proof:

Suppose the matrix A invertible $\Leftrightarrow$ there exists a matrix B, such that

$$AB = BA = I_n \Rightarrow |AB| = |I_n| \quad (\text{taking determinant for each side})$$

$$\Rightarrow |A| |B| = 1 \quad (\text{previous theorems } |AB| = |A| |B| \text{ and } \det(I_n) = 1)$$

$$\Rightarrow |A| \neq 0 \text{ and } |B| \neq 0$$

$$\text{So we get } |A| = \frac{1}{|B|} \Rightarrow |A| = \frac{1}{|A^{-1}|} \text{ or } |A^{-1}| = \frac{1}{|A|}$$

Remark: Sometimes invertible matrix is called (inverse matrix) or non-singular or has an inverse.

And not invertible matrix is called (non-inverse matrix), singular, or has no inverse.

Examples:

(1) $A = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$ not invertible matrix since $|A| = 0$.

(2) $B = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}$ invertible matrix since $|B| \neq 0$.

(3) Is the zero matrix has inverse (invertible)?

Solution: The zero matrix does not have an inverse because its determinant = zero.

Or If we suppose the matrix B is the inverse of the zero matrix O_n , then

$$O_n B = O_n \neq I_n \text{ which is a contradiction}$$

$\therefore$ The zero matrix does not have an inverse

(4) Is the identity matrix has inverse (invertible)?

Solution: Yes, the identity matrix has inverse which is the same matrix.

Suppose the matrix B is the inverse of the identity matrix I_n

$$I_n B = B I_n = I_n \quad (\text{the definition of an inverse matrix})$$

$$B = B I_n \quad (\text{previous theorem } I_n A = A)$$

$$\therefore B = I_n$$

Therefore, the inverse of the identity matrix is the identity matrix itself.

(5) If $A = A^{-1}$. Show that $\det(A) = \pm 1$.

$$\text{Solution: } A = A^{-1} \Rightarrow A A = I_n \quad (\text{the definition of an inverse matrix})$$

$$\Rightarrow (\det(A))^2 = \det(I_n) \quad (\text{taking determinant for each side})$$

$$\Rightarrow (\det(A))^2 = 1 \quad (\det(I_n) = 1)$$

$$\Rightarrow \det(A) = \pm 1$$

(6) If A is nonsingular matrix such that $A^2 = A$. Find the value of $\det(A)$?

Solution: Since A is nonsingular, so A^{-1} exists.

$$A^2 = A$$

$$A^2 A^{-1} = A A^{-1} \quad (\text{multiply each side by } A^{-1})$$

$$A A A^{-1} = A A^{-1} \quad (A^2 = A A)$$

$$A I_n = I_n \quad (A A^{-1} = I_n)$$

$$A = I_n \quad (\text{previous theorem } A I_n = A)$$

$$\det(A) = \det(I_n) \quad (\text{taking determinant for each side})$$

$$\det(A) = 1 \quad (\det(I_n) = 1)$$

Theorem: If A is an invertible matrix, then A^t is also an invertible matrix and $(A^t)^{-1} = (A^{-1})^t$.

Proof: Since A is invertible matrix, $\exists A^{-1}$ ($n \times n$) matrix) such that

$$A A^{-1} = A^{-1} A = I_n$$

$$(A A^{-1})^t = (A^{-1} A)^t = (I_n)^t \quad (\text{taking transpose for each side})$$

$$(A^{-1})^t A^t = A^t (A^{-1})^t = I_n \quad (\text{by previous theorems, } (I_n)^t = I_n \text{ and } (AB)^t = B^t A^t)$$

$$\therefore A^t \text{ is invertible and } (A^t)^{-1} = (A^{-1})^t.$$

Remark: The previous method of finding the inverse of a matrix (by definition) is impractical to find the inverse of a matrix of degrees higher than (2×2) . But there are other methods to do this.

Finding the inverse of a matrix (by the adjoint matrix method)

The Method of Adjoint Matrix

If $A = [a_{ij}]_{n \times n}$ such that $|A| \neq 0$ and $C = [c_{ij}]_{n \times n}$ represents the matrix of coefficients cofactor for the matrix A.

Theorem: without proof

The adjoint matrix for the matrix A is transposed matrix of the coefficients cofactor for the matrix A and denoted by $\text{adj}(A)$, i.e.

$$\text{adj}(A) = C^t(A) = [c_{ij}]_{n \times n}^t$$

$$\text{The inverse of the matrix A is } A^{-1} = \frac{\text{adj}(A)}{|A|} = \left[\frac{c_{ij}}{|A|} \right]^t = \left[\frac{c_{ij}}{|A|} \right]^t.$$

Examples:

(1) Find the inverse of the matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$ by adjoint matrix method?

Solution:

$$|A| = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2$$

$$c_{11} = (-1)^{1+1} |2| = 2, \quad c_{12} = (-1)^{1+2} |3| = -3$$

$$c_{21} = (-1)^{2+1} |2| = -2, \quad c_{22} = (-1)^{2+2} |4| = 4$$

$$C(A) = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} \Rightarrow \text{adj}(A) = C^t(A) = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$

The inverse of the matrix A is

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}}{2} = \begin{bmatrix} 1 & -1 \\ -\frac{3}{2} & 2 \end{bmatrix}$$

Investigation:

$$AA^{-1} = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -\frac{3}{2} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1}A = \begin{bmatrix} 1 & -1 \\ -\frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ by adjoint matrix method ?

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix} = (12 + 6 + 6) - (9 + 8 + 6) = 24 - 23 = 1$$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} = 6, \quad c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad c_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$c_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = -2, \quad c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1, \quad c_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$c_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3, \quad c_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0, \quad c_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$C(A) = \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \text{adj}(A) = C^t(A) = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The inverse of the matrix A is

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Theorem: without proof

If $A = [a_{ij}]_{n \times n}$ then $A (\text{adj}(A)) = (\text{adj}(A)) A = (\det(A)) I_n$.

Exercises:

(1) Find the inverse for each of the following matrices (if exists)

$$(a) A = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} \quad (b) A = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix} \quad (c) B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \quad (d) B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

(2) For any square matrix of degree $(n \times n)$ show that $\text{adj}(A^t) = (\text{adj } A)^t$.

(3) If A is a square matrix and $|A| \neq 0$. Prove that A has inverse and $A^{-1} = \frac{1}{|A|} \text{adj}(A)$.

Theorem: If A is invertible matrix, then αA invertible matrix for any scalar number $\alpha \neq 0$ and $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$.

Proof: We must prove $(\alpha A)(\frac{1}{\alpha} A^{-1}) = (\frac{1}{\alpha} A^{-1})(\alpha A) = I_n$

$$\begin{aligned} (\alpha A)(\frac{1}{\alpha} A^{-1}) &= (\alpha \frac{1}{\alpha})(AA^{-1}) && \text{(by previous theorem } (rA)(sB) = (rs)(AB)) \\ &= 1 \cdot I_n = I_n && \text{(A is invertible matrix so } AA^{-1} = I_n) \end{aligned}$$

In the same way we prove that $(\frac{1}{\alpha} A^{-1})(\alpha A) = I_n$.

$$\therefore (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

Theorem: If the matrix A has inverse, then this inverse is unique.

Proof: Let B and C are the inverses for the matrix A.

By the definition of an inverse matrix, we have

$$AB = BA = I_n \quad \dots(1)$$

$$AC = CA = I_n \quad \dots(2)$$

$$B = B I_n \quad \text{(by previous theorem } A = AI_n)$$

$$B = B(AC) \quad \text{(by (2))}$$

$$B = (BA)C \quad \text{(the multiplication of matrices is associative)}$$

$$B = I_n C \quad \text{(by (1))}$$

$$B = C \quad \text{(by previous theorem } AI_n = A)$$

$\therefore$ The inverse matrix is unique.

Theorem: If A is invertible matrix, then A^{-1} is invertible matrix. **(Home work)**

Theorem: If A is invertible matrix, then $(A^{-1})^{-1} = A$.

Or The inverse of the inverse matrix is equal to the matrix itself.

Proof:

$$A^{-1}(A^{-1})^{-1} = (A^{-1})^{-1} A^{-1} = I_n \quad \text{(the definition of an inverse matrix)}$$

$$\text{But } A^{-1}A = A A^{-1} = I_n$$

So each of A and $(A^{-1})^{-1}$ are inverse for the matrix A^{-1}

Since the inverse of the matrix is unique, so we get $(A^{-1})^{-1} = A$

Theorem: If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: We must prove that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$

$$(AB)(B^{-1}A^{-1}) = ((AB)B^{-1})A^{-1} \quad \text{(the multiplication of matrices is associative)}$$

$$= (A(BB^{-1}))A^{-1} \quad \text{(the multiplication of matrices is associative)}$$

$$= (AI_n)A^{-1} \quad \text{(B}^{-1} \text{ is the inverse matrix of the matrix B)}$$

$$= AA^{-1} \quad \text{(by previous theorem } AI_n = A)$$

$$= I_n \quad \text{(the definition of an inverse matrix)}$$

In the same way we prove $(B^{-1}A^{-1})(AB) = I_n$.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, prove that $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & -1 \\ 4 & -1 \end{bmatrix}$, $B^{-1}A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$,

$(AB)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$. So that $(AB)^{-1} = B^{-1}A^{-1}$

Corollary (1): If A, B and C are invertible matrices, then ABC invertible matrix.
i.e. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Proof:

$$\begin{aligned} (ABC)^{-1} &= ((AB)C)^{-1} && \text{(the multiplication of matrices is associative)} \\ &= C^{-1}(AB)^{-1} && \text{(by previous theorem } (AB)^{-1} = B^{-1}A^{-1} \text{)} \\ &= C^{-1}B^{-1}A^{-1} && \text{(by previous theorem } (AB)^{-1} = B^{-1}A^{-1} \text{)} \end{aligned}$$

Corollary (2): If $A_1, A_2, \dots, A_n$ are invertible matrices of the same degree, then $A_1A_2\dots A_n$ is invertible matrix where n is positive integer number and
 $(A_1A_2\dots A_n)^{-1} = A_n^{-1}\dots A_2^{-1}A_1^{-1}$ **(Home work)**

Corollary (3): If A is invertible matrix then A^n is invertible matrix for any positive integer number n and $(A^n)^{-1} = (A^{-1})^n$.

Proof: By using corollary (2)

Let $A_1 = A_2 = \dots = A_n = A$

Since $(A_1A_2\dots A_n)^{-1} = A_n^{-1}\dots A_2^{-1}A_1^{-1}$

So $(AA\dots A)^{-1} = A^{-1}\dots A^{-1}A^{-1}$

$(A^n)^{-1} = (A^{-1})^n$

Remark: We can prove Corollary (3) by using mathematical induction method

Definition: If A is invertible matrix, we define

$A^n = (A^{-1})^{-n}$ where n is a negative integer number

$$= \underbrace{A^{-1}A^{-1}\dots A^{-1}}_{(-n)\text{-times}}$$

$A^0 = I$

Examples:

(1) Let A square matrix of degree (n×n) such that $A^2 + 2A + I_n = O_n$. Show that A is invertible matrix and find its inverse?

Solution:

$$A^2 + 2A + I_n = O_n \quad (\text{given})$$

$$A^2 + 2A = O_n - I_n$$

$$A^2 + 2A = -I_n$$

$$A^2 + 2A I_n = -I_n \quad (\text{by previous theorem } AI_n = I_n A = A)$$

$$-A^2 - 2A I_n = I_n$$

$$A(-A - 2 I_n) = I_n$$

$$A(-A - 2 I_n) = (-A - 2 I_n) A = I_n$$

From that we have A is invertible matrix and $A^{-1} = -A - 2 I_n$

(2) If $A = \lambda I_n$ where λ is scalar number. Prove that

$$A \text{ is invertible matrix } \Leftrightarrow \lambda \neq 0 \text{ and } A^{-1} = \frac{1}{\lambda} I_n$$

Proof: $\Rightarrow$ Let A is invertible matrix. To prove $\lambda \neq 0$

$$\text{Suppose } \lambda = 0 \Rightarrow A = \lambda I_n$$

$$\Rightarrow A = O_n$$

But O_n not invertible matrix (since their determinant equal to zero)

$\therefore$ A is not invertible matrix which is a contradiction.

$\therefore$ Our hypothesis $\lambda = 0$ not true, so $\lambda \neq 0$.

$\Leftarrow$ Let $\lambda \neq 0$. To prove A is invertible matrix

$$A = \lambda I_n \quad (\text{given})$$

$$A^{-1} = \frac{1}{\lambda} I_n^{-1} \quad (\text{by previous theorem (if A invertible matrix then } \alpha A \text{ is invertible matrix and}$$

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}))$$

$$A^{-1} = \frac{1}{\lambda} I_n \quad (\text{The inverse matrix of identity matrix is the identity matrix itself})$$

(3) Let A be a square matrix such that $A^k = O$ for some positive integer values, this matrix is called nilpotent. Show that A is not invertible matrix.

Solution: Suppose A is invertible matrix

$\Rightarrow A^k$ is invertible matrix (by If A is invertible matrix then A^n is invertible matrix for any positive integer number n)

$\Rightarrow O$ is invertible matrix which is a contradiction

$\therefore$ A is not invertible matrix.

Exercise: Give example for nilpotent matrix of degree (2×2).

(4) If A and B are invertible matrices of any degree. Is $A + B$ invertible matrix?

If it is that, is $(A + B)^{-1} = A^{-1} + B^{-1}$?

Solution: If A and B are invertible matrices it is not necessary $A + B$ invertible matrix.

For example: let $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ are invertible matrices, but

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible matrix.

It may be A, B and $A + B$ are invertible matrices, but not necessary that $(A + B)^{-1} = A^{-1} + B^{-1}$.

For example: $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, $(A + B)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$

But $(A + B)^{-1} \neq A^{-1} + B^{-1}$, since

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \neq \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

(5) Let A + I invertible matrix. Show that $(A - I)$ and $(A + I)^{-1}$ commute matrices?

i.e. $(A - I)(A + I)^{-1} = (A + I)^{-1}(A - I)$

(Home work)

Remark: The matrix A which satisfy $A^2 = I$ is called involutory matrix.

(It is the matrix which is if it is multiply by itself the value equal to the identity matrix)

The involutory matrix is the inverse of itself.

The identity matrix is involutory matrix.

Example:

The matrix $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ is involutory matrix, since $A^2 = I$.

Exercises:

- (1) Let A be a square invertible matrix. Prove that $|A^k| = |A|^k$, where k is a negative integer number.
- (2) Let A be a square matrix and $A^2 = O$. Prove that $(I - A)$ is invertible matrix.
- (3) If A, B, C are square matrices of degree $(n \times n)$ and A is invertible matrix, then $AB = AC \Rightarrow B = C$

Definition: We say for the matrix of degree $m \times n$ is of **reduced echelon form** denoted by (r.e.f.) if satisfy the following conditions:

- (a) Rows consisting entirely of zeros, if exists, are appear at the bottom of the matrix.
- (b) The first non-zero entry in every row that is not completely composed of zeros is equal to 1 and it is called "the leading entry for that row".
- (c) If the sequential rows i and $i + 1$ not completely composed of zeros, the leading entry for the row $i + 1$ is appears on the right of the leading entry for the row i .
- (d) If there exist a column has a leading entry for a row, then all other entries for that column equal to zero.

Remark: The matrix of the reduced echelon form may be not containing entirely rows of zeros.

Examples:

(1) $A = \begin{bmatrix} 0 & 0 & \textcircled{2} & 9 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Matrix of echelon form (e.f.) but not of the reduced echelon form (r.e.f.) since the leading entry for the first row $= 2 \neq 1$ (the condition (b) not satisfy).

$$(2) \quad B = \begin{bmatrix} 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix not of the reduced echelon form (r.e.f.) since the leading entry for the second row not on the right of the leading entry for the first row (the condition (c) not satisfy).

$$(3) \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix of the reduced echelon form (r.e.f.)

$$(4) \quad D = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Matrix of the reduced echelon form (r.e.f.)

$$(5) \quad E = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Matrix of the reduced echelon form (r.e.f.)

$$(6) \quad F = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix of the reduced echelon form (r.e.f.)

$$(7) \quad \begin{bmatrix} 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Matrix not of the reduced echelon form (r.e.f.) since the condition (a) not satisfies.

$$(8) \quad \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & \textcircled{2} & -2 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Matrix not of the reduced echelon form (r.e.f.) since the condition (b) not satisfies.

$$(9) \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -5 & 3 \\ 0 & \textcircled{1} & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix not of the reduced echelon form (r.e.f.) since the condition (c) not satisfies.

$$(10) \begin{bmatrix} 1 & \textcircled{3} & \textcircled{3} & 1 \\ 0 & 1 & \textcircled{-4} & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix not of the reduced echelon form (r.e.f.) since the condition (d) not satisfies.

Linear Systems and Gaussian Elimination Method

Definition of the linear system: It is a set of (m) of linear equations and (n) of unknowns (variables), and it is expressed in the following form:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

It is written briefly as follows $AX = B$, where

A is the coefficients matrix (coefficients of the unknowns)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

X is the column of the unknowns $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$

B is the absolute quantities column $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

Examples:

(1) Consider the following linear system

$$\left. \begin{array}{l} 2x_1 - 3x_2 = 8 \\ 3x_1 + x_2 = 1 \end{array} \right\} \dots(1)$$

The absolute quantities column is $B = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$, the column of the unknowns is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

the coefficients matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}$, so we can write the system as follows:

$$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

When we compute the multiplication operator we get the system of the equations (1).

(2) $x + y + z = 1$

$$-y + 2z = 0$$

$$x + 2y = 2$$

The absolute quantities column is $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, the column of the unknowns is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

the coefficients matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, so we can write the system as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Definition the solution of the linear system: It is a set of the value of the unknowns which satisfy each equation in the system.

Remark: Each set of linear equation ($AX=B$) can be represent it by the matrix $[A:B]$, this matrix called augmented matrix.

Examples:

(1) Consider the following linear system

$$3x_1 + 5x_2 - 2x_3 = 5$$

$$4x_1 - x_2 + 3x_3 = 14$$

$$x_1 + x_2 + x_3 = 7$$

The augmented matrix is $\begin{bmatrix} 3 & 5 & -2 & \vdots & 5 \\ 4 & -1 & 3 & \vdots & 14 \\ 1 & 1 & 1 & \vdots & 7 \end{bmatrix}$

(2) If the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & -2 & 1 & \vdots & 3 \\ 3 & 0 & 4 & \vdots & 1 \end{bmatrix}$, so the set of the equations of the

system

$$x_1 + x_2 + x_3 = 2$$

$$-2x_2 + x_3 = 3$$

$$3x_1 + 4x_3 = 1$$

Remark: From above we note that the system of linear equations transfer to a system simple than it (equivalent system) by using some operations on equations. Also, the representation of the system by an augmented matrix means that each row in the matrix represents an equation, and for this the system can be solved by using the augmented matrix and these operations on the augmented matrix correspond to (similarity) operations on the equations. These operations on an augmented matrix are called **elementary row operations** and denoted by (e.r.o.) which are:

(1) We multiply the row (i) by a constant number α not equal to zero. It is denoted by $\{R_i = \alpha R_i\}$.

(2) We multiply the row (i) by a constant number α not equal to zero and add it to the row (j) where ($i \neq j$) is denoted by $\{R_j = r_j + \alpha r_i\}$.

(3) Replace row (i) with row (j) or vice versa and denote it by $\{r_j \leftrightarrow r_i\}$.

The above method which is used to solve any linear system is called Gaussian elimination method

Gauss-Jordan Reduction Method

Solving the system $AX = B$ in this method depends on converting the augmented matrix to the reduced echelon form.

Examples: (1) Solve the following linear equations systems by Gauss-Jordan reduction method

(a) $x - 4y = 11$
 $x - 2y = 7$

$$[A:B] = \begin{bmatrix} 1 & -4 & : & 11 \\ 1 & -2 & : & 7 \end{bmatrix} \xrightarrow{R_2 = r_2 - r_1} \begin{bmatrix} 1 & -4 & : & 11 \\ 0 & 2 & : & -4 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{2}r_2} \begin{bmatrix} 1 & -4 & : & 11 \\ 0 & 1 & : & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 = r_1 + 4r_2} \begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 1 & : & -2 \end{bmatrix}. \text{ The solution is } x = 3 \text{ and } y = -2.$$

(b) $2x + 3y = 0$
 $4x - y = 0$

$$[A:B] = \begin{bmatrix} 2 & 3 & : & 0 \\ 4 & -1 & : & 0 \end{bmatrix} \xrightarrow{R_2 = r_2 - 2r_1} \begin{bmatrix} 2 & 3 & : & 0 \\ 0 & -7 & : & 0 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{7}r_2} \begin{bmatrix} 2 & 3 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = r_1 - 3r_2} \begin{bmatrix} 2 & 0 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix} \xrightarrow{R_1 = \frac{1}{2}r_1} \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix}$$

The solution is $x = 0$ and $y = 0$

(2) Solve (if possible) the following system

$$x_1 + x_2 = 2$$

$$2x_1 + 4x_2 = -1$$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & \vdots & 2 \\ 2 & 4 & \vdots & -1 \end{bmatrix} \xrightarrow{R_2 = r_2 - 2r_1} \begin{bmatrix} 1 & 1 & \vdots & 2 \\ 0 & 2 & \vdots & -5 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{2}r_2} \begin{bmatrix} 1 & 1 & \vdots & 2 \\ 0 & 1 & \vdots & -5/2 \end{bmatrix}$$

$$\xrightarrow{R_1 = r_1 - r_2} \begin{bmatrix} 1 & 0 & \vdots & 9/2 \\ 0 & 1 & \vdots & -5/2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9/2 \\ -5/2 \end{bmatrix} \text{ or } x_1 = 9/2, x_2 = -5/2$$

(3) Solve (if possible) the following system

$$2x - 3y = 8$$

$$3x + y = 1$$

(Home work)

(4) Solve (if possible) the following system

$$x_1 + x_2 + x_3 = 1$$

$$2x_1 + 3x_2 + 3x_3 = 3$$

$$x_1 + 2x_2 + 2x_3 = 5$$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 3 & 3 & \vdots & 3 \\ 1 & 2 & 2 & \vdots & 5 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 = r_2 - 2r_1 \\ R_3 = r_3 - r_1 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 4 \end{bmatrix} \xrightarrow{R_3 = r_3 - r_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 3 \end{bmatrix}$$

The third row means $0x_1 + 0x_2 + 0x_3 = 3$ which implies that $0 = 3$ which is impossible so the system has no solution.

Remark: When solving this system by elimination method in solving equations, you find the same answer (that the system is inconsistent) that is, it has no solution.

Example: Solve (if possible) the following system

$$x + y + z = 1$$

$$2x + y + z = 1$$

$$3x + y + z = 1$$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 1 & \vdots & 1 \\ 3 & 1 & 1 & \vdots & 1 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 = r_2 - 2r_1 \\ R_3 = r_3 - 3r_1 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & -2 & -2 & \vdots & -2 \end{bmatrix}$$

$$\xrightarrow{R_3=r_3-2r_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow{R_1=r_1+r_2} \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

This mean $x = 0$

$$-y - z = -1$$

$$y + z = 1$$

Which mean when we take value for y we can find the value of z .

This mean the system has infinite solutions.

The solution is $x = 0, z = 1 - y, y = \text{any real number}$.

Or the solution $\{(0, a, 1 - a) : a \in \mathbb{R}\}$.

If $a = 1$, so $(0, 1, 0)$ is solution, or if $a = 2$, so $(0, 2, -1)$ is solution,and so on.

Equivalent Matrices

If the two matrices A and B are of the same degree, then A is a row equivalent with B If B can be obtained from A with an operation or (Elementary Row Operations) (e.r.o.), the equivalence is symbolized by $(\sim)$ we say $(A \sim B)$.

We can note that:

- (1) For any matrix A , then $A \sim A$.
- (2) For any matrices A and B , if $A \sim B$, then $B \sim A$.
- (3) For any matrices A, B and C , if $A \sim B$ and $B \sim C$, then $A \sim C$.

Examples for the equivalent matrices:

$$(1) \text{ If } A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3=r_3-r_2} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2=2r_2} \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 2 \\ -1 & 0 & 0 \end{bmatrix} = B$$

So $A \sim B$ (row equivalent).

(2) Show that $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \sim I_2$

Solution:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \xrightarrow{R_2 = r_2 + r_1} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 = r_1 + 2r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore A \sim I_2$

Exercise: Show that the matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & -2 \end{bmatrix}$ are row equivalent?

Theorem: (without proof)

The square matrix of degree $(n \times n)$ has inverse if it is row equivalent for the identity matrix.

For example the matrix $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ in example (2) above has inverse since it is row equivalent to I_2 .

Theorem: (without proof)

If A is a square matrix of degree $n \times n$, then the linear system $A_{n \times n} \cdot X_{n \times 1} = B_{n \times 1}$ has unique solution if and only if $|A| \neq 0$.

Remark: If A is a square matrix of degree $n \times n$, then the linear system $A_{n \times n} \cdot X_{n \times 1} = B_{n \times 1}$ has infinite number of the solutions or has no solution if and only if $|A| = 0$.

Examples:

(1) Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \Rightarrow |A| = 0$, so the linear system

$$2x - 3y = 8$$

$$2x - 3y = 3$$

————— by subtraction

$$0 = 6$$

Has no solution

(2) Let $A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \Rightarrow |A| = 0$, so the linear system

$$2x - 3y = 8$$

$$4x - 6y = 16$$

———— multiply the first equation by (2)

$$4x - 6y = 16$$

$$4x - 6y = 16$$

———— by subtraction

$$0 = 0$$

Has infinite number of solutions.

Exercises: Solve the following linear systems

(1) $x + y + z = 1$

$$x + 2y + 3z = -1$$

$$x + 4y + 4z = -9$$

(2) $x + 2y + 2z = 1$

$$x + 5y + 2z = 4$$

$$x + 8y + 2z = 8$$

(3) $x + y = 3$

$$2x - y = 1$$

(4) Find the value of a which make the following linear systems have no solution

(a) $x - 2y = 5$

$$3x + ay = 1$$

(b) $x - y + 2z = 3$

$$2x + ay + 3z = 1$$

$$-3x + 3y + z = 4$$

Cramer's Rule

This method using to find the solutions of the linear system which its coefficients matrix is square matrix and its determinant $\neq 0$.

Theorem: (without proof)

$$\text{Let } A_{n \times n} \cdot X_{n \times 1} = B_{n \times 1} \text{ and } |A| \neq 0, \text{ then } x_j = \frac{|A_j|}{|A|}, j = 1, 2, \dots, n, \text{ where}$$

A_j is the matrix obtained it by replace the column j for the matrix A by the absolute quantities column B .

Examples: Using the Gramer's rule to find the solution for the following systems:

(1) $x_1 - 2x_2 = 8$

$5x_1 + 2x_2 = 4$

Solution:

$$|A| = \begin{vmatrix} 1 & -2 \\ 5 & 2 \end{vmatrix} = 12, \quad |A_1| = \begin{vmatrix} 8 & -2 \\ 4 & 2 \end{vmatrix} = 24, \quad |A_2| = \begin{vmatrix} 1 & 8 \\ 5 & 4 \end{vmatrix} = -36$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{24}{12} = 2, \quad x_2 = \frac{|A_2|}{|A|} = \frac{-36}{12} = -3$$

The solution is $x_1 = 2, x_2 = -3$

(2) $-2x_1 + 3x_2 - x_3 = 1$

$x_1 + 2x_2 - x_3 = 4$

$-2x_1 - x_2 + x_3 = -3$

Solution:

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2, \quad x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{-2} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3, \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{-2} = \frac{-8}{-2} = 4$$

Exercises: Solve the following linear systems by Gramer's rule

(1) $2x + y + z = 6$

$3x + 2y - 2z = -2$

$x + y + 2z = -4$

(2) $2x + 4y + 6z = 2$

$x + 2z = 0$

$2x + 3y - z = -5$

Homogeneous Linear Systems

We say the linear system of the formula

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\
\vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0
\end{aligned}$$

It is a homogeneous system that can be written in the matrix form $AX = O$.

The solution $x_1 = x_2 = \dots x_n = 0$ is called the trivial solution for the homogeneous system.

If $x_1, x_2, \dots, x_n$ solution for the linear system where $x_i \neq 0$ for some values of i , then this solution is called non trivial.

$\therefore$ The homogeneous linear system always consistent (if it has some solutions), the trivial solution is one of their solutions.

Theorem: If A is a square matrix of degree $n \times n$, then the homogeneous system $AX = O$ has trivial solution if and only if $|A| \neq 0$.

Example:

$$3x + y = 0$$

$$2x + 4y = 0$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \Rightarrow |A| = 10 \neq 0.$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_1=r_1-r_2} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2=r_2-2r_1} \begin{bmatrix} 1 & -3 \\ 0 & 10 \end{bmatrix} \xrightarrow{R_2=\frac{1}{10}r_2} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1=r_1+3r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution is $x = y = 0$

$\therefore$ The trivial solution is the only solution for this system.

Remark: If A is a square matrix of degree $n \times n$, then the homogeneous system $AX = O$ has infinite number of the solutions if and only if $|A| = 0$.

Examples:

$$(1) x - 5y = 0$$

$$2x - 10y = 0$$

$$A = \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \Rightarrow |A| = 0$$

$$\begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \xrightarrow{R_2=r_2-2r_1} \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

$$x - 5y = 0 \Rightarrow x = 5y$$

$$0x + 0y = 0$$

The solution is the set $\{(5a, a); a \in \mathbb{R}\}$.

So we get this system has infinite number of the solution.

- (2) Find the value of α which make the linear system $(\alpha I - A)X = O$ has non trivial solution if $A = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$.

Solution: The homogeneous system $(\alpha I - A)X = O$ has non trivial solution if and only if $|\alpha I - A| = 0$

$$\alpha I - A = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \alpha - 2 & -6 \\ -2 & \alpha - 3 \end{bmatrix}$$

Since $|\alpha I - A| = 0$

$$\therefore \begin{vmatrix} \alpha - 2 & -6 \\ -2 & \alpha - 3 \end{vmatrix} = 0$$

$$(\alpha - 2)(\alpha - 3) - (-2)(-6) = 0$$

$$\alpha^2 - 5\alpha + 6 - 12 = 0$$

$$\alpha^2 - 5\alpha - 6 = 0$$

$$(\alpha - 6)(\alpha + 1) = 0 \Rightarrow \alpha = 6, \alpha = -1$$

Examples: Solve each of the following linear systems by Gauss-Jordan method

(1) $x + 2y + 3z = 0$

$$-x + 3y + 2z = 0$$

$$2x + y - 2z = 0$$

Note: since the absolute quantities column is zeros, so we can write the coefficients matrix only in augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array} \right] \xrightarrow[\begin{array}{l} R_2 = r_2 + r_1 \\ R_3 = r_3 - 2r_1 \end{array}]{\begin{array}{l} R_2 = r_2 + r_1 \\ R_3 = r_3 - 2r_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -3 & -8 & 0 \end{array} \right] \xrightarrow{R_2 = \frac{1}{5}r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -8 & 0 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{R_1=r_1-2r_2} \\ \xrightarrow{R_3=r_3+3r_2} \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{R_3=-\frac{1}{5}r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \xrightarrow{R_1=r_1-r_3} \\ \xrightarrow{R_2=r_2-r_3} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution is $x = y = z = 0$

$\therefore$ The trivial solution is the only solution for this system.

2. $x + 2y - z = 0$

$x + 3y + 2z = 0$

$3x + 8y + 3z = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 3 & 8 & 3 \end{bmatrix} \begin{array}{l} \xrightarrow{R_2=r_2-r_1} \\ \xrightarrow{R_3=r_3-3r_1} \end{array} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{array}{l} \xrightarrow{R_1=r_1-2r_2} \\ \xrightarrow{R_3=r_3-2r_2} \end{array} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - 7z = 0 \Rightarrow x = 7z$

$y + 3z = 0 \Rightarrow y = -3z$

$z = \text{any real number}$

$\therefore$ This system has infinite number of solutions.

$\therefore$ The solution is the set $\{(7a, -3a, a) : a \in \mathbb{R}\}$.

Exercises: Solve the following linear systems by Gauss-Jordan method

(1) $2x - 2y + 2z = 0$

$4x - 7y + 3z = 0$

$2x - y + 2z = 0$

(2) $x + 3y - 3z = 0$

$x + 3y - 2z = 0$

$2x + 6y - 3z = 0$

(3) $x + y + z + w = 0$

$x + w = 0$

$x + 2y + z = 0$

(4) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that the homogeneous system $AX = O$ has the only trivial solution if and only if $ab - bc \neq 0$.

Abstract for the method of transformations on rows (Gauss-Jordan method)

To find the inverse of the matrix A by the method (Gauss - Jordan), we write the matrix A with the identity matrix in the following form: $[A: I_n]$ then transformed the matrix A by the transformations of rows into I_n and thus transforms I_n to A^{-1} , (performed the transformations of rows operator in both matrices at the same time)

Examples: Find the inverse (if exists) for each of the following matrices

(1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 1 & 3 & 3 & : & 0 & 1 & 0 \\ 1 & 2 & 4 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\begin{matrix} R_2=r_2-r_1 \\ R_3=r_3-r_1 \end{matrix}]{\begin{matrix} R_2=r_2-r_1 \\ R_3=r_3-r_1 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1=r_1-2r_2} \begin{bmatrix} 1 & 0 & 3 & : & 3 & -2 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1=r_1-3r_3} \begin{bmatrix} 1 & 0 & 0 & : & 6 & -2 & -3 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix}$$

So the inverse of the matrix A is $A^{-1} = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 1 & -2 & 1 & : & 0 & 1 & 0 \\ 5 & -2 & -3 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\begin{matrix} R_3=r_3-5r_1 \end{matrix}]{\begin{matrix} R_2=r_2-r_1 \\ R_3=r_3-5r_1 \end{matrix}} \begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -4 & 4 & : & -1 & 1 & 0 \\ 0 & -12 & 12 & : & -5 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3=r_3-3r_2} \begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -4 & 4 & : & -1 & 1 & 0 \\ 0 & 0 & 0 & : & -2 & -3 & 1 \end{bmatrix}$$

At this point the matrix A is row equivalent to $B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$.

So the matrix A is singular (A has no inverse).

(3) Find the value(s) of a which make the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$ exists.

What is A^{-1} ?

Solution:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & a & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_2 = r_2 - r_1 \\ R_3 = r_3 - r_1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & a & -1 & 0 & 1 \end{array} \right] \\ & \xrightarrow{\substack{R_1 = r_1 - r_3 \\ R_2 = r_2 + r_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & -a & 2 & 0 & -1 \\ 0 & 0 & a & -2 & 1 & 1 \\ 0 & 1 & a & -1 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -a & 2 & 0 & -1 \\ 0 & 1 & a & -1 & 0 & 1 \\ 0 & 0 & a & -2 & 1 & 1 \end{array} \right] \end{aligned}$$

To be the third row not equal to zero, it must $a \neq 0$.

$$\xrightarrow{\frac{1}{a}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -a & 2 & 0 & -1 \\ 0 & 1 & a & -1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{array} \right] \xrightarrow{\substack{R_1 = r_1 + a r_3 \\ R_2 = r_2 - a r_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{array} \right]$$

$$\text{So } A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{bmatrix}.$$

Exercises: Find the inverse (if exists) for each of the following matrices

(1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

(3) $C = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$(2) \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & 4 \end{bmatrix}$$

$$(4) \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 5 & 5 & 1 \end{bmatrix}$$

Solving Linear Systems By Using The Inverse

Consider the linear system $A_{n \times n} X_{n \times 1} = B_{n \times 1}$, where the matrix $A_{n \times n}$ has inverse, then $X_{n \times 1} = A_{n \times n}^{-1} B_{n \times 1}$

Proof:

$$A X = B$$

$$A^{-1}(A X) = A^{-1} B \quad (\text{multiply each side by } A^{-1})$$

$$(A^{-1}A) X = A^{-1} B \quad (\text{the multiplication of the matrices is associative})$$

$$I_n X = A^{-1} B \quad (\text{the definition of the inverse } (A^{-1}A = I))$$

$$X = A^{-1} B$$

Remark: We use this method when the matrix is square and has inverse.

Examples:

(1) Solve the following linear system by using the inverse of the matrix

$$(a) \quad x + 2y = 4$$

$$3x + 4y = 5$$

Solution: The system can be written as follows $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We compute the inverse for the coefficients matrix as follows

$$[A \quad : \quad I_n] \rightarrow [I_n \quad : \quad A^{-1}]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right] \xrightarrow{R_2 = r_2 - 3r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right]$$

$$\xrightarrow{R_2 = -\frac{1}{2}r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right] \xrightarrow{R_1 = r_1 - 2r_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right] = [I_2 \quad : \quad A^{-1}]$$

$$\text{The inverse of the coefficients matrix is } A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix}.$$

The solution is $x = -3$ and $y = \frac{7}{2}$.

(b) $3x - 4y = -5$
 $-2x + 3y = 4$

Solution: The system can be written as follows $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

We compute the inverse for the coefficients matrix as follows

$$[A \quad \vdots \quad I_n] \rightarrow [I_n \quad \vdots \quad A^{-1}]$$

$$\begin{bmatrix} 3 & -4 & \vdots & 1 & 0 \\ -2 & 3 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow{R_1=r_1+r_2} \begin{bmatrix} 1 & -1 & \vdots & 1 & 1 \\ -2 & 3 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow{R_2=r_2+2r_1} \begin{bmatrix} 1 & -1 & \vdots & 1 & 1 \\ 0 & 1 & \vdots & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1=r_1+r_2} \begin{bmatrix} 1 & 0 & \vdots & 3 & 4 \\ 0 & 1 & \vdots & 2 & 3 \end{bmatrix}$$

The inverse of the coefficients matrix is $A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $x = 1$ and $y = 2$.

(3) Solve the linear system $-10x_1 + 5x_2 + 3x_3 = 1$
 $7x_1 - 3x_2 - 2x_3 = -2$
 $-4x_1 + 2x_2 + x_3 = 0$

Where $\begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & -5 \end{bmatrix}$

Solution: The system can be written as follows
$$\begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Since the inverse of the coefficients matrix given, then

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

The solution is $x_1 = -1$, $x_2 = -3$ and $x_3 = 2$.