Abbreviations and Symbols and Terms (English / Arabic)

Annual Payments (A.P): الدفعات السنوية
Annual Worth (A.W): القيمة السنوية
Book Value (B.V) : القيمة الدفترية
Borrow: يقترض من البنك ، يستعير
Breakeven Point: نقطة التعادل
Capital: رأس المال
Cash Flow Diagram: مخطط التدفق النقدي
Compound Interest: فائدة مركبة
Deposit: يوّدع في البنك، يضع
Depreciation: إندثار ، أهتلاك
Depreciation Rate: نسبة الإندثار
Fortune: ثروة
Future Amount: القيمة المستقبلية
Gradient Payment: دفعات منتظمة متدرجة
Interest: فائدة، ريع
Investment: إستثمار
Loan: قرض، سلفة
Owe: يدين
Owed: مستحق الدفع
Owning: مَدين، مطلوب ، مستحق الدفع
Principal: أساسي ، رأس المال
Principle: مبدأ ، الأصل ، القاعدة
Present Value (P.V): القيمة الحالية
Present Worth (P.W): القيمة الحالية
Rate of interest: نسبة الفائدة
Regular Payments : الدفعات المنتظمة
Salvage Value (S.V): القيمة الإستردادية
Scrape Value: قيمة السكراب (الخردة – آلية غير نافعة إقتصادياً)
Simple Interest: فائدة بسيطة
Single Payment: دفعة مفردة ، دفعة واحدة
Withdraw: يسحب مبلغاً من البنك، يسترد ، يسترجع

References

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Introduction

Engineering economics is the application of economic techniques to the evaluation of design and engineering alternatives.

The role of engineering economics is to assess the appropriateness of a given project, estimate its value, and justify it from an engineering standpoint.

This semester discusses the time value of money and other cash-flow concepts, such as compound and continuous interest. It continues with economic practices and techniques used to evaluate and optimize decisions on selection the best alternative. The final section expands on the principles of benefit-cost analysis.

When Must Interest and Profit be considered?

If capital for financing an enterprise must be borrowed, money that is paid for its use is called interest. On the other hand, if a person or corporation owns sufficient capital to finance a proposed project, the return that accrues is called profit. When owner's capital is the sole source of funds used to sponsor a project, there is no borrowed money in the true meaning of the term and there is no interest expense. However in this case if the owner of the capital decides to invest it in the proposed venture, he must forgo using it for some other profitable purpose - even if the other purpose is merely leaving it in a bank where it would draw interest.

In typical situations an investor must decide whether the expected profit, usually measured in terms of an annual rate of return on investment, is sufficient to justify investment of capital in the proposed venture. Although no interest cost is involved, if the capital is invested in the project the investor would expect, as a minimum, to receive profit at least equal to the amount the person has sacrificed by not using it in some other equally attractive and available opportunity. This profit, which is lost or forgone, is called the opportunity cost of using capital in the proposed venture.

To determine whether the expected capital return (profitability) is sufficient, it usually is necessary to compare the expected rate of profit with the rate that could be obtained from using the same capital in the some other manner.

Interest: The Cost of Money

Money left in a savings account earns interest so that the balance over time is greater than the sum of the deposits. In the financial world, money itself is a commodity, and like other goods that are bought and sold, money costs money. The cost of money is established and measured by an *interest rate,* a percentage that is periodically applied and added to an amount (or varying amounts) of money over a specified length of time. When money is borrowed, the interest paid is the charge to the borrower for the use of the lender's property; when money is loaned or invested, the interest earned is the lender's gain from providing a good to another. *Interest*, then, may be defined as the cost of having money available for use.

The operation of interest reflects the fact that money has a time value. This is why amounts of interest depend on lengths of time; interest rates, for example, are typically given in terms of a percentage per year. This principle of the time value of money can be formally defined as follows: the economic value of a sum depends on when it is received. Because money has earning power over time (it can be put to work, earning more money for its owner), a dollar received today has a greater value than a dollar received at some future time.

The changes in the value of a sum of money over time can become extremely significant when we deal with large amounts of money, long periods of time, or high interest rates. For example, at a current annual interest rate of 10%, $1 million will earn $100,000 in interest in a year; thus, waiting a year to receive $1 million clearly involves a significant sacrifice. In deciding among alternative proposals, we must take into account the operation of interest and the time value of money to make valid comparisons of different amounts at various times.

Time Value of Money

The following are reasons why \$1000 today is "worth" more than \$1000 one year from today:

1. Inflation
2. Risk
3. Cost of money

Of these, the cost of money is the most predictable, and, hence, it is the essential component of economic analysis. Cost of money is represented by (1) money paid for the use of borrowed money, or (2) return on investment.

Cost of money is determined by an interest rate.

Time value of money is defined as the time-dependent value of money stemming both from changes in the purchasing power of money (inflation or deflation) and from the real earning potential of alternative investments over time.

SIMPLE INTEREST

To illustrate the basic concepts of interest, an additional notation will be used:

$F(N)$ = Future sum of money after N periods

Then, for simple interest, $F(1) = P + (P)(i) = P(1 + i)$

and

$F(N) = P + (N)(P)(i) = P(1 + Ni)$

$$\textbf{Interest} = (\textbf{Principal})(\textbf{Number of Periods})(\textbf{Interest rate}) \quad \textbf{........(1)}$$

Example 1

\$100 at 10 percent per year for 5 years yields

Solution:

$F(5) = 100[1 + (5)(0.1)] = 100(1.5) =$ \$150

However, interest is almost universally compounded to include interest on the interest.

When the total interest earned or charged is directly proportional to the amount of the loan (principal), the interest rate, and the number of interest periods for which the principle is committed the interest and interest rate are said to be simple.

When simple interest is applicable, the total l interest, I, earned or paid may be computed by the formula:

I = P (N)(i)………………………………………………………..…… (1)

Where P = principle amount lent or borrowed

N = number of interest periods (e g , years)

i = interest rate per interest period

<u>Example 2</u>

$100 is loaned for 3 years at a simple interest rate of 10% per annum the interest earned will be

Solution:

I =$100x0.10 x 30 = $30

The total amount owed at the end of 3 years would be $100+ $ 30 = $ 130

Simple interest is not used frequently in commercial practice in modern times, but it is of importance to contrast simple interest with compound interest, which is explained below.

<u>Example 3</u>

An employer at Laser kinetics.com borrows $ 10,000 on last May and must repay a total of $ 10,700 exactly 1 year (May 2019). Determine the interest amount and the interest rate paid? Note use the simple interest law.

Solution:

The perspective here is that of the borrower since $ 10,700 repays a loan. Apply the following equation to determine the interest paid

$$\textbf{Interest} = \textbf{Amount owed now} - \textbf{Original Amount} \;\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Interest = $ 10,700 - $ 10,000 = $ 700

The next equation can use for determining the interest rate paid for 1 year.

$$\textbf{Interest Rate } (\%) = \frac{\textit{Interest Accured Per Time Unit}}{\textit{Original Amount}}\, x\; 100\% \;\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{Percent Interest Rate } (\%) = \frac{\$\,700}{\$\,10{,}000}\, x\; 100\% = 7\%\; per\; year.$$

Example 4

A contractor plans to borrow $ 20,000 from a bank for 1 year at 9% interest for new shovel equipment

a- Compute the interest and the total amount due to after 1 year.

b- Construct a column graph that shows the original amount and total amount due to after one year used to compute the loan interest rate 9% per year.

Solution:

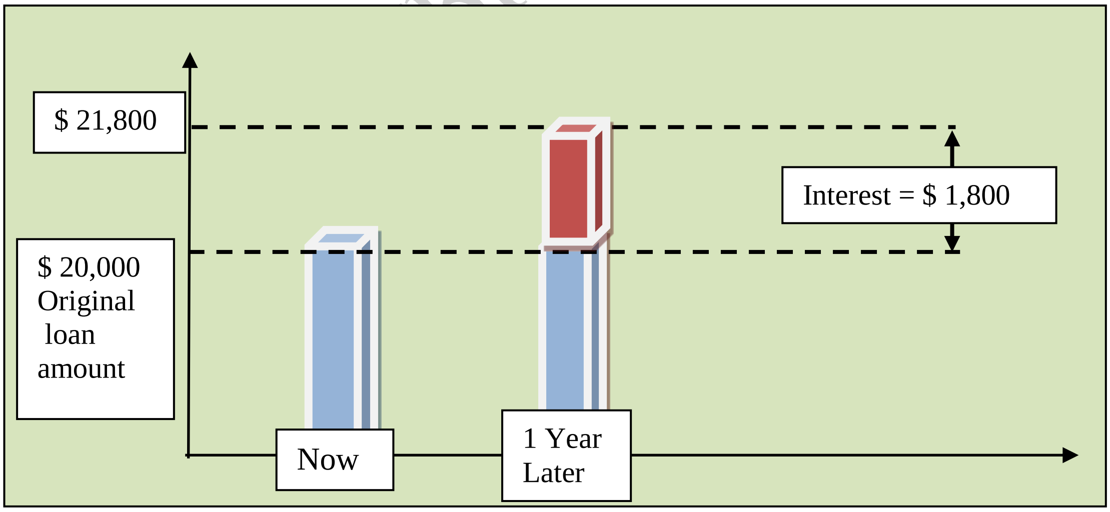
a- Compute the total interest accrued by solving Eq. (1) for interest accrued.

$$Interest = \$\,20{,}000\; x\; 0.09 = \$\,1{,}800$$

The total amount due is the sum of principal and interest.

$$Total\ due = \$\,20{,}000 + \$\,1{,}800 = \$\,21{,}800$$

b- The figure below shows the values used in equation (1): $ 1,800 interest. $ 20,000 original loan principal, 1- year interest period.



Example 5

a- Calculate the amount deposited 1 year ago to have $ 1,000 now at an interest rate of 5% per year?

b- Calculate the amount of interest earned during this time period?

Solution:

a- The total amount accrued is the sum of the original deposit and the earned interest. If X is the original deposit

$$Total\ accrued = original + original\ (interest\ rate)$$

$$\$\ 1{,}000 = X + X(0.05) = X(1 + 0.05) = 1.05\ X$$

The original deposit is

$$X = \frac{\$\ 1{,}000}{1.05} = \$\ 952.38$$

b- Apply the following equation:

$$\boldsymbol{Interest = Total\ amount\ now - Original\ amount} \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{4})$$

Interest = \$ 1,000 – 952. 38= \$ 47.62

Example 6

A contractor loaned money to an engineering staff member for a design to the new dam. The loan is \$ 1,000 for 3 years at 5% per year simple interest. How much money will the contractor repay at the end of 3 years? Tabulate the results.

Solution:

The interest for each of the 3 years is

$$Interest\ per\ year = \$\ 1{,}000\ x\ 0.05 = \$\ 5$$

Total interest = $\$\ 1{,}000\ x\ 3\ x 0.05 = \$\ 150$

The amount due after 3 years is

$$Total\ due = \$\ 1{,}000 + 150 = \$\ 1{,}150$$

Simple interest computation

1 End of Year	2 Amount Borrowed	3 Interest	4 Amount Owed	5 Amount Paid
0	\$ 1,000	---------		
1	---------	50	\$ 1,050	\$ 0
2	---------	50	1,100	0
3	---------	50	1,150	\$ 1,150

Compound Interest

Wherever the interest charge for any interest period (a year, for example) is based on the remaining principle amount plus any accumulated interest charges up to the beginning of that period, the interest is said to be *Compound*. The effect of compounding of interest can be shown in the following calculations.

F (1) = P + (P) (i) = P (1 + i)

is the same as simple interest

F (2) = F (1) + F (1) (i)

Interest is applied to the new sum:

= (F) (1) (1 + i) = P $(1 + i)^2$

F (3) = F(2)(1 + i) = $P(1 + i)^3$

and by mathematical induction,

$F(n) = P\ (1 + i)^n$

$$\boldsymbol{Interest = (Principal + all\ accrued\ interest\ ) x\ (interest\ rate) .. (5)}$$

Example 7

An Engineer deposited amount of money ($ 1,000) in the bank for three years at an interest rate of 10%. Calculate the cumulative amount of money for each year (compound per period).

Solution:

The table below describes the cumulative the amount of money for each year.

End of year	(1) amount owed at beginning of year	(2) = (1) x 10% Interest charge for year	(3) = (1)+(2) Amount owed at end of year
1	$ 1,000	$ 100	$ 1,100
2	1,100	$ 110	$ 1,210
3	1,210	$ 121	$ 1,331

Example 8

What will $100 yields at 10 percent per year for 5 years?

F (5) = $100(1 + 0.1)^5$

$=100(1.1)^5$

$=100(1.61051)$

= \$161.05

Which is over 7 percent greater than with simple interest.

Example 9

In 1626 Willem Verhulst bought Manhattan Island from the Canarsie Indians for 60 florins (\$24) worth of merchandise (a price of about 0.5 cents per hectare [0.2 cents per acre]). At an average interest rate of 6 percent, what is the present value (2001) of the Canarsies' \$24?

Solution

$F = P(1 + i)^n$

= \$24(1 + 0.06)375

$= \$\ 7.4\ x\ 10^{10}$

Seventy-four billion dollars is a reasonable approximation of the present land value of the island of Manhattan.

Compound Amount Factor

In the formula for finding the future value of a sum of money with compound interest, the mathematical expression

$(1 = i)^N$ is referred to as the compound amount factor, represented by the functional format (F/P, i, n). Thus,

F = P (F/P, i, n)

Interest tables: Values of the compound amount, present worth, and other factors that will be discussed shortly, are tabulated for a variety of interest rates and number of periods in most texts on engineering economy.

Example tables are presented in Appendix C to this chapter.

Although calculators and computers have greatly reduced the need for such tables, they are often still useful for interpolations.

Whenever the interest charge for any interest period (a year, for example) is base d on the remaining principle amount plus any accumulated interest charges up to the beginning of that period, the interest is said to be compound.

Example 10

The effect of compounding of interest can be shown by the following table for $ 100 loaned for three periods at an interest rate of 10% compounded per period.

Table (1) compounded interest

Period	(1) Amount Owed at Beginning of period	(2)= 1 x 10% interest Charge for period	(3) = (1) + (2) Amount Owed at End of Period
1	$ 100.00	$ 10.00	$ 110.00
2	$ 110.00	$ 11.00	$ 121.00
3	$ 121.00	$ 12.10	$ 133.10

Thus $ 133.10 would be due for repayment at the end of the third period. If the length of a period is 1 year, the $ 133.10 at the end of three periods (years) can be compared directly with the $ 130.00 given earlier for the same problem with simple interest. The difference is due to the effect of compounding which essentially is the calculation of interest on previously earned interest. This difference would be much greater for larger amounts of money, higher interest rates, or greater numbers of years. Compound interest is much more common in practice than is simple interest and is used throughout the remainder of this text.

Equivalence Calculations

Economic equivalence refers to the fact that a cash flow — whether it is a single payment or a series of payments — can be said to be converted to an *equivalent* cash flow at any point in time; thus, for any sequence of cash flows, we can find an equivalent single cash flow at a given interest rate and a given time.

Equivalence calculations can be viewed as an application of the compound interest relationships developed in Equation $F = P (1 + i)^n$. The formula developed for calculating compound interest, $F = P\,(1+1)^n$, expresses the equivalence between some present amount, P, and a future amount, F, for a given interest rate, i, and a number of interest periods, n.

Example 11

What will the $1000 be at the end of a 3-year investment period at 8%.

$ 1000 $(1+0.08)^3$ = $ 1259.71.

To understand better the mechanics of interest and to expand on the notion of economics equivalence, we consider a situation in which we borrow $8,000 and agree to repay it in 4 years at annual interest rate of 10%. There are many ways in which the principal of this loan (***i.e.***….. $8,000) and the interest on it can be repaid. For simplicity, we have selected four plans to demonstrate the idea of economic equivalence. In each the interest rate is 10% and the original amount borrowed is $ 8,000; thus primary differences among plans rest with items (3) and (4) above. The four plans are shown in table below, and it will soon be apparent that all are equivalent at an interest rate of 10% per year.

It can be seen in plan 1 that $ 2,000 of the loan principal is repaid at the end of year 1 through 4. As a result, the interest we repay at the end of a particular year is affected by how much we still owe on the loan at the beginning of that year. Our end –of-year payment is just the sum of $ 2,000 and interest paid on the beginning –of –year amount owed.

Table (2) Plans for Interest Payments

1 Year	(2) Amount Owed at Beginning of Year	(3) = 10% X (2) Interest Owed for Year	(4) = (2)+(3) Total Money Owed at End of Year	(5) Principal Payment	(6) = (3) + (5) total End – of –Year Payment
PLAN 1 : AT END OF EACH YEAR PAY $ 2,000 PRINCIPAL PLUS INTEREST DUE					
1	$ 8,000	$ 800	$ 8,800	$ 2,000	$ 2,800
2	$ 6,000	$ 600	$ 6,600	$ 2,000	$ 2,600
3	$ 4,000	$ 400	$ 4,400	$ 2,000	$ 2,400
4	$ 2,000	$ 200	$ 2,200	$ 2,000	$ 2,200
		$ 2,000 (Total interest)		$ 8,000	$ 10,000 (total amount repaid)
PLAN 2: PAY INTEREST DUE AT END OF EACH YEAR AND PRINCIPAL AT END OF 4 YEARS					
1	$ 8,000	$ 800	$ 8,800	$ 0	$ 800
2	$ 8,000	$ 800	$ 8,800	$ 0	$ 800
3	$ 8,000	$ 800	$ 8,800	$ 0	$ 800
4	$ 8,000	$ 800	$ 8,800	$ 8,000	$ 8,800
		$ 3,200 (total interest)		$ 8,000	$ 11,200 (total amount repaid)
PLAN 3: PAY IN 4 EQUAL END OF YEAR PAYMENT					
1	$ 8,000	$ 800	$ 8,800	$ 1,724	$ 2,524
2	$ 6,276	$ 628	$ 6,904	$ 1,896	$ 2,524
3	$ 4,380	$ 438	$ 4,818	$ 2,086	$ 2,524
4	$ 2,294	$ 230	$ 2,524	$ 2,294	$ 2,524
		$ 2,096 (total interest)		$ 8,000	$ 10,096 (total amount repaid)
PLAN 4: PAY PRINCIPAL AND INTEREST IN ONE PAYMENT AT END OF 4 YEARAS					
1	$ 8,000	$ 800	$ 8,800	$ 0	$ 0
2	$ 8,800	$ 880	$ 9,680	$ 0	$ 0
3	$ 9,680	$ 968	$ 10,648	$ 0	$ 0
4	$ 10,648	$ 1,065	$ 11,713	$ 8,000	$ 11,713
		$ 3,713 (total interest)		$ 8,000	11,713 (total $amount repaid)

Plan 2 indicates that none of the loan principal is repaid until the end of the fourth year. Our interest cost each year is $ 800, and it is repaid in the end of years 1 through 4. Since interest does not accumulated in either plan 1 or plan 2, compounding of interest is not present. Notice that $ 3,200 in interest is paid with plan 2, whereas only $ 2,000 is paid in plan 1. The reason of course, is that we had use of the $8,000 principal for 4 years in plan 2 but on average had use of much less than $8,000 in plan 1.

Plan 3 requires that we repay equal end –of- year amount of 2,524 each. We will show how the $ 2,524 per year is computed. For purpose here, the student should observe that the four end – of –year payments in plan 3 completely repay the $ 8,000 loan principal with interest at 10%.

Finally, plan 4 shows that no interest and no principal are repaid for the first 3 years of the loan period. Then at the end of the fourth year, the original loan principal plus accumulated interest for the 4 years is repaid in a single lump-sum amount of $ 11,712 .80 (rounded in table above to $ 11,713), both plans 3 and 4 involve compound interest, even though it may not be apparent in plan 3. The total amount of interest repaid in plan 4 is the highest of all plans considered. Not only was the principal repayment in plan 4 deferred until the end of year 4, but we also deferred all interest until that time. If annual interest rates are rising above 10%, can you see than plan 4 cases bankers to turn gravy haired rather quickly?

Cash-Flow Diagrams

Cash flow is the sum of money recorded as receipts or disbursements in a project's financial records.

A cash flow diagram presents the flow of cash as arrows on a time line scaled to the magnitude of the cash flow, where expenses are down arrows and receipts are up arrows.

Year-end convention ~ expenses occurring during the year are assumed to occur at the end of the year.

The cash flow diagram employs several conventions:

1- The horizontal line is a time scale with progression of time moving from left to right. The period (or year) labels are applied to intervals of time rather than points on time scale. Note, for example, that the end of period 2 is coincident with the beginning of period 3. Only if specific dates are employed should the points in time rather than intervals be labeled.

2- The arrows signify cash flows. If a distinction needs to be made, downward arrows represent disbursements (negative cash flows or cash outflows) and upward arrows represent receipts (positive cash flows or cash inflows).

3- The cash flow diagram is dependent on point of view. For example, the situations shown in figures below were based on cash flow as seen by the lender. If the direction of all arrows had been reversed, the problem would have been diagramed from the borrower's viewpoint.

Example 12

Determine the present amount that is economically equivalent to $3,000 in five years, given the investment potential of 8% per year.

Given: F = $3,000, N = 5 years, i = 8% per year. Find: P.

$$F = P(l + i)^{n} \quad \ldots\ldots (6)$$

Rearranging Equation above to solve for P, we obtain

$$P = \frac{F}{(1+i)^{n}}$$

Substituting the given quantities into the equation yields

$$P = \frac{\$\,3,000}{(1+0.08)^{5}} = \$\,2,042$$

Single Amounts: Find *F*,

Given *P*, *i*, and *N*. If you had $2,000 now and invested it at 10% interest compounded annually, how much would it be worth in eight years (Figure 1)?

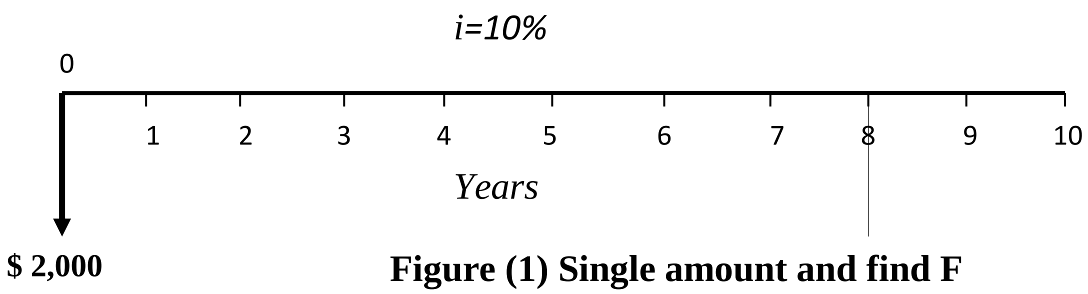


Figure (1) Single amount and find F

Example 13

Given: P = \$2,000, i = 10% per year, and n = 8 years. **Find:** F.

We can solve this problem in any of three ways:

1. Using a calculator: You can simply use a calculator to evaluate the $(1 + i)^N$ term (financial calculators are preprogrammed to solve most future-value problems):

F = \$2,000(1 + 0.10)^8

= \$4,287.18.

2. Using compound-interest tables: The interest tables can be used to locate the compound-amount factor for i = 10% and n = 8. The number you get can be substituted into the equation. Compound-interest tables are attached in these papers. Using this method, we obtain

F = \$ 2,000 (F/ P, 10%, 8) = \$ 2,000 (2.1436) = \$ 4,287.20

This amount is essentially identical to the value obtained by direct evaluation of the single cash flow compound-amount factor. This slight deviation is due to rounding differences.

3. Using a computer: Many financial software programs for solving compound interest problems are available for use with personal computers, many spreadsheet programs such as Excel also provide financial functions to evaluate various interest formulas. With Excel, the future worth calculation looks like the following: = FV (10%, 8, 2000,0).

Example 14

A person wants to receive amount of money equal to \$ 8,500 after 5 years. How much money has to deposit now in bank with compound rate interest 7%?

First Method

$F = P(1 + i)^n$

$8{,}500 = P(1 + 0.07)^5$

$P = \frac{8,500}{(1+0.07)^5} = \$\ 6,060.38$

<u>Or Second Method</u>

Can use the Interest Table with $i = 7\%$, and the deposit period is = 5 years the factor for this rate is = 0.7130

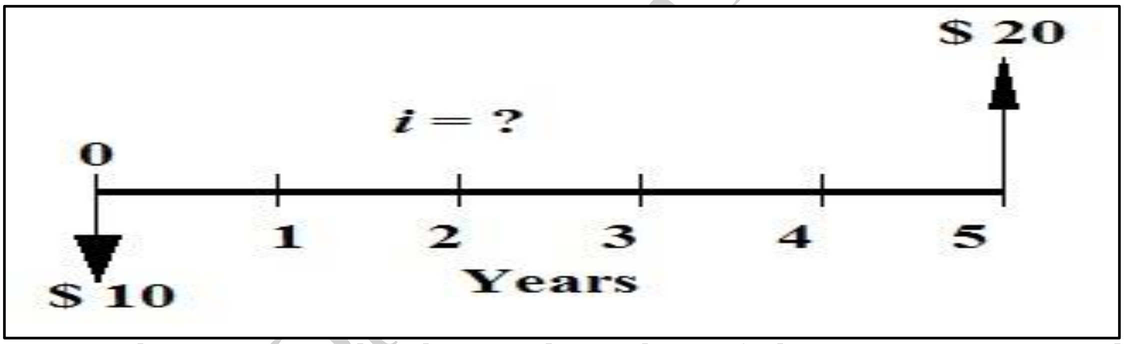
$P = 8,500\ x\ 0.7130 = \$\ 6060.5$

<u>Example 15</u>

<u>Solving for *i*</u>

Suppose you buy a share of stock for $10 and sell it for $20; your profit is thus $10. If that happens within a year, your rate of return is an impressive 100% ($10/$10 = 1). If it takes five years, what would be the rate interest on your investment?

Given: *P* = $10, F = $ 20, and ***n*** = *5*. **Find: *i*.**



Here we know P, F, and N, but we do not know ***i*,** the interest rate you will earn on your investment. This type of rate of return calculation is straightforward, since you make only a one-time lump-sum investment. We start with the following relationship:

$F = P\ (1+i)^n$ --- (2)

We then substitute in the given values:

$\$\ 20 = \$\ 10\ (1+i)^5$

Next, we solve for i by one of the following methods:

Method 1: Go through a trial-and-error process in which you insert different values of i into the equation until you find a value that "works" in the sense that the right-hand side of the equation equals \$ 20.The solution value is i = 14.87 %. The trial-and-error procedure is extremely tedious and inefficient for most problems, so it is not widely practiced in the real world.

Method 2: You can solve the problem by using the interest tables attached to these papers.

Start with the equation

$\$\ 20 = \$\ 10\ (1 + i)^5$

Which is equivalent to

$2 = \ 1\ (1 + i)^5 = (F/P,\ i, 5).$

Now look across the n = *5* row under the *(F/P,* i, *5)* column until you can locate the value of 2.This value is approximated in the 15% interest table at *(FIP,* 15%, 5) = 2.0114, so the interest rate at which \$10 grows to \$20 over five years is very close to 15%. This procedure will be very tedious for fractional interest rates or when *N* is not a whole number, as you may have to approximate the solution by linear interpolation.

Method 3: The most practical approach is to use either a financial calculator or an electronic spreadsheet such as Excel. A financial function such as *RATE (N,* 0, *P,* ***F)*** allows us to calculate an unknown interest rate. The precise command statement would be as follows:

= RATE (5.0, -10, 20) = 14.87%

Note that we enter the present value (P) as a negative number in order to indicate a cash outflow in Excel.

Example 15

A contractor wants to borrow (\$ 20,000) for buying a new equipment but he needs to select the bank which imposes a small rate interest, he likes to repay this loan after 10 years equal to \$ 35,000. Which is the best i interest for this decision?

Use the trial and error, assume the $i = 8\%$

$35{,}000 = 20{,}000(1 + 0.08)^{10} = \$\ 43{,}178.5 \neq \$\ 35{,}000$

Use a lower rate interest; let = 6%

$35{,}000 = 20{,}000\ (1 + 0.06)^{10} = \$\ 35{,}816.95$

$\therefore i = 5.755\% \ \cong 6\%$

Use the Excel sheet and use insert function RATE;

= RATE (10,0, -20, 35) = 6%

Example 16

Single Amounts: Find *n*, Given *P*, *F*, and *i*

You have just purchased 100 shares of General Electric stock at $30 per share. You will sell the stock when its market price doubles. If you expect the stock price to increase 12% per year, how long do you expect to wait before selling the stock (Figure 3)?

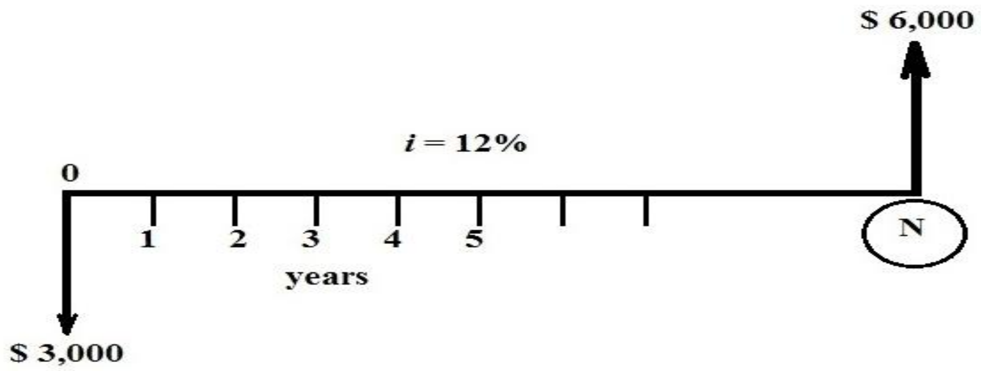


Figure (3) find (n)

Given: P = \$3,000, F = \$6,000, and i = 12% per year. **Find:** n (years).

Using the single-payment compound-amount factor, we write

$F = P\ (1+i)^n = P\ (F/P, i, n).$ Use the trial and error

Which in this case is

$\$\ 6{,}000 = \$\ 3{,}000\ (1+0.12)^n = \$\ 3{,}000\ (F/P, 12\%, n).$

or

$2 = (1.12)^n = (F/P, 12\ \%, n).$

Again, we could use a calculator or a computer spreadsheet program to find N.

1. **Using a calculator:** We start with

Log 2 = n log 1.12.

Solving for ***n*** gives

$$n = \frac{\log 2}{\log 1.12} = 6.11 \approx 6 \text{ years}$$

2. **Using a spreadsheet program:** Within Excel, the financial function NPER (i, 0, P, *F)* computes the number of compounding periods it will take an investment ***P*** to grow to a future value ***F,*** earning a fixed interest rate i per compounding period. In our example, the Excel command would look like this:

= NPER (12%, 0, -3,000, 6,000)

The calculated result is 6.1163.

The rule states that, to find the time it takes for the present sum of money to grow by a factor of two, we divide 72 by the interest rate. For our example, the interest rate is 12%. Therefore the Rule of 72 indicates that it will take $\frac{72}{12} = 6$ years for a sum to double. This result is, in fact, relatively close to our exact solution.

Example 16

Present Values of an Uneven Series by Decomposition into Single Payments

Wilson Technology, a growing machine shop, wishes to set aside money now to invest over the next four years in automating its customer service department.

The company can earn 10% on a lump sum deposited now, and it wishes to withdraw the money in the following increments:

Year 1: $25,000 to purchase a computer and database software designed for customer service use;

Year 2: $3,000 to purchase additional hardware to accommodate anticipated growth in use of the system;

Year 3: No expenses: and

Year 4: $5,000 to purchase software upgrades.

How much money must be deposited now in order to cover the anticipated payments over the next four years?

This problem is equivalent to asking what value of P would make you in different in your choice between P dollars today and the future expense stream of ($25,000, $3,000, $0, $5,000). One way to deal with an uneven series of cash flows is to calculate the equivalent present value of each single cash flow and then to sum the present values to find P. In other words, the cash flow is broken into three parts as shown in Figure below.

Given: Uneven cash flow; i = 10% per year. **Find:** P.

We sum the individual present values as follows:

P = \$25,000 (P/F. 10%. I) + \$3,000(P/F, 10%. 2) + \$5.000(P/F, 10%,4)

= \$28,622.

To see if \$28,622 is indeed a sufficient amount, let's calculate the balance at the end of each year. If you deposit \$28,622 now, it will grow to (1.10)(\$28,622), or \$31,484, at the end of year one. From this balance, you pay out \$25.000. The remaining balance, \$6,484, will again grow to (1.10) (\$6,484), or \$7.132, at the end of year two. Now you make the second payment (\$3,000) out of this balance, which will leave you with only \$4,132 at the end of year two.

Since no payment occurs in year three, the balance will grow to $\$(1.10)^2$ (\$4.132), or \$5,000, at the end of year four. The final withdrawal in the amount of \$5,000 will deplete the balance completely.

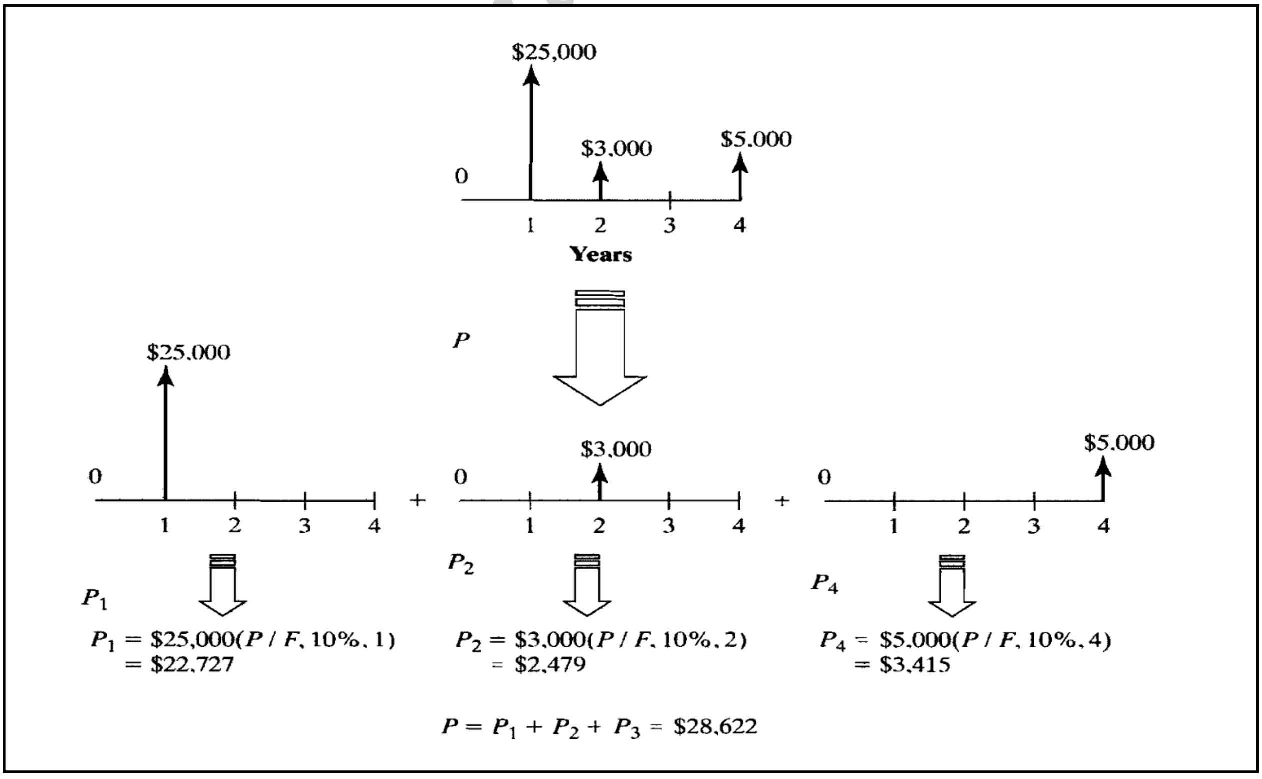


Figure (4) Decomposition of uneven cash flow series

Compound-Amount Factor: Find F, Given A, i, and n

Suppose we are interested in the future amount F of a fund to which we contribute A dollars each period and on which we earn interest at a rate of i per period. The contributions are made at the end of each of the N periods. These transactions are graphically illustrated in Figure 2.15. Looking at this diagram, we see that, if an amount A is invested at the end of each period for N periods. The total amount F that can be withdrawn at the end of *n* periods will be the sum of the compound amounts of the individual deposits.

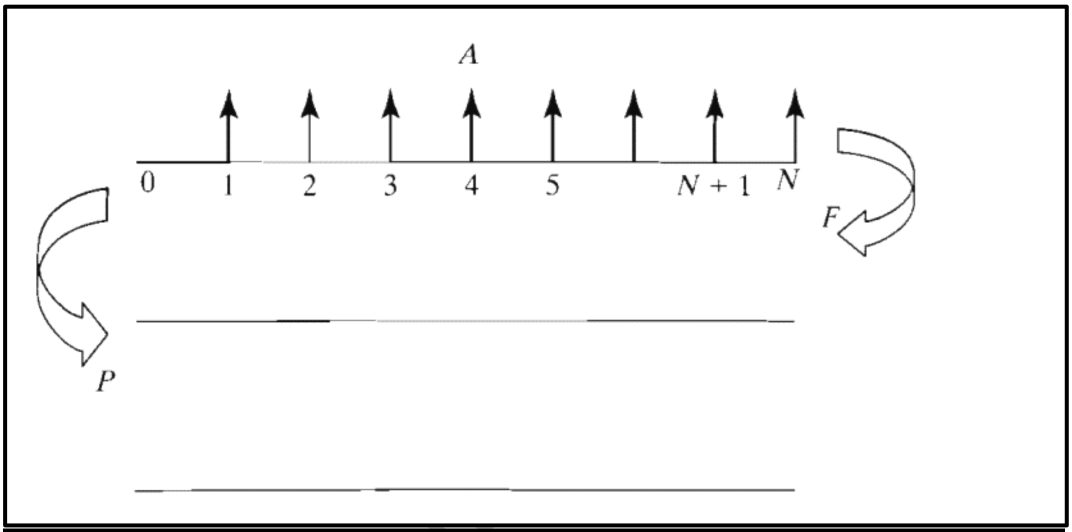


Figure (5) Equal series find equivalent P or F

As shown in Figure (5), the A dollars we put into the fund at the end of the first period will be worth $A(1+i)^{N-1}$ at the end of N periods. The A dollars we put into the fund at the end of the second period will be worth $A(1+i)^{N-2}$, and so forth. Finally, the last A dollars that we contribute at the end of the N^{th} period will be worth exactly A dollars at that time. This means we have a series in the form

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \ldots\ldots\ldots\ldots + A(1+i) + A$$

or, expressed alternatively,

$$F = A + A(1+i) + A(1+i)^2 + \ldots\ldots\ldots + A(1+i)^{N-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(7)}$$

Multiplying Eq. (7) by $(1+i)$ results in

$(1+i)$ F = A$(1+i)$ + A $(1+i)^2$+ ………. + A $(1+i)^N$ **………………. (8)**

Subtracting Eq. (7) from Eq. (8) to eliminate common terms gives us

$(1+i)$ F - F = - A + A $(1+i)^N$

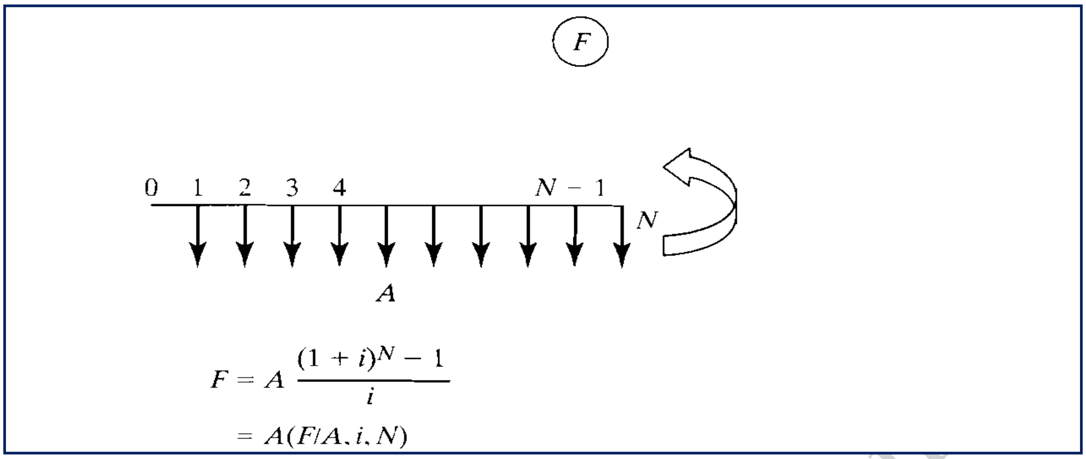


Figure (6) Cash flow diagram of the relationship between A and F

Solving for F yields

$$\mathbf{F = A\left[\frac{(1+i)^n - 1}{i}\right] = A\ (F / A,\ i,\ n)} \text{ ……………………………………(9)}$$

The bracketed term in Eq. (4) is called the **equal-payment-series compound amount factor,** or the **uniform-series compound-amount factor;** its factor notation is ***(F/A.*** i, n). This interest factor has been calculated for various combinations of i and N in the tables attached with this lecture.

Equal -Payment Series: Find F, Given i, A, and n

<u>Example 17</u>

Suppose you make an annual contribution of $5,000 to your savings account at the end of each year for five years. If your savings account earns 6% interest annually, how much can be withdrawn at the end of five years?

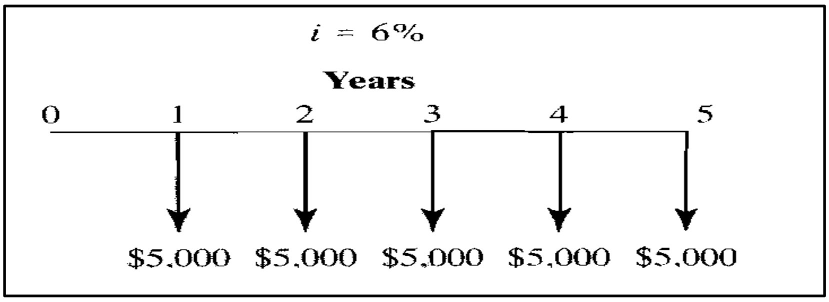


Figure (7) Equal – Payment series find F

Given: A = \$5.000, n = 5 years, and ***i*** = 6% per year.

Find: F.

By using the Equation (3) to obtain the future value for the series payments.

$$\mathbf{F = A\left[\frac{(1+i)^n - 1}{i}\right] = A\,(F / A, \mathit{i}, n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(10)}$$

$$F = \$\,5{,}000\left[\frac{(1+0.06)^5 - 1}{0.06}\right] = \$\,5{,}000\ x\ 5.63709 = \$\,28{,}185.46$$

Or we can find by using the equal-payment-series compound-amount factor, we obtain

F = \$5,000(F/A, 6%. 5)

= \$5,000(5.6371)

= \$28,185.46.

To obtain the future value of the annuity on Excel, we may use the following financial command:

= FV (6%, 5, 5000, 0).

Handling Time Shifts in a Uniform Series

Previously: In Example, the first deposit of the five-deposit series was made at the end of period one, and the remaining four deposits were made at the end of each following period. Suppose that all deposits were made at the ***beginning*** of each period instead. How would you compute the balance at the end of period five?

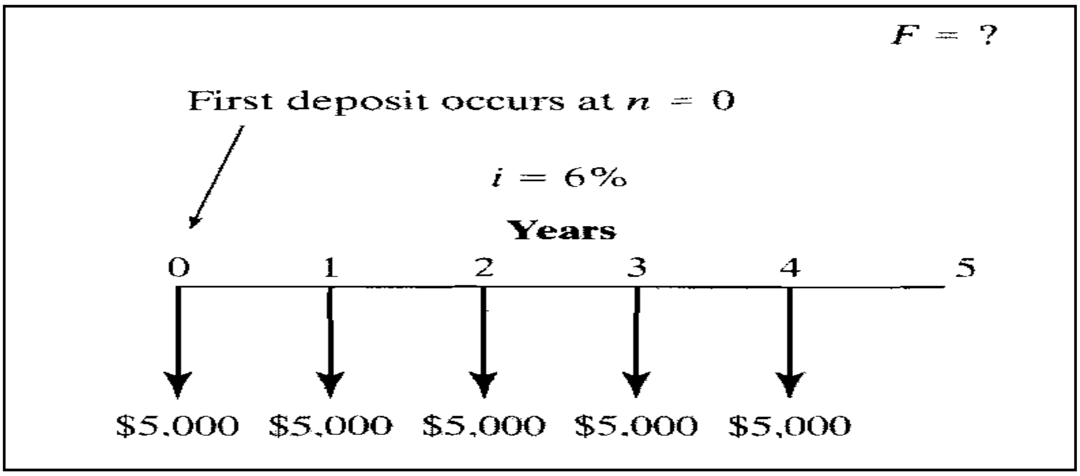


Figure (8) Handling Time Shifts in a Uniform Series

Given: in Figure 8; $\boldsymbol{i}$ = 6% per year. **Find:** $\boldsymbol{F_5}$.

Compare Figure above with the previous Figure: Each payment in Figure above has been shifted one year earlier; thus, each payment is compounded for one extra year.

Note that with the end-of-year deposit, the ending balance F was $28,185.46.

With the beginning-of-year deposit, the same balance accumulates by the end of period four. This balance can earn interest for one additional year. Therefore, we can easily calculate the resulting balance as

F_5 = $ 28,185.46 (1.06) = $ 29,876.59

Annuity due can be easily evaluated using the following financial command available on Excel:

= FV (6%, 5, 5000, 1).

Another way to determine the ending balance is to compare the two cash flow patterns. By adding the $5,000 deposit at period zero to the original cash flow and subtracting the $5,000 deposit at the end of period five, we obtain the second cash flow. Therefore, the ending balance can be found by making an adjustment to the $28,185.46:

F_5 = $ 28,185.46 + $ 5,000 (F / P, 6%, 5) - $ 5,000 = $ 29,876.59.

Sinking-Fund Factor: Find A, Given F, *i*, and n

If we solve Eq. (3)

$$\mathbf{F} = \mathbf{A}\left[\frac{(1+i)^n - 1}{i}\right] = \mathbf{A}\ (\mathbf{F} / \mathbf{A}, i, \mathbf{n})$$ for A, we obtain

$$\mathbf{A} = \mathbf{F}\left[\frac{i}{(1+i)^n - 1}\right] = \mathbf{F}\ (\mathbf{A} / \mathbf{F}, i, \mathbf{n}) \quad \text{..} \quad (11)$$

The term within the brackets is called the **equal-payment-series sinking-fund factor,** or just **sinking-fund factor,** and is referred to with the notation ***(A/F,*** i, n). A sinking fund is an interest-bearing account into which a fixed sum is deposited each interest period; it is commonly established for the purpose of replacing fixed assets.

Example 18

Find A, Given *F*, n, and i. You were asked by your manager want to set up a savings plan for your organization. Your organization needs a comprehensive maintenance and will be executed after 8 years from now. The maintenance plan will need at least $100,000 in the bank. How much do you need to save each year in order to have the necessary funds if the current rate of interest is 7%? Assume that end-of-year payments are made. (see figure below).

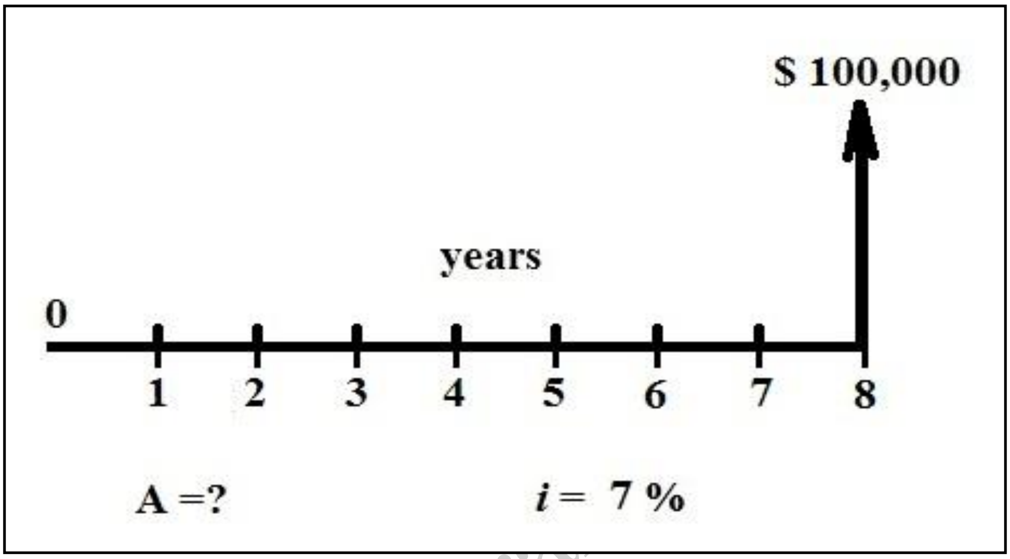


Figure (9) Sinking-Fund Factor, find A

Using Eq. (4)

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = \mathbf{F\ (A\ /\ F,}\ \boldsymbol{i}\mathbf{,\ n)}$$

$$A = \$\ 100{,}000\left[\frac{0.07}{(1+0.07)^8 - 1}\right] = \$\ 9{,}746.77$$

Or

Using the sinking-fund factors, we obtain

A = *$100,00O (A/F,* 7%, 8)

= $9,746.78.

Example 19

You wants to purchase a new shovel, there is two alternatives the first one is to pay the present cost equal to $ 45,000 and the second one is to pay the annual payments to collect the future amount is \$ 50,000 after 5 years with rate interest 6%. Which alternative will be selected? Discuss them.

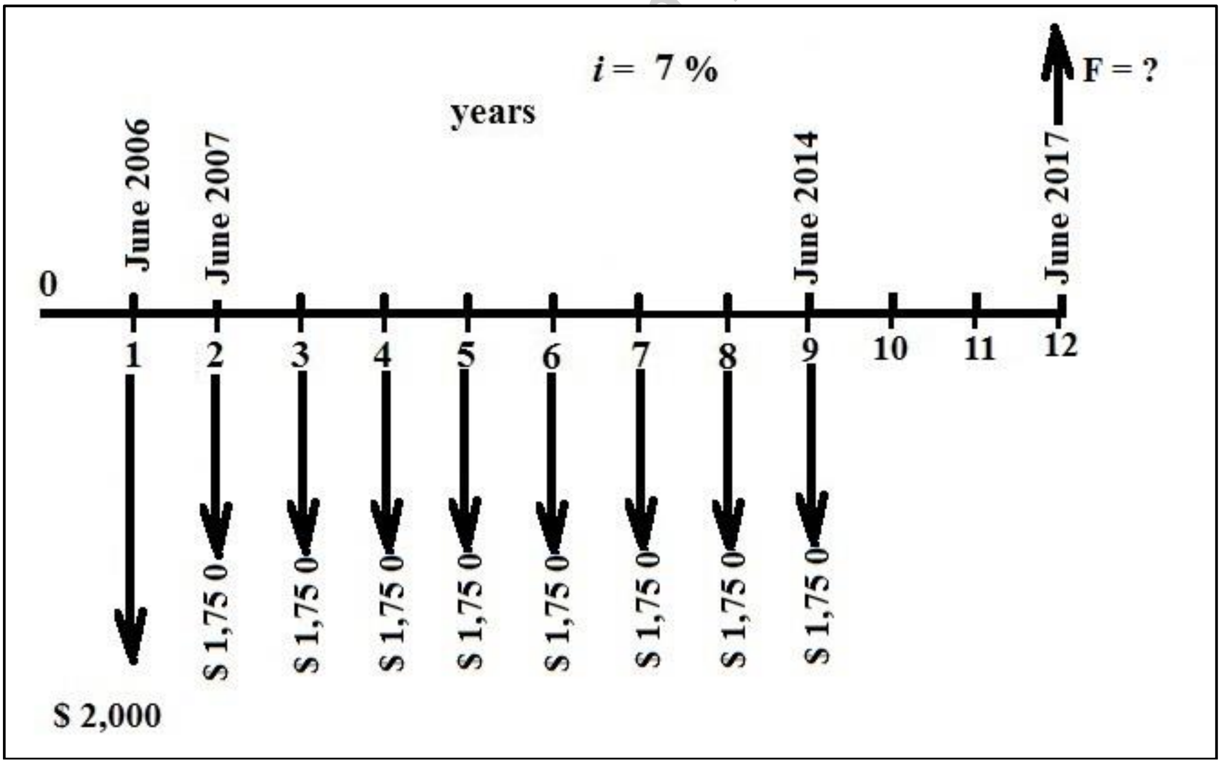
$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = \mathbf{F\ (A / F,}\ \boldsymbol{i}\mathbf{,\ n)}$$

$$A = \$\ 50{,}000\left[\frac{0.06}{(1+0.06)^5 - 1}\right] = \$\ 50{,}000 \text{ x } 0.17739 = \$\ 8869.5$$

The second alternative will be selected for saving the initial price although it pays \$ 8869.5 each year in second alternative.

<u>Example 20</u>

A contractor deposited in the bank at June of 2006 amount of money equal \$ 2,000, and in the next year June of 2007 he began to deposit every year amount of money for seven year (till to 2014) a uniform series payment equal to \$ 1,750, then he planned to withdraw the total money at the June 2017. How much money he can receive? The rate interest of the bank = 7%.



Solution

$F_1 = P\ (1+i)^n = 2{,}000\ x(1+0.07)^8 = 2{,}000x\ 1.7181 = \$\ 3{,}436.2$

$F_2\ = A\ \left(\frac{(1+i)^n - 1}{i}\right) = 1{,}750\ x(\frac{(1+0.07)^8 - 1}{0.07}) = 1{,}750\ x\ 102598 = \$\ 17{,}954.65$

$F_1 + F_2 = 3{,}436.2 + 17{,}954.65 = \$\ 21{,}390.85$

$F_{Total} = P\,(1+i)^n = 21{,}390.85\ x\ (1+0.07)^3 = 21{,}390.85\ x\ 1.2250$
$= \$\ 26{,}204.71$

Capital-Recovery Factor (Annuity Factor): Find A, Given P, *i*, and n

We can determine the amount of a periodic payment, A, if we know P, i, and n. Figure below illustrates this situation. To relate P to A, recall the relationship between P and Fin Eq. (1):

F = P (l + *i*) ^{n}**.** By replacing (F) by P (1 + *i*)^{n}**,** we get

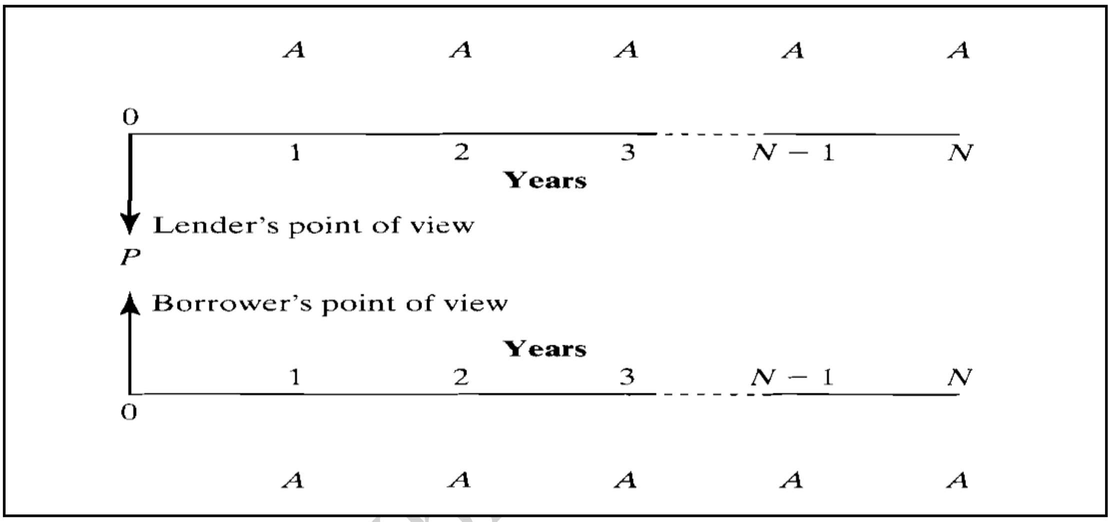


Figure (10) Capital-Recovery Factor (Annuity Factor)

$$\boldsymbol{F = P\,(1+i)^n}$$

$$\boldsymbol{F = A\ x\ \left[\frac{i}{(1+i)^n - 1}\right]}$$

$$\boldsymbol{P\,(1+i)^n = A\ x\ \left(\frac{i}{(1+i)^n - 1}\right)}$$

$$\mathbf{A = P\ (1+i)^n}\ \left[\frac{\boldsymbol{i}}{\boldsymbol{(1+i)^n - 1}}\right]$$

$$\mathbf{A = P}\left[\frac{\boldsymbol{(1+i)^n\, i}}{\boldsymbol{(1+i)^n - 1}}\right] \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(12)}$$

$$P = A\,x \left[\frac{(1+i)^n - 1}{i\,(1+i)^n}\right]$$

Now we have an equation for determining the value of the series of end-of-period payments, A, when the present sum *P* is known. The portion within the brackets is called the **equal-payment-series capital-recovery factor,** or simply **capital-recovery factor,** which **is** designated *(A / P, i*, n). In finance, this *A / P* factor is referred to as the **annuity factor.** The annuity factor indicates a series of payments of a fixed, or constant, amount for a specified number of periods.

Paying Off an Educational Loan: Find A, Given *P / i*, and *n*

<u>Example 21</u>

You borrowed $21,061.82 to finance the educational expenses for your senior year of college. The loan will be paid off over five years. The loan carries an interest rate of 6% per year and is to be repaid in equal annual installments over the next five years. Assume that the money was borrowed at the beginning of your senior year and that the first installment will be due a year later. Compute the amount of the annual installments (Figure below).

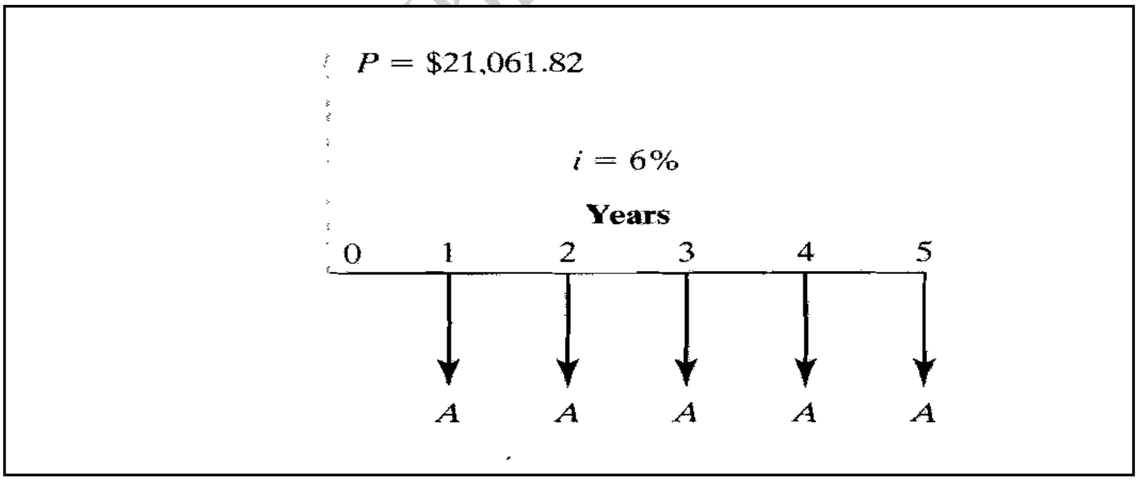


Figure (11) Paying Off an Educational Loan

Given: ***P*** = $21,061.82, i = 6% per year, and n = *5* years. **Find:** A.

Using the Eq. (6) $\mathbf{A} = \mathbf{P}\left[\frac{(1+i)^n i}{(1+i)^n - 1}\right]$

A = \$ 21,061.82 $\left[\frac{(1+0.06)^5\ 0.06}{(1+0.06)^5\ -1}\right]$ = \$ 5,000.0

Or using the capital-recovery factor from table, we obtain

A = \$21,061.82(A / P, 6%, *5)*

= \$21,061.82(0.2374)

= \$5,000.

The Excel solution using annuity function commands is as follows:

= PMT (i, N, P)

= PMT (6%, 5, 21,061.82)

The result of this formula is \$5,000.

Example 22

The primary calculations – performed by contractor- of the expenses for leasing a construction plant result equal to \$ 850 per year this expenses involved the rent and maintenance for 5 years (duration of project life) with rate of interest is 4%. Now the contractor has amount of money \$ 3,000. How much the total of the present amount has to have which be equivalent to annual payments?

Solution

$$P = A\ x\,\frac{(1+i)^n-1}{ix\ (1+i)^n} = 850\ x\,\frac{(1+0.04)^5-1}{0.04\ x\ (1+0.04)^5} = 850\ x4.4518 = \$\ 3{,}784.03$$

The additional amount should be existing with contractor adding to present value:

3,784.03 -3,000 =\$ 784.03

Example 23

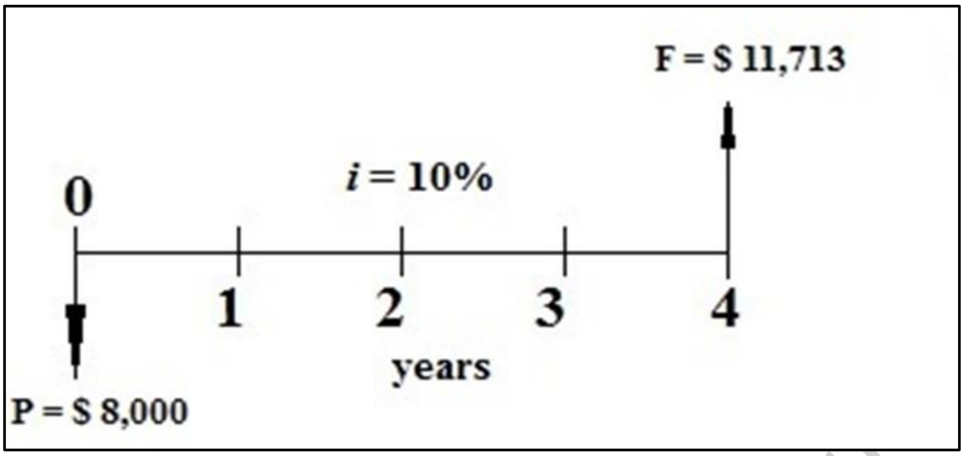
Below the different payments were deposited in the bank with their dates, what is the total future amount for them at the end of 9th year the compound rate interest is 5%. Draw also the cash flow diagram:

Solution

{500 + 750 (P/A, 5%, 5) + 1000 (P/ F, 5%, 6)} [F/P, 5%, 9]

$= \{500 + [750 \frac{(1+0.05)^5 - 1}{0.05\ (1+0.05)^5}] + \left[1000\ x\ (\frac{1}{(1+0.05)^6})\right]\}\ x\ (1+0.05)^9$

$= (500 + 750 \text{ x } 4.3294 + 1000 \text{x } 0.74621)\ x\ 1.5513 = \$\ 6{,}970.30$



Nominal *i* and Effective *i*

In this section, we will consider several concepts crucial to managing money. In previous times, we examined how time affects the value of money, and we developed various interest formulas for that purpose. Using these basic formulas, we will now extend the concept of equivalence to determine interest rates implicit in many financial contracts. To this end, we will introduce several examples in the area of loan transactions. For example, many commercial loans require that interest compound more frequently than once a year-for instance, monthly or daily. To consider the effect of more frequent compounding, we must begin with an understanding of the concepts of nominal and effective interest.

As briefly mentioned, the market interest rate is defined as the interest rate quoted by the financial market, such as by banks and financial institutions. This interest rate is supposed to consider any anticipated changes in earning power as well as purchasing power in the economy. In this section. We will review the nature of this interest rate in more detail.

Nominal Interest Rates

Many banks. For example, state the interest arrangement for credit cards in the following manner:

"%18"compounded monthly".

This statement means simply that each month the bank will charge (1.5% interest 12months per year = 18% per year) on the unpaid balance.

We say that 18% is the nominal interest rate or annual percentage rate (APR) and that the compounding frequency is monthly (12 times per year) . Although the APR is commonly used by financial institutions and is familiar to many customers, it does not explain precisely the amount of interest that will accumulate in a year. To explain the true effect of more frequent compounding on annual interest amounts, we will introduce the term effective interest rate, commonly known *as annual effective yield*, or annual percentage yield (APY).

Annual Effective Yields

The annual effective yield (or effective annual interest rate) is the one rate that truly represents the interest earned in a year. On a yearly basis, you are looking for a cumulative rate-1.5% each month for 12 times. This cumulative rate predicts the actual interest payment on your outstanding credit card balance.

We could calculate the total annual interest payment for a credit card debt of 1,000 by using the formula given in the following equation.

If **P** = $1,000, ***i*** = 18 %, and ***n*** = 12, the rate interest compounded monthly, we obtain

$i_{effective} = \frac{0.18}{12} = 0.015$

$F = P \, x \, (1 + i)^n$

$= \$ \, 1{,}000 \, x \, (1 + 0.015)^{12}$

= $ 1,195.62

Clearly, the bank is earning more than 18% on your original credit card debt. In fact, you are paying $195.62.The implication is that, for each dollar owed, you are paying an equivalent annual interest of 19.56 cents. In terms of an effective annual interest rate (i,), the interest payment can be rewritten as a percentage of the principal amount:

Time Period: The basic time unit of the interest rate. This is the ***n*** statement of ***i%*** per time period. The time unit of 1 year is by far the most common. It is assumed when not stated otherwise.

Compounding Period (CP): The time unit used to determine the effect of interest. It is defined by the compounding term in the interest rate statement. If it is not stated, it is assumed to be 1 year.

$$\boldsymbol{Effective\ rate\ per\ (CP) = \frac{i\,\%\ per\ time\ period\ n}{m\ compounding\ periods\ per\ n} = \frac{i}{m}} \text{ .... } \mathbf{(13)}$$

Example 24

The different bank loan rates for three separate equipment projects are listed below. Determine the effective rate on the basis of the compounding period for each quote.

a) 9% per year compounded quarterly
b) 9% per year, compounded monthly
c) 9 % per year, compounded weekly

Solution:

Apply the equation $\boldsymbol{Effective\ rate\ per\ (CP)} = \frac{\boldsymbol{i\ \%\ per\ time\ period\ n}}{\boldsymbol{m\ compounding\ periods\ per\ n}} = \frac{\boldsymbol{i}}{\boldsymbol{m}}$ to determine the effective rate per CP for different compounding frequencies. The accompanying graphic indicates how the interest rate is distributed over.

Nominal i % per n	Compounding Period	m	*Effective Rate Per CP*	Distribution Over Time Period n
a) 9% per year	Quarrter	4	2.25 %	2.25 %	2.25 %	2.25 %	2.25 % 1 2 3 4
b) 9% per year	Month	12	0.75 %	1 2 3 4 5 6 7 8 9 10 11 12 — 0.75 %
c) 4.5 % per 6- months	Week	26	0.173 %	0.173 % 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Example 25

If amount of money equal to $ 1,000 is deposited in a bank with rate interest 18% yearly, find the effective rate interest if compounded

a- Semiannually b- monthly

Solution

a- There are 2 compounding periods per year.

Thus, $m = 2$ and $i_{effective} = \frac{i}{m} = \frac{0.18}{2} = 0.09$

$i_{effective\ rate\ interest\ per\ year} = (1 + 0.09)^2 - 1 = .1881 = 18.81\%$

$F = 1{,}000x\ (1 + .1881)^1$= 1,000 x 1.1881= $ 1,188.1

b- There are 12 compounding periods per year.

Thus, $m = 12$ and $i_{effective} = \frac{i}{m} = \frac{0.18}{12} = 0.015$

$i_{effective\ rate\ interest\ per\ year} = (1 + 0.015)^{12} - 1 = .19562$

$= 19.56\%$

$F = 1{,}000x\ (1 + .19562)^1$= 1,000 x 1.19562= $ 1,195.61

Example 26

What is the largest Rate of interest: 3 % compounded monthly or the Rate interest of 3.5% compounded semiannually?

Solution

1- i= 3% compounded monthly

$i = \frac{3}{12} = 0.25\%\ \ per\ period$

F= $P(1 + i)^n$

Assume that P= 1

$i_{effective\ rate\ interest\ per\ year} = 1\ (1 + 0.0025)^{12} - 1 = 0.0304 = 3.04\%$

2- i =$3\frac{1}{2}$ % compounded semiannually

$i = \frac{3.5}{2} = 1.75\%\ \ per\ period$

F= $P(1 + i)^n$

Assume that P = 1

$i_{\,effective\ rate\ interest\ per\ year} = 1\,(1 + 0.0175)^2 - 1 = 0.0353 = 3.53\%$

$\therefore i = 3.5\ \%\ compounded\ semiannually\ > i = 3\%\ compounded\ monthly$

Example 27

A man deposited a $ 2,000 in the bank with rate interest 8% compounded daily calculate the cumulative amount after one year?

Solution

$i = 8\%\ compounded\ daily$

$i = \frac{8}{365} = 0.02191\ \%\ \ per\ period$

F= $P(1 + i)^n$

P= $ 2,000

$i_{\,effective\ rate\ interest\ per\ year} = (1 + 0.0002191)^{365} - 1 = 0.08327$

$= 8.327\%$

F= $P(1 + i)^n = 2{,}000\ x\ (1 + 0.08327)^1 = \$\ 2{,}166.54$

Example 28

A fund is setup with an initial investment of $ 1,000 to provide uniform year-end payment each year for six year. How much will the payment amount to; if the investment is made at an interest rate of 3% and fund is to be completely exhausted by the sixth payment?

Solution

$A = P\ \left[\frac{i\ x\ (1+i)^n}{(1+i)^n - 1}\right] = 1{,}000\ x\ \left[\frac{0.03\ x\ (1+0.03)^6}{(1+0.03)^6 - 1}\right]$= $ 184.60

Example 29

A uniform annual investment is to be made into a sinking fund with a view to providing the capital at the end of 7 years for the replacement of a tractor. An interest rate of $2\frac{1}{2}\%$ to available what is the annual investment needed to provide $ 15,000?

Solution

$A = F\ \left[\frac{i}{(1+i)^n - 1}\right] = 15{,}000\ x\ \left[\frac{0.025}{(1+0.025)^7 - 1}\right]$= $ 1,987.5

Example 30

If the initial purchase cost of a machine is $ 10,000 and the annual maintenance cost is $ 200 for 8 years (its useful life). If the annual interest rate is i = 6 % and the machine has no salvage value, what is the cost of the equivalent annual mechanism. Ignore the values of fees and labor costs?

Solution

$A = P\left[\frac{(1+i)^n i}{(1+i)^n -1}\right] = 10{,}000\left[\frac{(1+0.06)^8 \ x\ 0.06}{(1+0.06)^8 -1}\right] = \$\ 1{,}610.30$

The total equivalent annual cost = $ 1,610.30 + 200 = $ 1810.30 / year

Example 31

A man deposits amount of $ 500 every 6 months for a period of 7 years. What is the amount will be accumulated after the last payment if the interest rate is 8% which compounded quarterly?

Solution

$i = 8\ \%\ compounded\ quarterly$

$i = \frac{8}{4} = 2\ \% = 0.02\ \ per\ period$

Because the deposit every 6 months or two times in one year

$n\ period = \frac{No.of\ interest\ during\ the\ year}{No.\ of\ Payments\ during\ the\ year} = \frac{4}{2} = 2$

$F = P\ (1+i)^{n = \frac{No.of\ interest\ during\ the\ year}{No.\ of\ Payments\ during\ the\ year} = \frac{4}{2} = 2}$

$F = 1(1+0.02)^2 = 1.0404$

$i_{\ effective\ rate\ interest\ per\ year} = (1+0.02)^2 - 1 = 0.0404 = 4.04\%$

There is two payments each year (there is two deposits every year), so; No. of periods = No. of years x No. of payments (deposits) every year.

$n = 7\ x\ 2 = 14$

$F = A\left[\frac{(1+i)^n - 1}{i}\right] = 500\left[\frac{(1+0.0404)^{14} - 1}{0.0404}\right] = \$\ 9{,}172.12$

Example 32

A contractor wants to purchase a new construction equipment by $ 100,000; there is an agreement with the supplier side for covering this cost by paying a uniform series each 3 months for (10) years. Find the uniform payments if the rate interest 5% compounded monthly?

Solution

$i = 5\ \%\ compounded\ monthly$

$i = \frac{5}{12} = 0.4166\ \% = 0.004166\ \ per\ period$

Because the payments every 3 months

$n\ period = \frac{No.of\ interest\ during\ the\ year}{No.\ of\ Payments\ during\ the\ year} = \frac{12}{4} = 3$

$F = P\ (1+i)^{n = \frac{No.of\ interest\ during\ the\ year}{No.\ of\ Payments\ during\ the\ year} = \frac{12}{4} = 3}$

$F = 1(1 + 0.004166)^3 = 1.01255$

$i_{\ effective\ rate\ interest\ per\ year} = (1 + 0.004166)^3 - 1 = 0.01255$

$= 1.255\ \%$

There is four (4) payments each year, so;

No. of periods = No. of years x No. of payments every year.

$n = 10\ x\ 4 = 40$

$A = P\ \left[\frac{i\ x\ (1+i)^n}{(1+i)^n - 1}\right] = 100{,}000\ x \left[\frac{0.01255\ x\ (1+0.01255)^{40}}{(1+0.01255)^{40} - 1}\right] = \$\ 3{,}192.4$

<u>Example 33</u>

What is the effective interest rate of a nominal interest rate of 6 % compounded quarterly for one year?

Solution

$i = 6\ \%\ compounded\ quarterly$

$i = \frac{6}{4} = 1.5\ \% = 0.015\ \ per\ period$

F= $P(1+i)^n$

F= $1(1 + 0.015)^4 = 1.06136$

100x.015=101.5

101.5x.015=1.5225+101.5= 103.0225

103.0225x0.015=1.5453375+103.0225= 104.5678375

104.5678375x0.15=1.568517555 +104.56778375

= 106.13630

$i_{\ effective\ rate\ interest\ per\ year}$ =106.1363-100
= 6.136%

$i_{\ effective\ rate\ interest\ per\ year} = (1 + 0.015)^4 - 1 = 0.06136 = 6.136\ \%$

Example 34

A man has deposited \$ 1,000 at now and after 4 years he also deposited \$ 3000 and after 6 years he deposited again \$ 1500, with a rate of interest of 6%, compounded semiannually. How much will the amount become after 10 years?

Solution

$i = 6\ \%\ compounded\ quarterly$

$i = \frac{6}{2} = 3\ \% = 0.03\quad per\ period$

F= $P(1+i)^n$

F= $1(1+0.03)^2 = 1.0609$

$i_{\,effective\ rate\ interest\ per\ year} = (1+0.03)^2 - 1 = 0.0609 = 6.09\ \%$

$F_1 = 1{,}000(1+0.0609)^{10} = \$\ 1{,}806.11$

$F_2 = 3{,}000(1+0.0609)^{6} = \$\ 4{,}277.28$

$F_3 = 1{,}500(1+0.0609)^{4} = \$\ 1{,}900.15$

$The\ Total\ F = 1{,}806.11 + 4{,}277.28 + 1{,}900.15 = \$\ 7{,}983.54$

Example 35

A savings and loan offers a 5.25% rate per annum compound daily over 365 days per year. What is the effective annual rate?

Solution

$i_e = \left(1 + \frac{0.0525}{365}\right)^{365} - 1 = 0.0\ 539 = 5.39\%$

Deferred Annuities (Uniform Series) or Deferred Loan Repayment

All annuities (uniform series) discussed to this point involved the first cash flow being made at the end of the first period, and they are called ***ordinary annuities***. If the cash flow does not being until some later date, the annuity is known as a ***deferred annuity.*** If the annuity is deferred ***J*** periods, the situation is as portrayed in (figure below). It should be noted in this figure that the entire framed ordinary annuity has been moved forward from "time present" or "time 0" by ***J*** periods. It must be remembered that in an annuity deferred for J periods the first payment is made at the end of the (J+1) period, assuming that all periods involved are equal in length.

The present worth at the end of period J of an annuity with cash flow of amount A is from Eq. $P = A(P/A\ , i\,\%, N)$. The present worth of the single amount A (P/A, *i*%, N) as of time 0 will then be $A(P/A\ , i\,\%, N)\ (P/F\ , i\,\%, J)$.

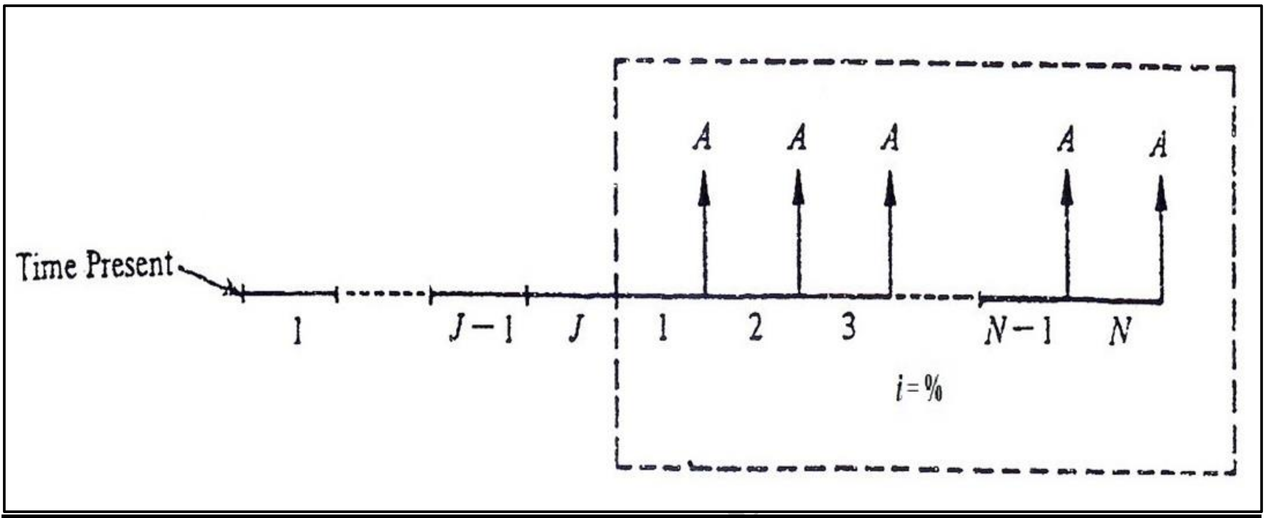


Figure (16) General Cash flow representation of a Deferred Annuity (Uniform Series)

Example 36

Suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest at 12% compounded annually to provide payments of $ 2,000 on each of the son's 18th, 19th, 20th and 21st birthdays.

Solution

The problem is represented in (figure below). One should first recognize that an ordinary annuity of four payments of $ 2,000 each is involved, and that the present worth of this annuity occurs at the 17th birthday when a (*P/A, i%, N*) is utilized. It often is helpful to use a subscript with P or F to denote the point in time. Hence

P_{17} = *A (P/A, 12%,4)* = $ 2,000 (3.0373)= $ 6,074.60

Note the dashed arrow in figure denoting P_{17}. Now that P_{17} is known, the next step is to calculate P_0. With respect to P_0, P_{17} is a future worth, and hence it could also be denoted F_{17}. Money at a given point in time, such as end of period

17, is the same regardless of whether it is called a present worth or future worth. Hence

$P_0 = F_{17}$ (P/F, 12%, 17) = \$ 6,074.60 (0.1456) = \$ 834.46

Which is the amount that the father would have to deposit.

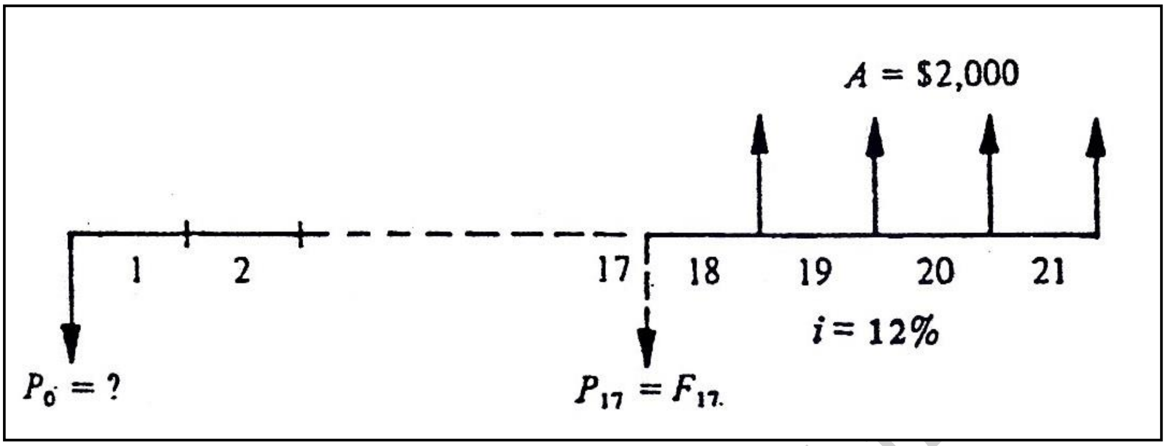


Figure (17) Cash flow diagram of Deferred Annuity

Example 37

As an addition to the previous problem, suppose that it is desired to determine the equivalent worth of the four \$ 2,000 payments as of the son's 24th birthday. Physically, this could mean that the four payments never were withdrawn or that possibly the son took them and immediately re-deposited them in an account also earning interest at 12% compounded annuity. Using our subscript system, we desire to calculate F_{24} as shown in figure (18).

Solution

One way to work this is to calculate

F_{21} = A (F / A, 12%, 4) = \$ 2,000 (4.7793) = \$ 9,558.60

To determine F_{24}, F_{21}becomes P_{21}, and

$F_{24} = P_{21}$ (F / P, 12%, 3) = \$ 9,558.60 (1.4049) = \$ 13,428.88

Another quicker way to work the problem is to recognize that the

P_{17} = \$ 6,074.60 and P_0 = \$ 884.46 are each equivalent to the four \$ 2,000 payments. Hence one can find F_{24} directly given P_{17} or P_0, we obtain

F_{24} = P_0 (F/P, 12%, 24) =\$ 884.48 (15.1786) = \$ 13,424.68 which checks closely with the previous answer. The two numbers differ by \$ 4.02, which can be attributed to round –off error in the interest factors.

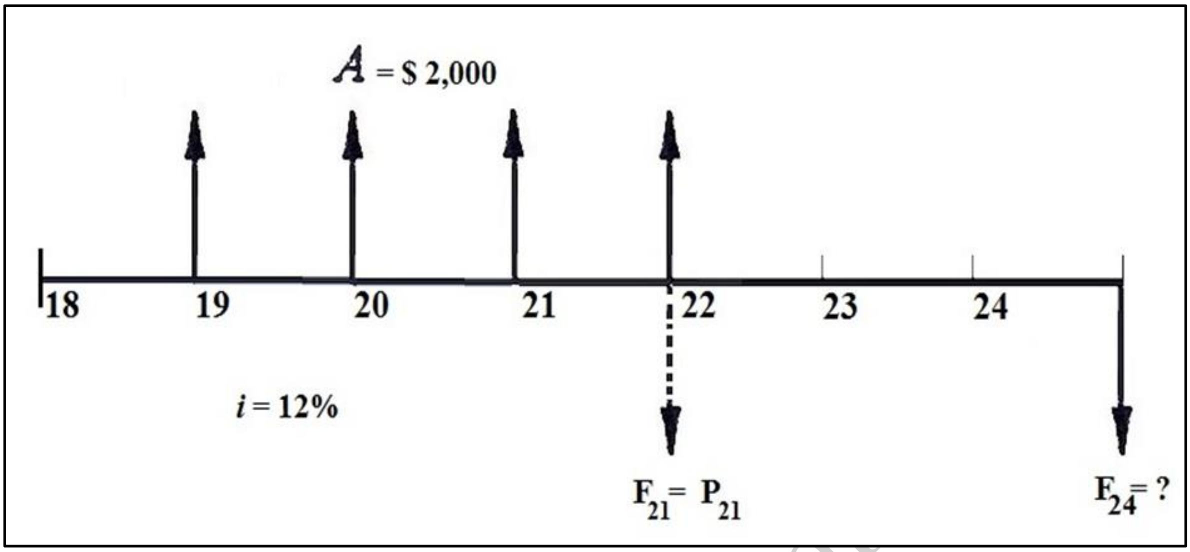


Figure (18) Cash flow diagram for the Deferred Annuity

Example 38

You borrowed \$21,061.82 to finance the educational expenses for your senior year of college. The loan will be paid off over five years. The loan carries an interest rate of 6% per year and is to be repaid in equal annual installments over the next five years. Assume that the money was borrowed at the beginning of your senior year and that the first installment will be due a year later. Compute the amount of the annual installments. Now; Suppose that you had wanted to negotiate with the bank to defer the first loan installment until the end of year two (but still desire to make five equal installments at 6% interest). If the bank wishes to earn the same profit as in the previous example, what should be the annual installment?

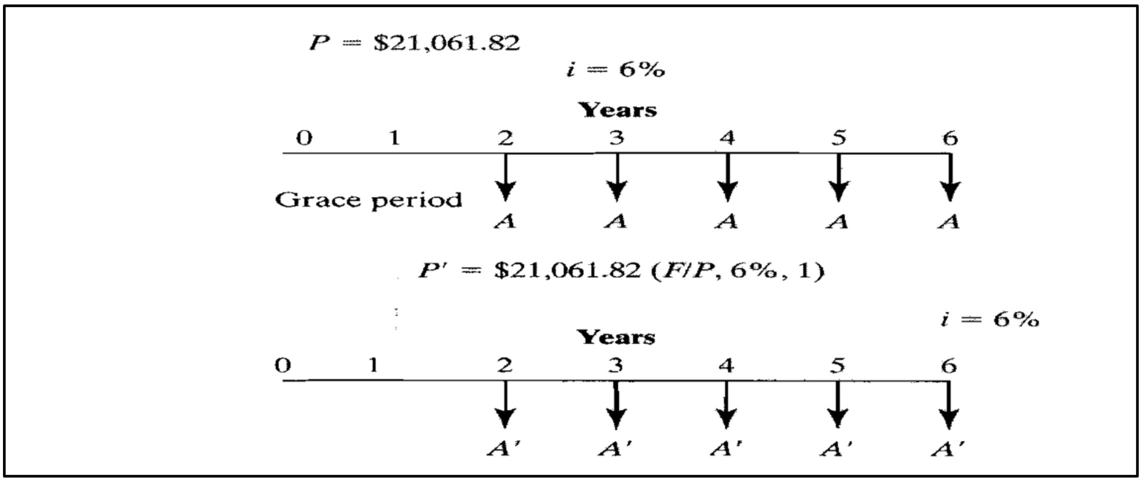


Figure (19) Find A annual payment for life (5) years

Given: P = $21,061.82, i = 6% per year, and *N* = 5 years, but the first payment occurs at the end of year two.

Find: A.

Solution

In deferring one year, the bank will add the interest accrued during the first year to the principal. In other words, we need to find the equivalent worth of $ 21,061.82 at the end of year 1, P':

Thus, you are borrowing $22,325.53 for five years. To retire the loan with five equal installments, the deferred equal annual payment, **A',** will be

A′ = $ 22,325.53 (A / P, 6%, 5)

$$A' = \$\,22{,}325.53\ x\left(\frac{0.06\ x(1.06)^5}{(1.06)^5-1}\right) = \$\,5{,}300.00$$

By deferring the first payment for one year, you need to make an additional **$5300** in payments each year.

Net Cash Flow

In many economic calculations, we need to evaluate expenditures or expenses values against with revenues, returns or interest. This is done by unifying these financial values according to the years of their expenditures or revenues, and then the compulsory collection is made; one of them is positive and the other negative.

Example 39

Before evaluating the economic merits of a proposed investment, the XYZ Corporation insists that its engineers develop a cash flow diagram of the proposal. An investment of $ 10,000 can be made that will produce uniform annual revenue of $ 5,310 for 5 years and then have a positive salvage value of $ 2,000 at the end of year 5. Annual disbursements will be $ 3,000 at the end of each year for operating and maintaining the project. Draw a cash flow diagram and determine the cumulative cash flow over the 5 – year life of the project.

Solution

The initial investment of $ 10,000 and annual disbursement of $ 3,000 are cash outflows, while annual revenues and the salvage value are cash inflows (figure

Notice that the beginning of a given year is the end of the preceding year, for example, the beginning of year 1is the end of year 0. Cumulative cash flow is shown in this tabulation.

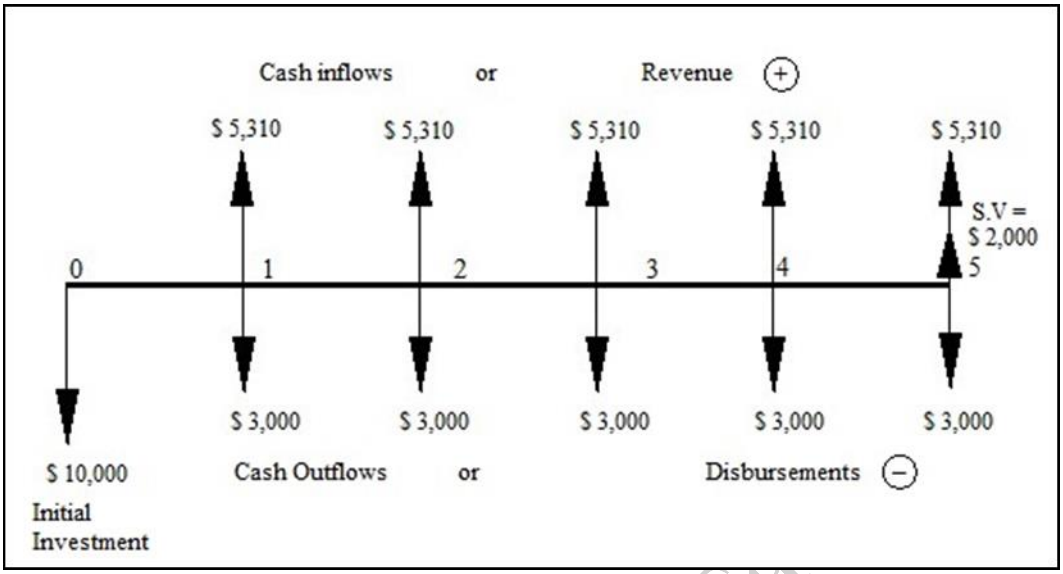


Figure (20) Cash Flow for payments and revenues

End of Year	Net Cash Flow	Cumulative Cash Flow
0	- $ 10,000	- $ 10,000
1	+ 2,310	- 7,960
2	+ 2,310	- 5,300
3	+ 2,310	- 3,070
4	+ 2,310	- 760
5	+ 4,310	+ 3,550

Table (4) Cumulative Cash flow

When monetary returns each year to owner of the $ 10,000 capital are considered, the economic attractiveness of the project can be determined. This general subject involves different methods of compounding interest (or forgone profits) and also different conventions and assumptions concerning the timing of cash flows.

Example 40

A mechanical device will cost $20,000 when purchased. Maintenance will cost $1000 per year. The device will generate revenues of $5000 per year for 5 years. The salvage value is $7000.It is difficult to solve a problem if you cannot see it. The easiest way to approach problems in economic analysis is to draw a picture. The picture should show three things:

1. A time interval divided into an appropriate number of equal periods
2. All cash outflows (deposits, expenditures, etc.) in each period
3. All cash inflows (withdrawals, income, etc.) for each period

Unless otherwise indicated, all such cash flows are considered to occur at the end of their respective periods.

Figure (15) is a cash-flow diagram showing an outflow or disbursement of $1000 at the beginning of year 1 and an inflow or return of $2000 at the end of year 5.

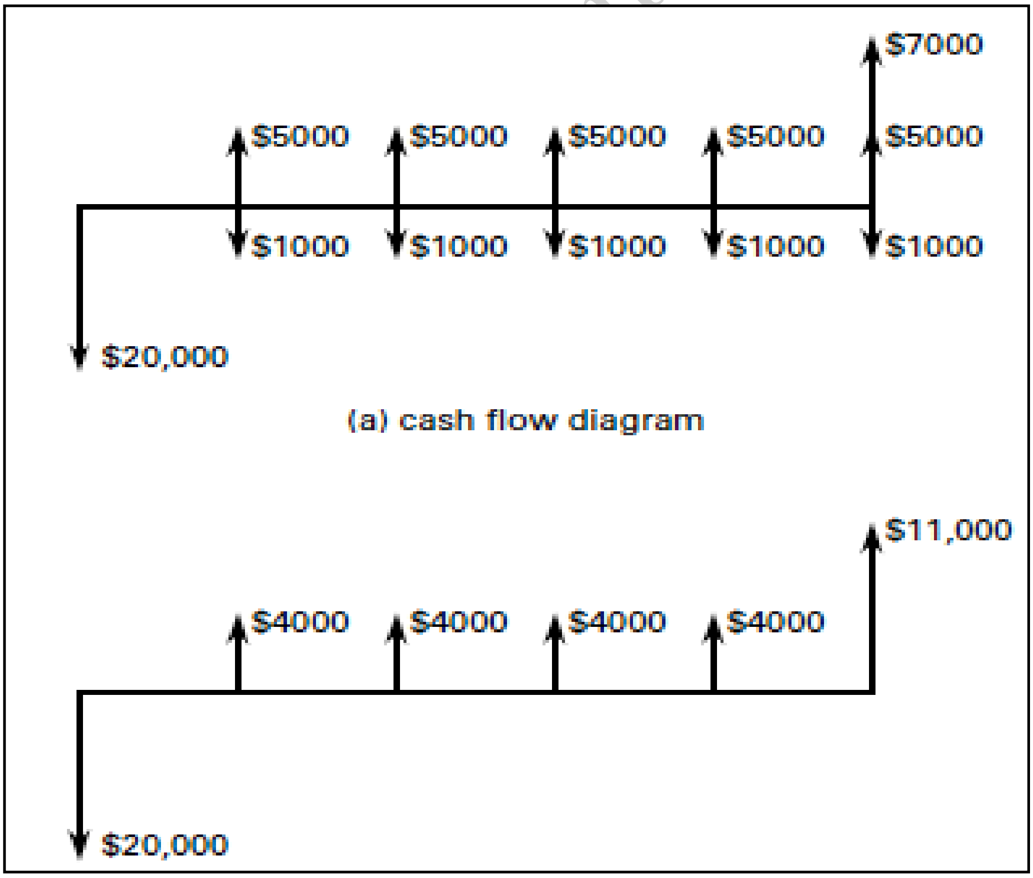


Figure (21) Simplified cash flow diagram

Notation

To simplify the subject of economic analysis, symbols are introduced to represent types of cash flows and interest factors. The symbols used in this chapter conform to ANSI Z94;4 however, not all practitioners follow this standard convention, and care must be taken to avoid confusion when reading the literature. The following symbols will be used here:

P = Present sum of money ($)

F = Future sum of money ($)

n = Number of interest periods

i = Interest rate per period (%)

A complete list of the ANSI Z94 symbols is given in Appendix A to this chapter.

Handling Linear Gradient Series

Sometimes cash flows will vary linearly, that is, they increase or decrease by a set amount, G, the gradient amount. This type of series is known as a **strict gradient series,** as seen in Figure below. Note that each payment is A, = $(n$ - 1) G. Note also that the series begins with a zero cash flow at the end of period zero. If G > 0, the series is referred to as an *increasing* gradient series. If G < 0, it is referred to as a *decreasing* gradient series.

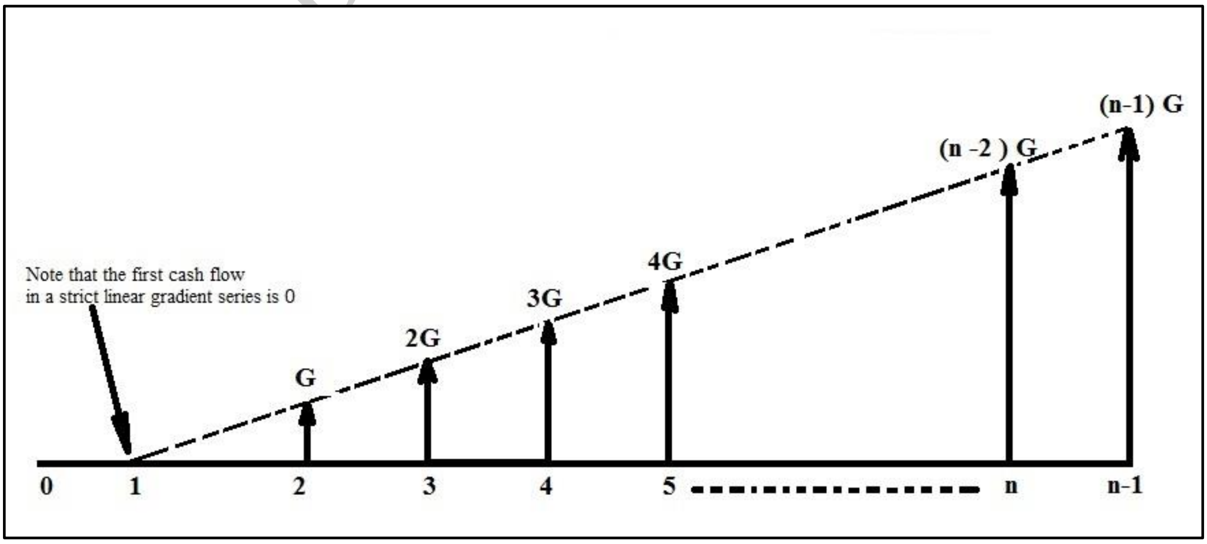


Figure (22) Cash flow diagram of a strict gradient series

Arithmetic Gradient Factors (P/G and A/G)

An *arithmetic gradient* is a *cash flow series* that either increase or decrease by a constant amount. The cash flow, whether income or disbursement, changes by the same arithmetic amount each period. The *amount* of the increase or decrease is the *gradient*. For example, if a manufacturing engineer predicts that the cost of maintaining a robot will increase by $ 500 per year until the machine is retired, a gradient series is involved and the amount of the gradient is $ 500.

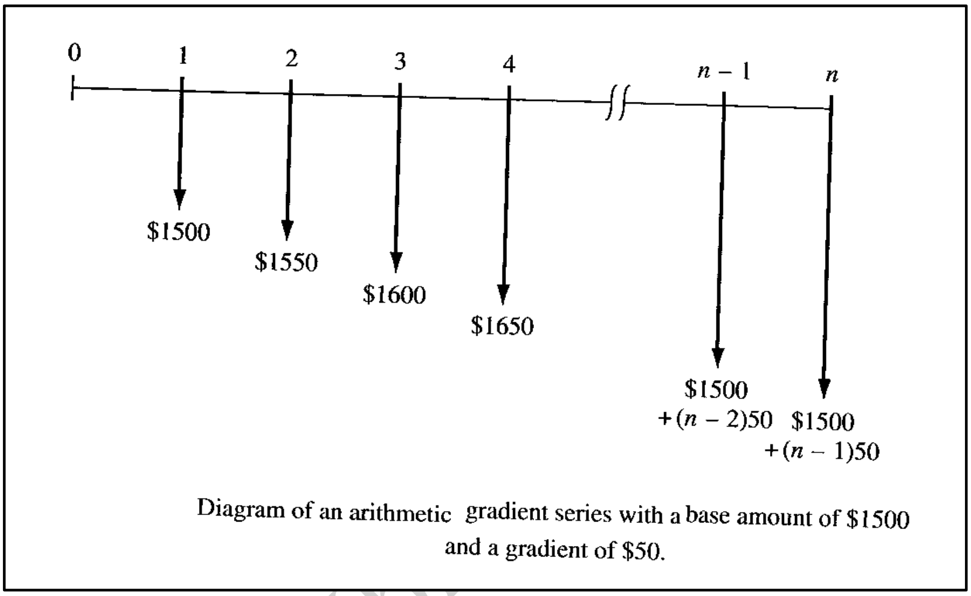
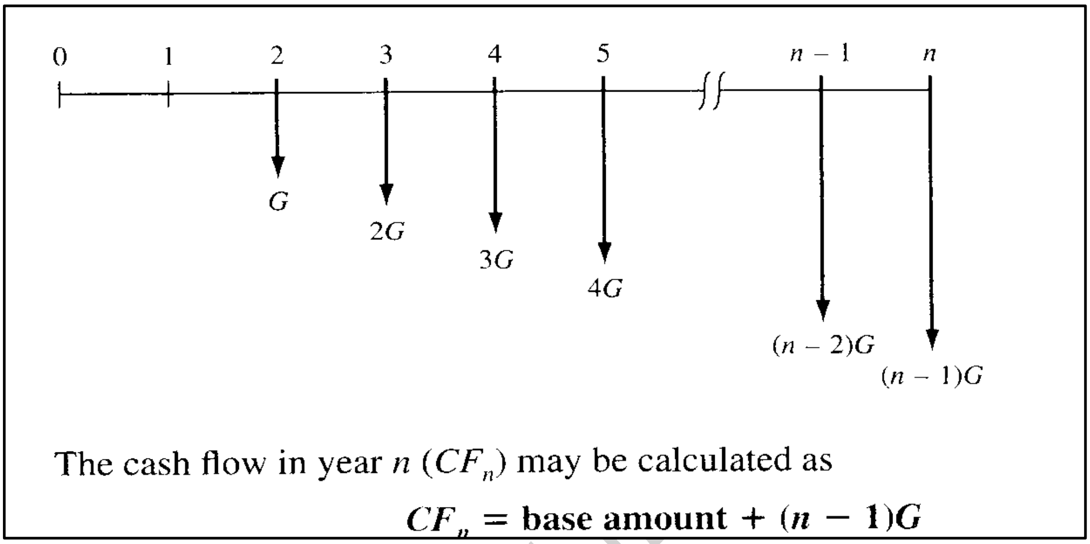


Figure (23) Cash flow diagram of a strict gradient series

Formulas previously developed for an A series have year-end amounts of equal value. In the case of a gradient, each year-end cash flow is different, so new formulas must be derived. First, assume that the cash flow at the end of year 1 is not part of the gradient series, but is rather a base amount. This is convenient because in actual applications, the base amount is usually larger or smaller than the gradient increase or decrease. For example, if you purchase a used car with a 1 year warranty, you might expect to pay the gasoline and insurance costs during first year of operation. Assume these cost $1500; that is, $1500 is the base amount. After the first year you absorb the cost of repairs, which could reason - ably be expected to increase each year. If you estimate that total costs will increase by $50 each year, the amount the second year is $1550, the third $1600, and so on to year n, when the total cost is 1500 + (n - 1)50. The cash flow diagram is shown in figure below. Note that the gradient ($50) is first observed between year 1 and year 2, and the base amount is not equal to the gradient.

Define the symbol G for gradients as

G= constant arithmetic change in the magnitude of receipts or disbursements from one time period to the next; G may be positive or negative.



The cash flow in year n (CF_n) may be calculated as

CF_n= base amount + (n -1) G

If the base amount is ignored, a generalized arithmetic (increasing) gradient cash flow diagram is as shown in figure above. Note that the gradient begins between years 1 and 2. This is called a conventional gradient.

Example 41

A manufacturing of Equipment Company has initiated a new program for renting the equipment. It expects to realize a revenue of $ 80,000 in fees next year. Fees are expected to in increase uniformly to a level of $ 200,000 in 9 years. Determine the arithmetic gradient and construct the cash flow diagram.

Solution

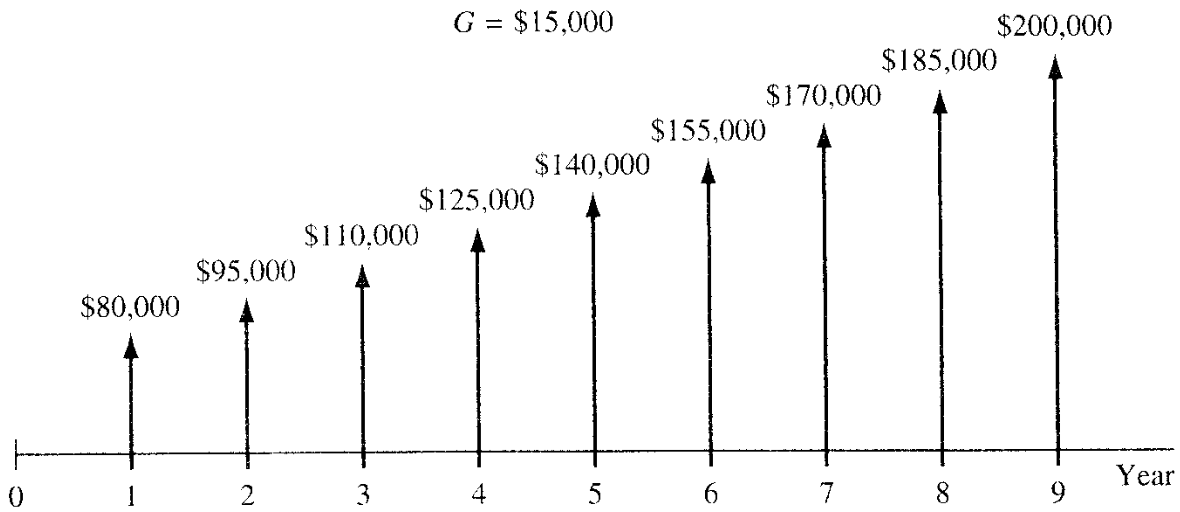
The base amount is $80,000 and the total revenue increase is

$$\text{Increase in 9 years} = 200{,}000 - 80{,}000 = 120{,}000$$

$$\text{Gradient} = \frac{\text{increase}}{n-1}$$

$$= \frac{120{,}000}{9-1} = \$15{,}000 \text{ per year}$$

The cash flow diagram is shown in Figure 2–12.



In this text, three factors are derived for arithmetic gradients: the P/ G factor for present worth, A / G factor for annual series and the F / G factor for future worth. There are several ways to derive them. We use the single payment present worth factor (P/F, i, n), but the same result can be obtained using the F / P, F/ A, factor.

In figure above the present worth at year 0 of only the gradient is equal to the sum of the present worths of the individual values, where each value is considered a future amount.

$$\boldsymbol{P = G(P/F, i, 2) + 2G(P/F, i, 3) + 3G\,(P/F, i, 4) + \cdots \ldots \ldots}$$

$$\boldsymbol{+[(n-2)G](P/F, i, n-1) + [(n-1)G]\,(P/F, i, n)}$$

Factor out G and use the P / F formula.

$$P = G[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \frac{3}{(1+i)^4} + \cdots \ldots .. + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}] \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(14)}$$

Multiplying both sides of above Equation by $(1+i)^1 yields$

$$P\,(1+i)^1 = G\left[\frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \frac{3}{(1+i)^3} + \cdots \ldots .. + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}}\right] \ldots \ldots .(\mathbf{15})$$

Subtract Equation (14) from Equation (15) and simplify.

$$i\,P = G\left[\frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \cdots \ldots .. + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] - \boldsymbol{G}\left[\frac{\boldsymbol{n}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right] \ldots . \ldots \ldots ..(\mathbf{16})$$

The left bracketed expression is the same as that contained in following Equation:

$$P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots \ldots .. + \frac{1}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}\right]$$

Where the P / A factor was derived. Substitute the closed- end form of the P/A factor from the following Equation $P = A\left[\frac{(1+i)^n - 1}{i\,(1+i)^1}\right]$ into Equation **(16)** and solve for P to obtain a simplified relation.

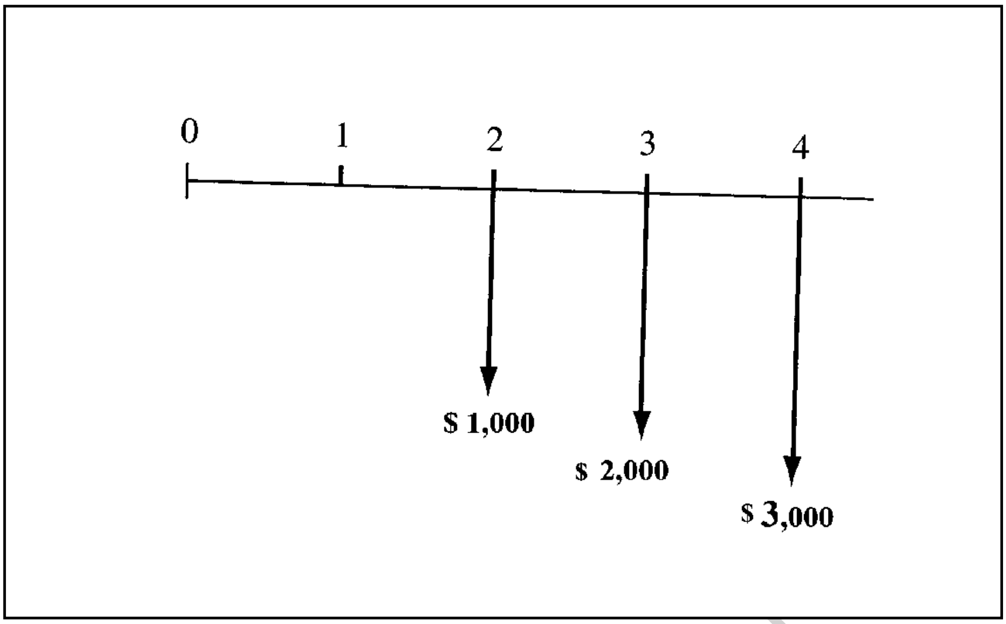
$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i\,(1+i)^n} - \frac{\boldsymbol{n}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(\mathbf{17})$$

<u>Example 42</u>

It is expected that maintenance costs for the first year equal to zero of the construction equipment, at the end of the second year will be \$ 1,000 and \$ 2,000 for the third year and \$ 3,000 for the fourth year. The annual rate interest is 15%. Calculate

1- The present value of these expenses
2- The annual amount of the uniform series for the four years

Solution



Cash flow Diagram

1-

$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i\,(1+i)^n} - \frac{\boldsymbol{n}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right]$$

$$P = \frac{1{,}000}{0.15}\left[\frac{(1+0.15)^4 - 1}{0.15\,(1+0.15)^4} - \frac{\mathbf{4}}{(\mathbf{1+0.15})^{\mathbf{4}}}\right] = \$\ \mathbf{3,790}$$

2-

$$A = G\left[\frac{1}{i} - \frac{\boldsymbol{n}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right] = \mathbf{1,000}\ \boldsymbol{x}\ \left[\frac{1}{0.15} - \frac{\mathbf{4}}{(\mathbf{1+0.15})^{\mathbf{4}}}\right] = \$\ \mathbf{1,326.27}$$

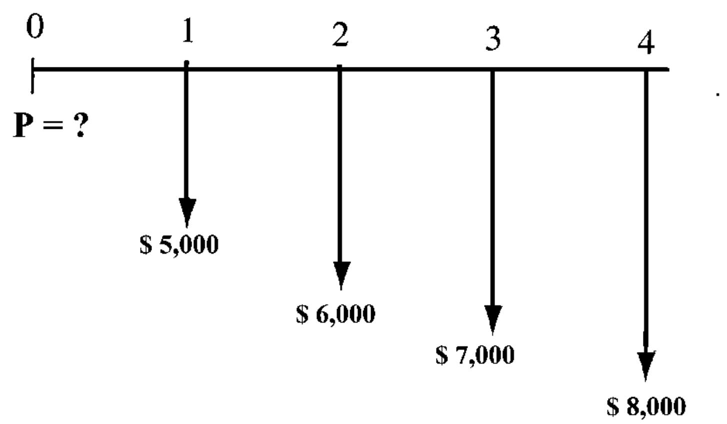
Example 43

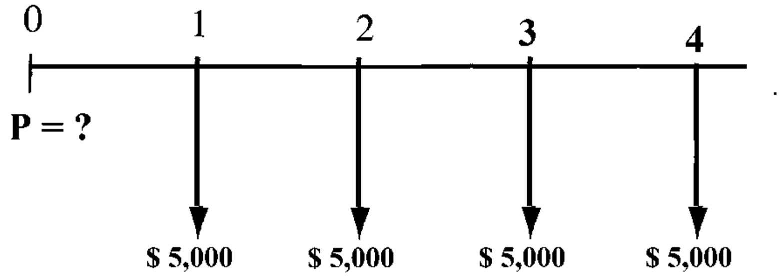
Below the following payments are the installments for purchasing the new shovel, if the contractor wants to pay these installments at now; how much cost will be? With rate interest equal to 15%, for 4 years.

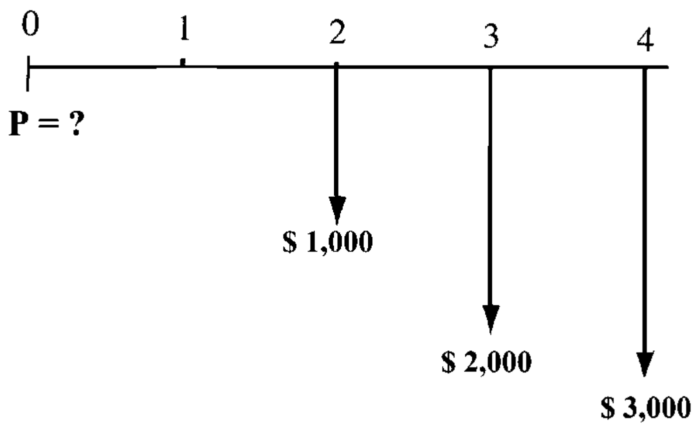
End of Year	Payment
1	$ 5,000
2	$ 6,000
3	$ 7,000
4	$ 8,000

Solution

To solve this problem with gradient form, it needs to divide to 2 two parts for confirming this.







Total P amount = the amount of regular payments + the amount of gradient

$$P_A = A\ x\ \left[\frac{(1+i)^n-1}{i\ x\ (1+i)^n}\right] = 5{,}000\ x\ \left[\frac{(1+0.15)^4-1}{0.15\ x\ (1+0.15)^4}\right] = \$\ 14{,}274.89$$

$$P_G = \frac{G}{i}x\left[\frac{(1+i)^n-1}{i\ x\ (1+i)^n} - \frac{n}{(1+i)^n}\right] = \frac{1{,}000}{0.15}x\left[\frac{(1+0.15)^4-1}{0.15\ x\ (1+0.15)^4} - \frac{4}{(1+0.15)^4}\right] = \$\ 3{,}824$$

$$P_{total} = P_A + P_G = 14{,}274.89 + 3{,}824 = \$\ 18{,}098.82$$

For find the equivalent payments as annual payments use the following equation:-

$$A_{annual} = P\ x\ \left[\frac{i\ x\ (1+i)^n}{(1+i)^n-1}\right] = 18{,}098.82\ x\ \left[\frac{0.15\ x(1+0.15)^4}{(1+0.15)^4-1}\right] = \$\ 6{,}339.42$$

Example 44

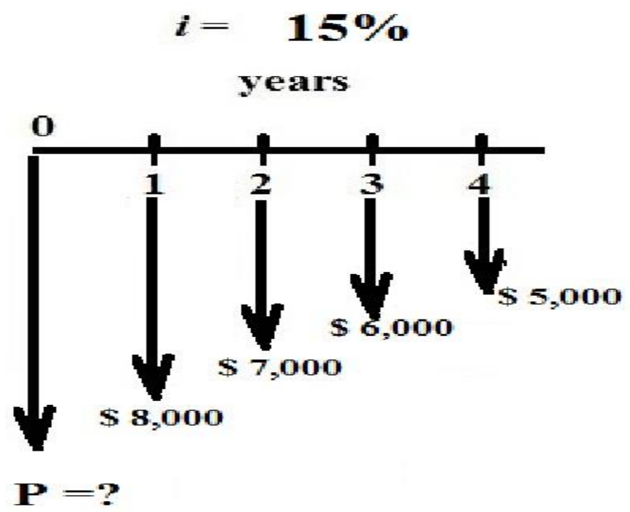
Assume that the amount of payments for renting the equipment are decreasing in gradient annually; similar to the following table:-

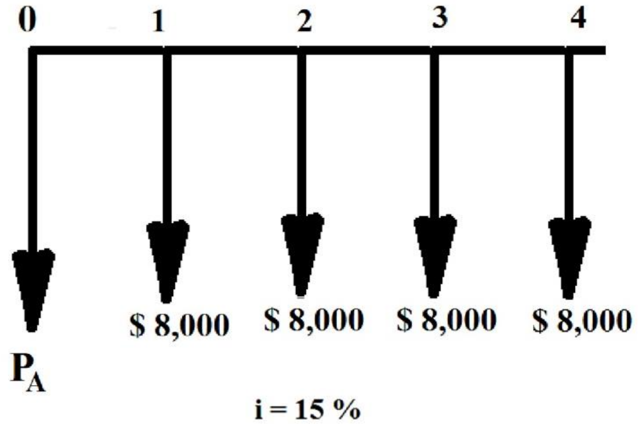
End of Year	Payments
1	$ 8,000
2	$ 7,000
3	$ 6,000
4	$5,000

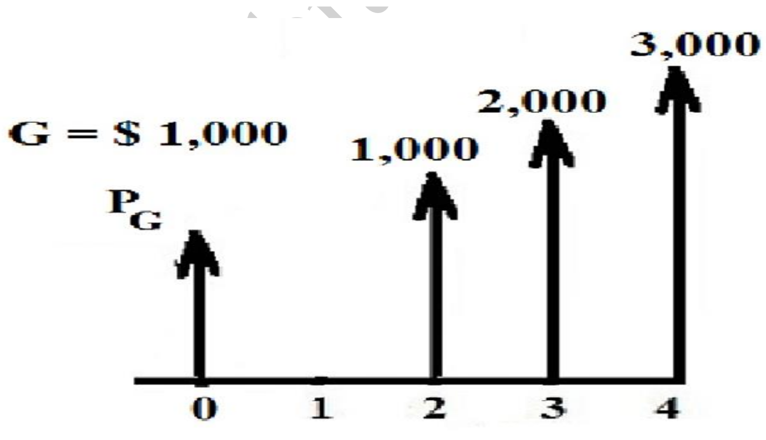
Solution

To solve this problem it should be divided into two parts; the first part as regular payments and the second like the gradient payments:-

$$P_{NET} = P_A - P_G$$







$$P_A = A\ x\ \left[\frac{(1+i)^n-1}{i\ x\ (1+i)^n}\right] = 8{,}000\ x\ \left[\frac{(1+0.15)^4-1}{0.15\ x\ (1+0.15)^4}\right] = \$\ 22{,}874.6$$

$$P_G = \frac{G}{i} x \left[\frac{(1+i)^n-1}{i\ x\ (1+i)^n} - \frac{n}{(1+i)^n}\right] = \frac{1{,}000}{0.15} x \left[\frac{(1+0.15)^4-1}{0.15\ x\ (1+0.15)^4} - \frac{4}{(1+0.15)^4}\right] = \$\ 3{,}824$$

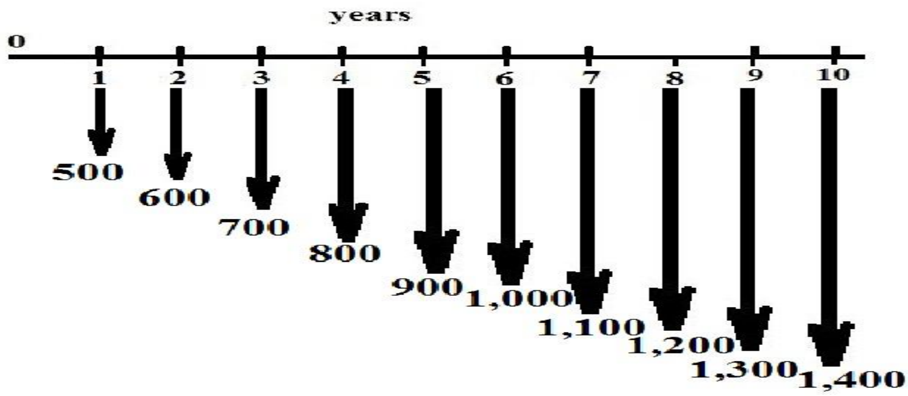
$$P_{NET} = P_A - P_G = \$\ 22{,}874.6 - \$\ 3{,}826.7 = \$\ 19{,}047.87$$

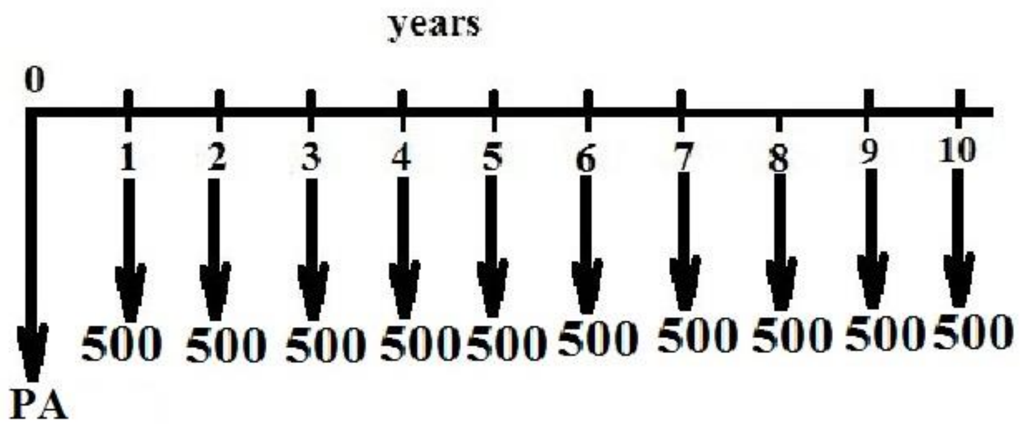
The sign negative for algebraic sum process

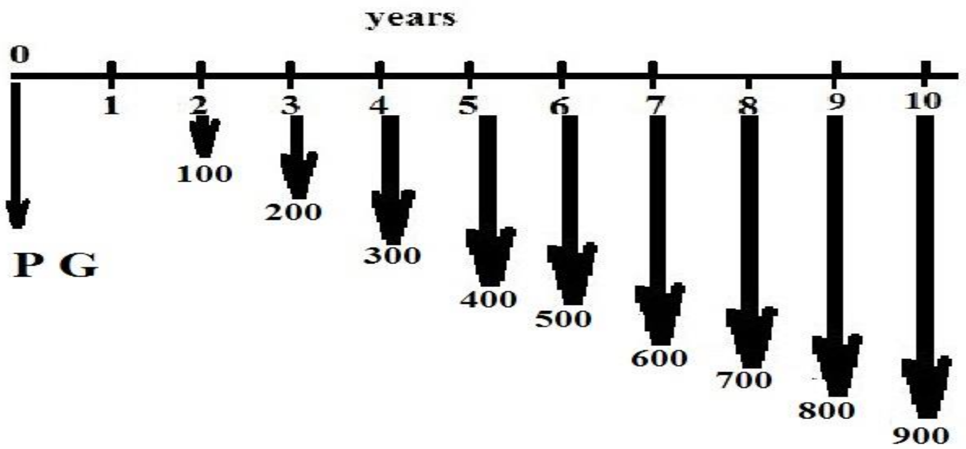
Example 45

A man plans to invest the money by depositing $ 500 / year from now. He has ensured that this deposit will increase by $ 100 yearly for ten years. What is the present value of this investment and the rate of interest 5 % per year? What is the value of the annual amounts equivalent to this annually investment?

Solution







$$P_A = A\ x\ \left[\frac{(1+i)^n - 1}{i\ x\ (1+i)^n}\right] = 500\ x\ \left[\frac{(1+0.05)^{10} - 1}{0.05\ x\ (1+0.05)^{10}}\right] = \$\ 3{,}861.26$$

$$P_G = \frac{G}{i} x \left[\frac{(1+i)^n - 1}{i\ x\ (1+i)^n} - \frac{n}{(1+i)^n}\right] = \frac{100}{0.05} x \left[\frac{(1+0.05)^{10} - 1}{0.05\ x\ (1+0.05)^4} - \frac{10}{(1+0.05)^{10}}\right] = \$3{,}168$$

$$P_{Total} = P_A + P_G = \$\ 3{,}861.26 + \$\ 3{,}168 = \mathbf{\$\ 7{,}029.26}$$

The value of the annual equivalent amounts is the result of sum of an equal amount to (A) and an equivalent amount to (G);

$$A = A_1 + A_G$$

$$A_1 = \$\ 500$$

$$A_G = Gx\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = 100\ x\left[\frac{1}{0.05} - \frac{10}{(1+0.05)^{10} - 1}\right] = \$\ 410.2$$

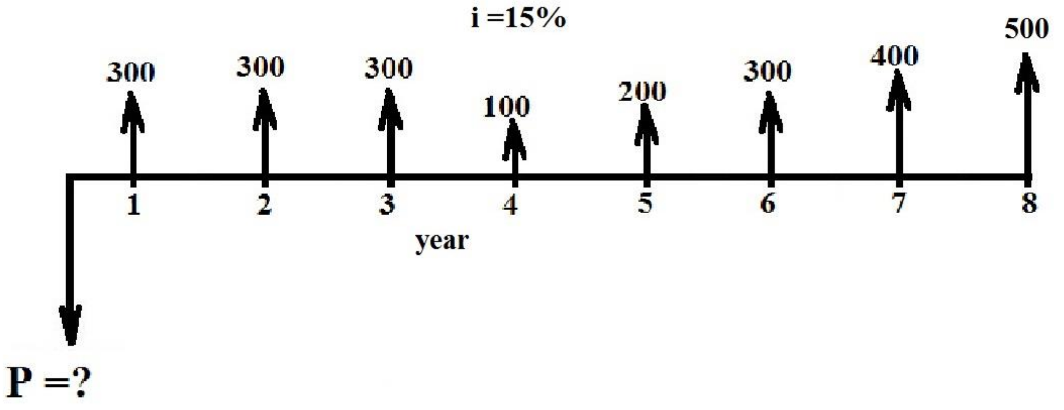
$$A = 500 + 410.2 = \mathbf{\$\ 910.2}$$

The total value of P is **$ 7,029.26** then apply the law A= P $x\ [\frac{i\ x\ (1+i)^n}{(1+i)^n - 1}]$, and the output will be **$ 910.2** It is the annual equivalent of the current effective amount.

<u>Example 46</u>

Find the present value for the following cash flow below with rate interest is

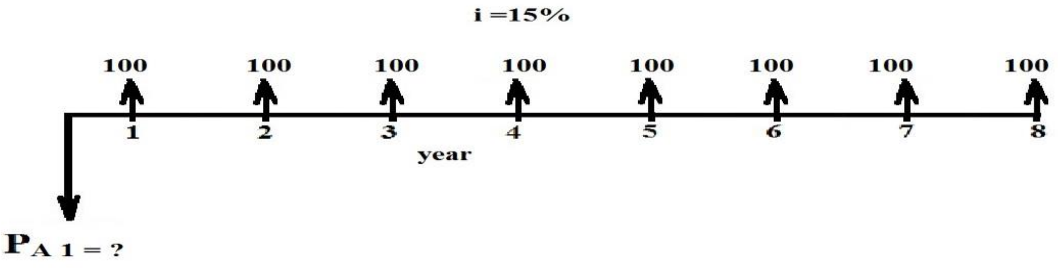
i= 15%



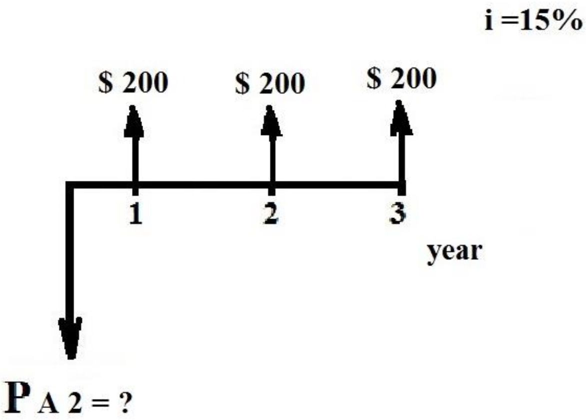
<u>Solution</u>

To solve this question we need to dived the payments into three parts

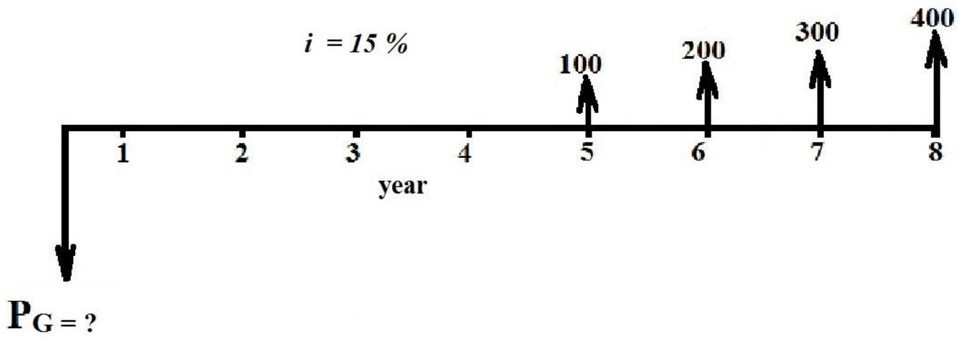
1- The first Part as shown in figure below:



2- The second part as shown below:-



3- The third part as shown below



$$P_{A1} = A\,x \left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n}\right] = 100\,x \left[\frac{(1+0.15)^8-1}{0.15\,x\,(1+0.15)^8}\right] = \$\,448.73$$

$$P_{A2} = A\,x \left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n}\right] = 200\,x \left[\frac{(1+0.15)^3-1}{0.15\,x\,(1+0.15)^3}\right] = \$\,1{,}156.14$$

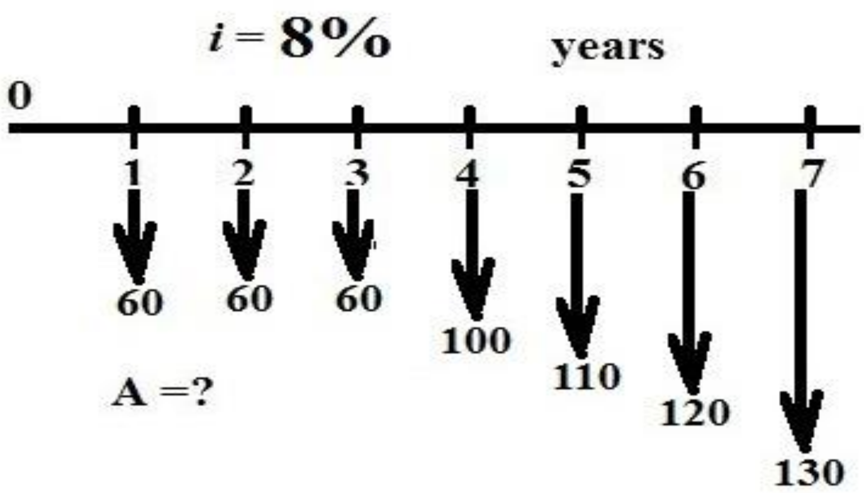
$$P_G = \frac{G}{i} x \left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n} - \frac{n}{(1+i)^n}\right] = \frac{100}{0.15} x \left[\frac{(1+0.15)^5-1}{0.15\,x\,(1+0.15)^5} - \frac{5}{(1+0.15)^5}\right] = \$\,577.5 \text{ at year 3}$$

$$P = F\,x \left[\frac{1}{(1+i)^n}\right] = 577.5\,x \left[\frac{1}{(1+0.15)^3}\right] = \$\,379.86$$

$$\boldsymbol{P_{Total} = P_{A1} + P_{A2} + P_G = 448.73 + 1{,}156.14 + 379.73 = \$\,1{,}282.73}$$

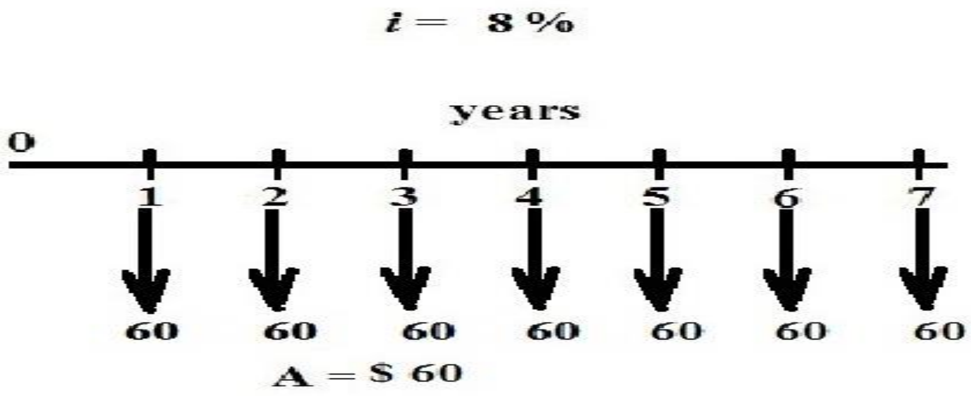
Example 47

Find the annual payments which equivalent to the cash flow in below and find the equivalent the present value with rate interest is 8%?

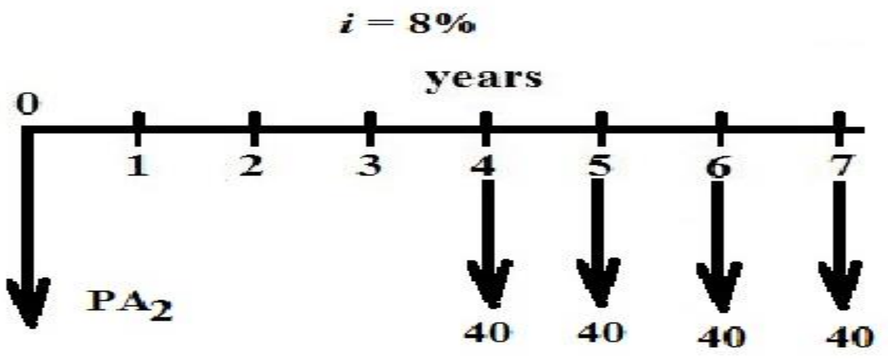


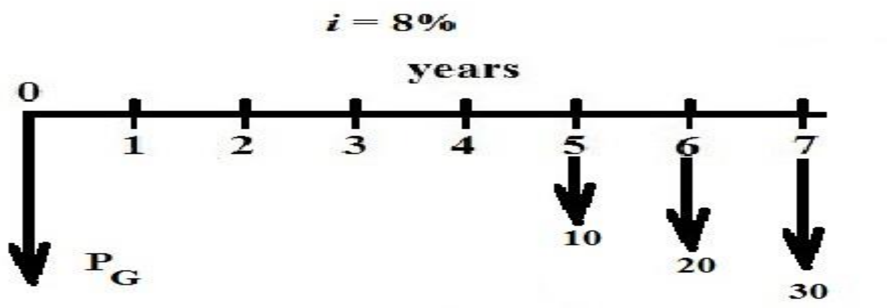
Solution

To solve this question it can be divided to three parts as flowing:-



PA1





$$P_{A1} = A\,x\,\left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n}\right] = 60\,x\,\left[\frac{(1+0.08)^7-1}{0.15\,x\,(1+0.08)^7}\right] = \$\,312.42$$

$$P_{A2} = A\,x\,\left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n}\right] = 40\,x\,\left[\frac{(1+0.08)^4-1}{0.15\,x\,(1+0.08)^4}\right] = \$\,132.35\ at\ year\ 3$$

$$P_{A2} = F\,x\,\left[\frac{1}{(1+i)^n}\right] = 132.35\,x\,\left[\frac{1}{(1+0.08)^3}\right] = \$\,105.04$$

$$P_G = \frac{G}{i}x\left[\frac{(1+i)^n-1}{i\,x\,(1+i)^n} - \frac{n}{(1+i)^n}\right] = \frac{10}{0.08}x\left[\frac{(1+0.08)^4-1}{0.15\,x\,(1+0.08)^4} - \frac{4}{(1+0.08)^4}\right] = \$\,46.50$$

$$P_G = F\,x\,\left[\frac{1}{(1+i)^n}\right] = 46.50x\,\left[\frac{1}{(1+0.08)^3}\right] = \$\,36.91$$

$$\boldsymbol{P_{Total} = P_{A1} + P_{A2} + P_G = 312.42 + 105.04 + 36.91 = \$\,454.37}$$

ELEMENTARY COST CALCULATIONS

COSTS AS DECISION INPUTS

Business costs frequently consist of a fixed component plus a variable cost which varies with sales volume or physical output. Fixed cost would include fixed charges for equipment and other overhead; variable costs per unit consist mainly of direct labor, direct materials and other directly allocable inputs like power. Let F = fixed cost, v= variable cost per unit and X = number of items produced. The cost C for an output X is then

$$\boldsymbol{C = F + vX} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(18)}$$

Example 48

What are s the costs for 100,000 items when the fixed costs are $ 10,000 and the variable costs 60¢ per unit? Here,

Solution

$F = \$10{,}000, \quad v = 0.6 \quad X = 100{,}000. \quad from\ equation\ (18)$

$C = 10{,}000 + (0.6)x\ (100{,}000) = \$\ 70{,}000$ Ans.

The variable cost u is the slope of C. Differentiating C with respect to X gives:

$dc/dx = v$

Sometimes F and v are not known explicitly and have to be computed from two values of C; i.e. we know total cost C_1 for quantity $X_1$and C_2 for X_2. Assuming a straight – line relationship, the cost function looks like figure below. Then F is the intercept on the vertical (C) axis and v is the slope. From the geometry,

$$\boldsymbol{v = \frac{C_2 - C_1}{X_2 - X_1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(19)}$$

$$\boldsymbol{F = C_1 - v\,X_1 = C_2 - v\,X_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(20)}$$

Example 49

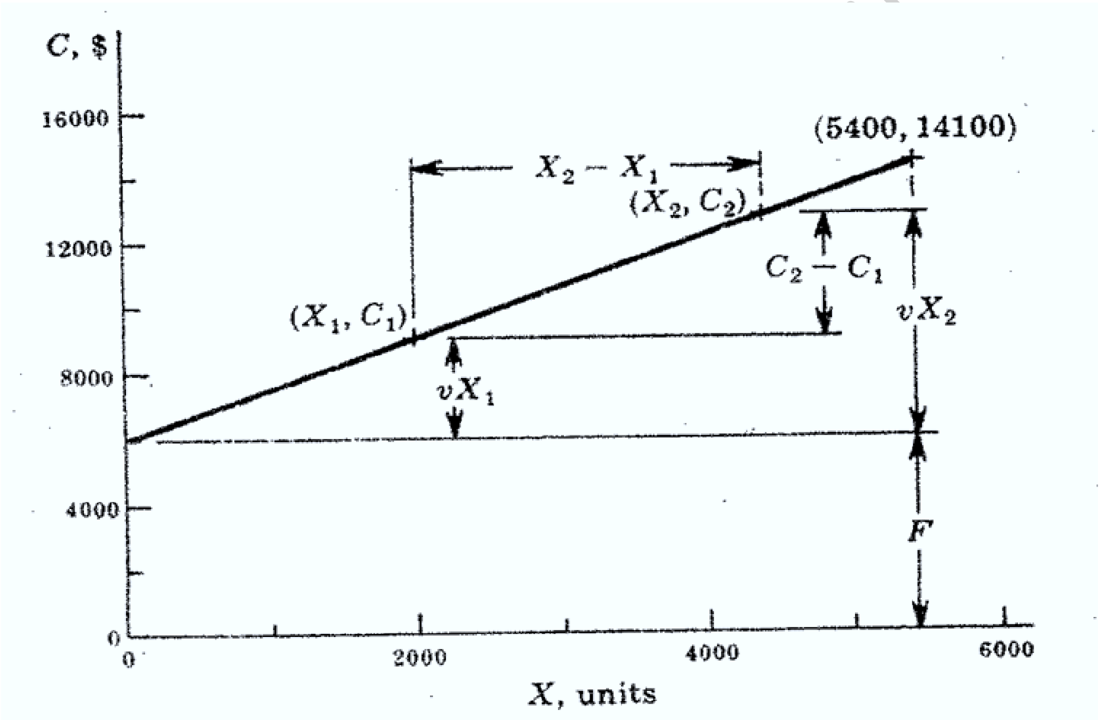
A cost function is linear: when 2,000 items are made, total costs are $ 9,000; when 4,400 items are made, they are $12,600. (a) Find the fixed cost and the variable cost per unit. (b) Find the cost for 5,400 units.

Solution

$$v = \frac{12{,}600-9{,}000}{4{,}400-2{,}000} = \$\ 1.5\ \ per\ unit$$

$$F = 9{,}000 - (1.5)x\ (2{,}000) = \$\ 6{,}000$$

$$C\ = 6{,}000 - (1.5)x\ (5{,}400) = \$\ 14{,}400$$



BREAKEVEN ANALYSIS

Profit Calculations.

When an item is sold for an amount $\boldsymbol{r}$ each, the total revenues $\boldsymbol{R}$ are

$\boldsymbol{R = rX}$ ……………………………………………………………………(21)

$and\ the\ profit\ is$

$\boldsymbol{P = R - C}$ ……………………………………………………………....(22)

Plotting R and C on the same graph (Fig. below) shows a crossover or breakeven point for which $P = R - C = 0$; i.e. $R = C$. This breakeven point X_b is given by

$\boldsymbol{rX_b = F + vX_b}$ ……………………………………………………………(23)

Or

$$\boldsymbol{X_b = \frac{F}{r-v}} \quad \text{……………………………………………………(24)}$$

For $X > X_b$ there is a profit; for $X < X_b$ profit is negative i.e. there is a loss. The quantity $r - v$ is called the $profit\ contribution$.

Example 50

In a manufacturing process, fixed cost is \$ 10,000. The variable cost per unit is 40¢ and the item is sold for 60¢ each. Find (a) the breakeven point, (b) the breakeven volume?

Solution

a) F= \$ 10,000, $v = 0.4, \quad r = 0.6$

$X_b = \frac{10,000}{0.6-0.4} = 50,000\ units.$

b) The corresponding volume R_b is obtained by substituting $X = X_b$

$R_b = (0.6)x\ (50,000) = \$\ 30,000.$

$R_b = 10,000 + (0.4)x\ (50,000) = \$\ 30,000.$

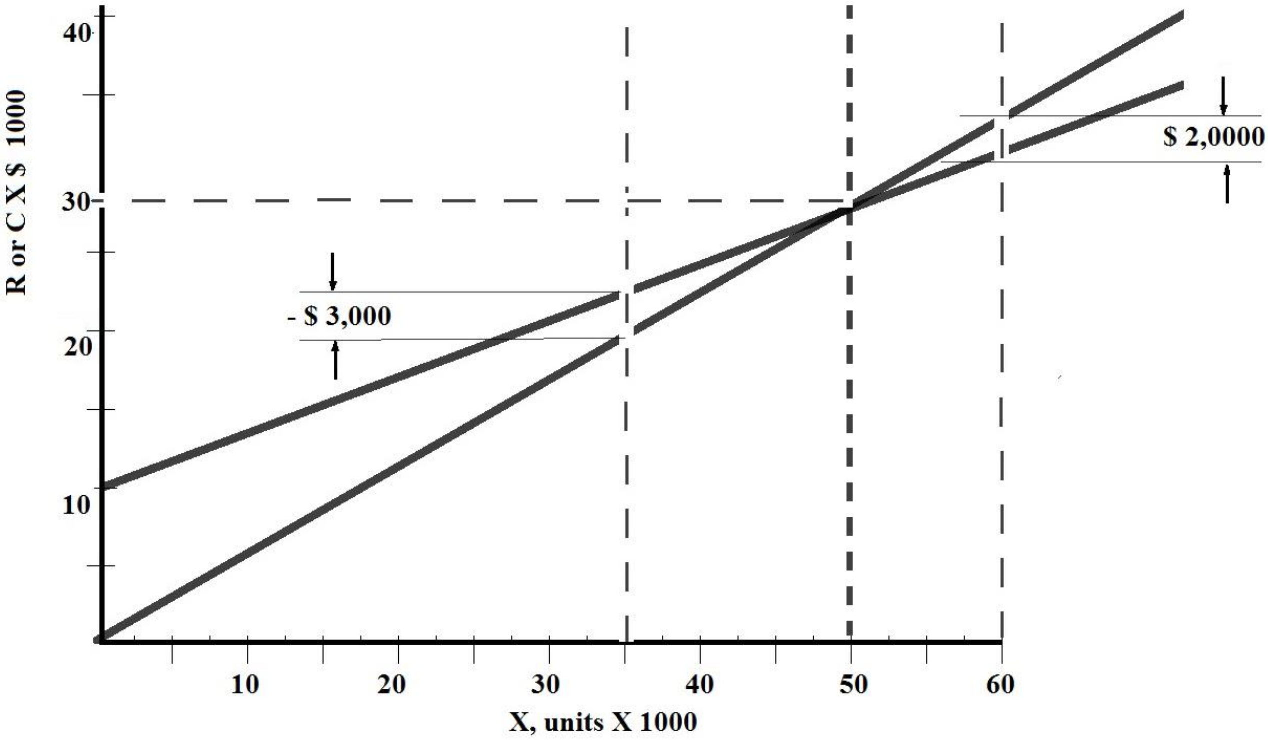
The results are plotted in figure below;

Example 51

In Example 50, find the profit (loss) for volumes of (a) 60,000, (b) 35,000 units.
Solution

$P = R - C = (0.6)(60{,}000) - [10{,}000 + (0.4)(60{,}000)]$ =\$ 2,000

$P = R - C = (0.6)(35{,}000) - [10{,}000 + (0.4)(35{,}000)]$ =\$ -3,000
(a loss)



Assessing Two Alternatives

Given two processes, 1 and 2, with fixed costs F_1 and F_2 and variable costs per unit v_1 and v_2, breakeven analysis can be used to determine at what volume X it pays to shift from one process to the other, or whether it pays at all. The breakeven point X_b is given by equating the two cost functions:

$$\boldsymbol{F_1 + v_1 X_b = F_2 + v_2 X_b}$$

$$X_b = \frac{F_1 - F_2}{v_2 - v_1}$$

If X_b is to be positive, $F_1 > F_2$ and $v_1 < v_2$ this is the typical case which arises from the use of a more expensive automatic machine which, however, saves on direct labor. If $F_1 > F_2$ and $v_1 > v_2$, the cost lines would diverge and process l would never be economically justified.

Example 52

A factory has two machines with the following costs:

	Machine A	Machine B
Fixed Cost, $	1200	2000
Variable Costs per piece, $ / unit	0.75	0.5

Find (a) the breakeven point and (b) the production cost at breakeven.

Solution

(a) $X_b = \frac{F_1 - F_2}{v_2 - v_1} = \frac{2{,}000 - 1{,}200}{0.75 - 0.5} = 3{,}200$

If the order quantity $Q < 3{,}200$ use machine A; if $Q > 3{,}200$ use machine B.

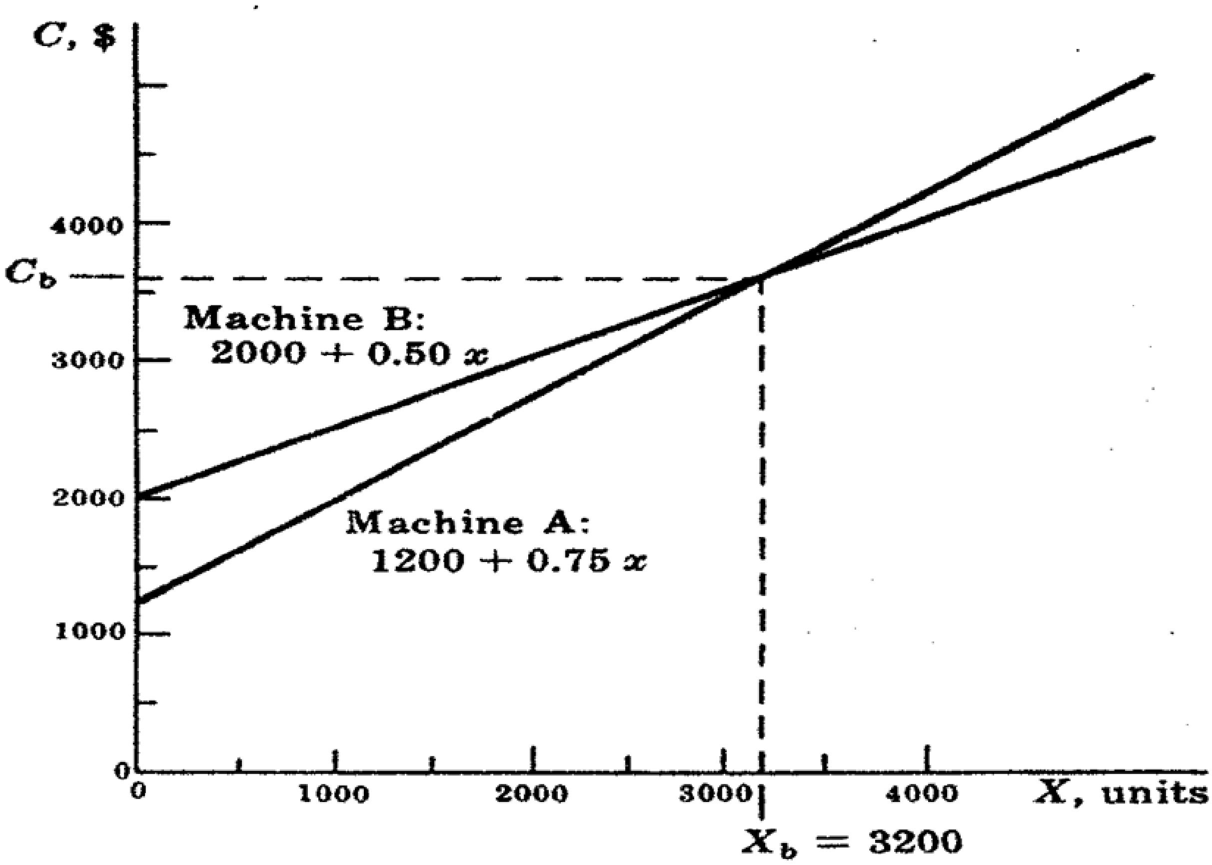
(b) The cost C_b of the 3,200 pieces at breakeven is obtained by substituting either the data for machine A or B into the following equation:-

$$C_b = 2{,}000 + (0.5)x\,(3{,}200) = \$\,3{,}600$$

Or

$$C_b = 1{,}200 + (0.75)x\,(3{,}200) = \$\,3{,}600$$

Figure below is drawn to scale for this example.



DEPRECIATION

DEPRECIATION TERMINOLOGY

Depreciation is defined as the decline in value of physical assets, such as buildings and equipment, over their estimated useful lives. In the context of tax liability, depreciation is the amount allowed as a deduction in computing taxable income and, hence, income tax. It is a bookkeeping entry that does not involve an outlay of cash, and the amount of depreciation allowed by laws and regulations is referred to as the depreciation allowance.

It is important to differentiate between the estimated useful life used in depreciation computation and the actual useful life of a physical asset. The former is often an arbitrary length of time, specified in the regulations for computing federal income taxes, while the latter refers to the economic life of the physical asset which may be terminated because of deterioration or technological obsolescence. The estimated useful life used in depreciation computation is also referred to as depreciable life.

To understand the depreciation allowance, it is necessary to retrace the cycle of the acquisition, use, and disposal of a physical asset. When the physical asset is first purchased, an expenditure is made, but the expenditure cannot be deducted as an expense in computing federal income taxes because no expense is incurred when one asset is exchanged for another, e.g., cash for a physical asset. The outflow of resources comes at the end of the useful life of the physical asset when it is disposed of. The depreciation allowance is a way of recognizing that this outflow did not happen all at once but was in the process of taking place over a period of years. So, the depreciation allowance is a systematic allocation of the cost of a physical asset between the time it is acquired and the time it is disposed of.

Primary terms used in depreciation are defined here. Most terms are applicable to corporations as well as individuals who own depreciable assets.

Depreciation is the reduction in value of an asset. The method used to depreciate an asset is a way to account for the decreasing value of the owner and to represent the diminishing value (amount) of the capital funds invested in it. The annual depreciation amountD_t, does not represent an actual cash flow, nor does it necessarily reflect the actual usage pattern of the asset during ownership.

Book depreciation and **tax depreciation** are terms used to describe the purpose for reducing asset value. Depreciation may be performed for two reasons:

1- Use by a corporation or business for internal financial accounting. This is book depreciation.

2- Use in tax calculation per government regulations. This is tax depreciation.

The methods applied for these two purposes may or may not utilize the same formulas, as is discussed later. Book depreciation indicates the reduced investment in an asset based upon the usage pattern and expected useful life of the asset. There are classical depreciation methods used to determine book depreciation: straight line, declining balance, and the infrequently used sum-of-year digits method.

Terms and Concepts

First cost or unadjusted basis is the delivered and installed cost of asset including purchase price, delivery and installation fees, and other depreciable direct costs incurred to prepare the asset for use. The term unadjusted basis B, or simply basis, is used when the asset is new, with the term adjusted basis used after some depreciation has been charged.

Book value represents the remaining, undepreciated capital investment on the books after the total amount of depreciation charges to date have subtracted from the basis. The book value BV, is usually determined at the end of each year, which is consistent with the end-of-year convention.

Recovery period is the depreciable life n of the asset in years. Often there are different n values for book and tax depreciation. Both of these values may be different from the asset's estimated productive life.

Market value, a term also used in replacement analysis, is the estimated amount realizable if the asset were sold on the open market. Because of the structure of depreciation laws, the book value and market value may be substantially different. For example, a commercial building tends to increase in market value, but the book value will decrease as depreciation charges are taken. However, a computer workstation may have a market value much lower than its book value due to rapidly changing technology.

Salvage value is the estimated trade-in or market value at the end of the asset's useful life. The salvage value, S expressed as an estimated dollar amount or as a percentage of the first cost, may be positive, zero, or negative due to dismantling and carry-away costs.

Depreciation rate or **recovery rate** is the fraction of the first cost removed by depreciation each year. This rate, denoted by d_t, may be the same each year, which is called the straight-line rate, or different for each year of the recovery period.

Personal property, one of the two types of property for which depreciation is allowed, is the income-producing, tangible possessions of a corporation used to conduct business. Included is most manufacturing and service industry property-vehicles, manufacturing equipment, materials handling devices, computers and networking equipment, telephone equipment, office furniture, refining process equipment, and much more. **Real Property** includes real estate and all improvements-office buildings, manufacturing structures, test facilities, warehouses, apartments, and other structures. Land itself is considered real property, but it is not depreciable.

Half - year convention assumes that assets are placed in service or disposed of in mid-year, regardless of when these events actually occur during the year.
This convention is utilized in this text and in most U.S.-approved tax depreciation methods. There are also midquarter and midmonth conventions.

Methods of Depreciation

There are different methods to calculate depreciation. This diversity is related to the main factor that causes the depreciation. There are methods that have been taken by passing the time (or the effect of the time) as a basis for depreciation. While the use function is mainly used in the calculation of depreciation in other methods, and there are combined methods between them which use to calculate the value of depreciation.

1- STRAIGHT-LINE DEPRECIATION

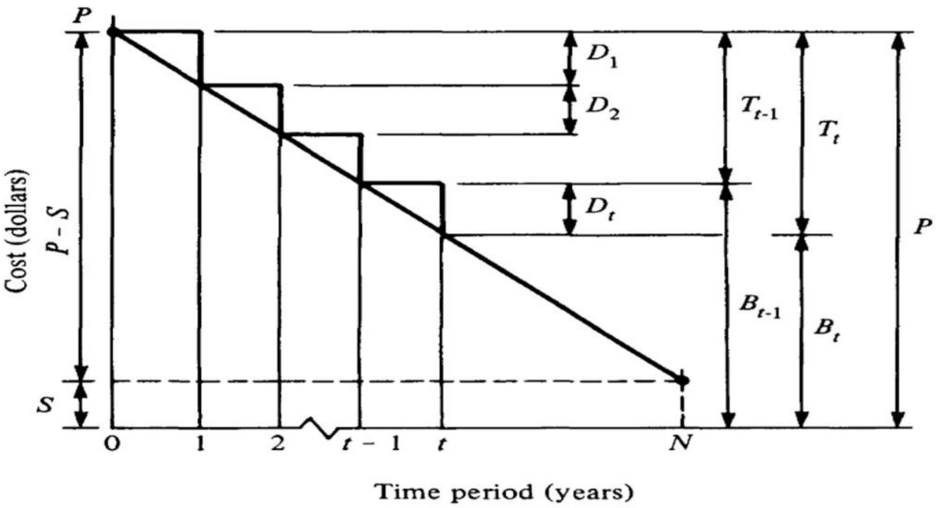
The most common form of depreciation, and the one easiest to comprehend, is *straight-line depreciation.* Under straight-line (SL) depreciation, the cost of a physical asset, less its estimated salvage value, is allocated uniformly to each year of the estimated useful life. Thus, D_1 is constant for all values of t and is given by:

$$\boldsymbol{D_t = \frac{P-S}{N}} \quad \ldots\ldots\ldots\ldots (\mathbf{25})$$

The book value at the end of year *t* is

$$\boldsymbol{B_t = P - \left(\frac{P-S}{N}\right) t} \quad \ldots\ldots\ldots\ldots (\mathbf{26})$$

<u>Example 53</u> If an asset has a first cost of $50,000 with, a $10,000 estimated salvage value after 5 years, (a) Calculate the annual depreciation and (b) compute and plot the book value of the asset after each year, using straight line depreciation.



Salvage value of $2,000 and an estimated useful life of 7 years. Determine the depreciation allowance for each year and the book value at the end of each year, using straight-line depreciation.

<u>Solution</u>

For P = $16,000, S = $2,000, and N = 7, we have for each year *(t* = 1, 2 . . . or7)

$D_t = \frac{16{,}000-2{,}000}{7}$ = $ 2,000

Furthermore,

$B_t = 16{,}000 - 2{,}000 = \$\ 14{,}000$

Hence, B_1 = $14,000, B_2= $12,000, B_3= $10,000, B_4 = $8,000,
B_5= $6,000, B_6 = $4,000, and B_7= $2,000.

2- SUM-OF-THE-YEARS-DIGITS DEPRECIATION

Several other methods of depreciation are known as *accelerated depreciation methods.*

Under an accelerated depreciation method, the depreciation allowance for an asset will be greater in the earlier years of its life and less in the later years of its life than under the straight-line method. The total depreciation allowance would be the same under the accelerated method as under the straight-line method, and, hence, total taxable income would be the same for the whole period; however, greater allowances in the earlier years also mean less tax in the earlier years. When the time value of money is taken into consideration, there is a substantial advantage in having the smaller amounts of taxable income in earlier years and larger amounts of taxable income in later years. Thus, using a method of accelerated depreciation amounts to getting an interest-free loan from the government.

One form of accelerated depreciation is the *sum-of-the-years'-digits (SOYD) method.* Under the SOYD method, the annual depreciation allowance is obtained by multiplying the net depreciable value *(P — S)* by a fraction which has as its numerator the number of years of remaining useful life, and its denominator the sum of all the digits from 1 to *N*. Let Z be the sum of the years' digits such that:

$$\boldsymbol{S.O.Y.D = 1 + 2 + 3 + \cdots \ldots \ldots \ldots (N-1) + N = \frac{N(N+1)}{2}} \text{……….} \textbf{(27)}$$

Then, the depreciation allowance for any year *t* is

$$\boldsymbol{D_t = \frac{N-(t-1)}{S.O.Y.D}(P-S)} \ldots\ldots\ldots\ldots \textbf{(28)}$$

Hence, the book value at the end of year *t* is given by

$$\boldsymbol{B_t = P - \frac{(2N-t+1)t}{N\,(N+1)}(P-S)} \ldots\ldots\ldots\ldots \textbf{(29)}$$

Or from the following Equation,

$$\boldsymbol{B_t = B_{t-1} - D_t} \ldots\ldots\ldots\ldots \textbf{(30)}$$

Or from the following Equation,

$$\boldsymbol{BV_t = B - \sum_{j=1}^{t} dj} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{31})$$

Example 55

Find the depreciation allowance for each year and the book value at the end of each year for the asset in the previous example (54), using the SOYD method.

Solution

In this problem, S.O.Y.D = 1 + 2 + 3 + · · · + 7 = 28

Or S.O.Y.D = $\frac{N\ (N+1)}{2} = \frac{7\ x\ (7+1)}{2} = 28$

P - S = (16,000- 2,000) = 14,000.

$D_1 = \frac{N-(t-1)}{S.O.Y.D}(P-S) = \frac{7-(1-1)}{28}\ x\ 14{,}000 = 3{,}500$

$D_1 = \frac{7}{28} x\ (P-S) = \frac{7}{28} x\ (14{,}000) = \$\ 3{,}500$

$\mathrm{B}_1 = 16{,}000 - 3{,}500 = \$\ 12{,}500$

$D_2 = \frac{6}{28} x\ (P-S) = \frac{6}{28} x\ (14{,}000) = \$\ 3{,}000$

$\mathrm{B}_2 = 12{,}500 - 3{,}000 = \$\ 9{,}500$

$D_3 = \frac{5}{28} x\ (P-S) = \frac{5}{28} x\ (14{,}000) = \$\ 2{,}500$

$\mathrm{B}_3 = 9{,}500 - 2{,}500 = \$\ 7{,}000$

$D_4 = \frac{4}{28} x\ (P-S) = \frac{4}{28} x\ (14{,}000) = \$\ 2{,}000$

$\mathrm{B}_4 = 7{,}000 - 2{,}000 = \$\ 5{,}000$

$D_5 = \frac{3}{28} x\ (P-S) = \frac{3}{28} x\ (14{,}000) = \$\ 1{,}500$

$B_5 = 5{,}000 - 1{,}500 = \$\ 3{,}500$

$D_6 = \frac{2}{28} x\ (P-S) = \frac{2}{28} x\ (14{,}000) = \$\ 1{,}000$

$B_6 = 3{,}500 - 1{,}000 = \$\ 2{,}500$

$D_7 = \frac{1}{28} x\ (P-S) = \frac{1}{28} x\ (14{,}000) = \$\ 500$

$B_7 = 2{,}500 - 500 = \$\ 2{,}000$

3- DECLINING BALANCE DEPRECIATION

The *declining balance depreciation* is another form of accelerated depreciation in which the depreciation allowance for any year *t* is obtained by multiplying the book value of the previous year (t-1) by a *constant depreciation rate r.* Since the book value at year 0 is the original cost of the asset, i.e., B0 = P, we have

$D_1 = B_0 \, x \, r, \quad D_2 = B_1 x \, r, \ldots\ldots \, D_{N-1} = B_{N-2} x \, r, \; D_N = B_{N-1} \, x \, r$

Thus, for year t,

$$\boldsymbol{D_t = B_{t-1} \; x \, r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(32)}$$

The book value at the end of any year *t* can be obtained by noting that, from the following Equations,

$$\boldsymbol{B_t = P - T_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(33)}$$

$$\boldsymbol{B_t = B_{t-1} - D_t = P\,(1 - r)^t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(34)}$$

The advantage of the declining balance depreciation method is that the salvage value *S* need not be estimated in the beginning years when the method is used for filing annual income tax returns. But there is no assurance that the accumulated depreciation at the end of year *N* will equal the net depreciable value (P — 5) if the constant depreciation factor *r* is arbitrarily chosen. In order to ensure that

TN = *P* — *S* or *BN* = *S* for *S* > 0, we find the value of *BN* by letting *t* = *N* in the following Eq. such that

$P\,(1 - r)^N = S$

From which we obtain

$$\boldsymbol{r = 1 - \sqrt[N]{\frac{S}{P}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(35)}$$

<u>Example 56</u>

A compressor was purchased at a cost of $16,000 with an estimated salvage value of $2,000 and an estimated useful life of 7 years. Determine the depreciation allowance for each year and the book value at the end of each year, using Declining Balance depreciation

Solution

For P = \$16,000, S = \$2,000, and N = 7,

we have for each year (t = 1, 2. . . or7)

$r = 1 - \sqrt[N]{\frac{S}{P}}$ = 1-$\sqrt[7]{\frac{2,000}{16,000}}$ = 1- 0.743 = 0.257 = 25.7%

D_1 =16,000 x 0.257 = 4,112

$B.V_1 = 16,000 - 4,112 = 11,888$

$D_2 = 11,888\,X0.257 = 3,055.21$

$B.V_2 = 11,888 - 3,055.216 = 8,832.79$

$D_3 = 8,832.78\,X0.257 = 2,270.02$

$B.V_3 = 8,832.79 - 2,270.02 = 6,562.77$

$D_4 = 6,562.77\,X0.257 = 1,686.63$

$B.V_4 = 6,562.77 - 1,686.62 = 4,876.15$

$D_5 = 4,876.15\,X0.257 = 1,253.17$

$B.V_5 = 4,876.15 - 1,253.17 = 3,622.98$

$D_6 = 3,622.98\,X0.257 = 931.10$

$B.V_6 = 3,622.98 - 931.10 = 2,691.88$

$D_7 = 2,691.88\,X0.257 = 691.81$

$B.V_7 = 2,691.88 - 691.81 = 2,000.06$

4- SINKING FUND DEPRECIATION

The sinking fund formula assumes that a sinking fund is established in which funds will accumulate for replacement purpose. The total depreciation that has taken place up to any given time is assumed to be equal to the accumulated value of the sinking fund at that time. In this manner the invested capital is presented.

With this formula, if the estimated life, salvage value and interest rate on the sinking fund are known, a uniform yearly deposit can be computed. This deposit is the annual cost of depreciation. Thus,

$$\boldsymbol{D = (P - S) X\ Factor\ (\frac{i}{(1+i)^{L}-1})}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \textbf{(36)}$$

$$\boldsymbol{B.V_1 = P - D_1}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \textbf{(37)}$$

$$\boldsymbol{D_t = (P - S)x(A/F, i\%,\ N)x(F/A, i\%, t)}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \textbf{(38)}$$

$$\boldsymbol{D_t = D_{1\,x}\ ((1+i)^{t-1})}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \textbf{(39)}$$

$$\boldsymbol{B.V_t = B.V_{t-1} - D_t}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \textbf{(40)}$$

$$\boldsymbol{B.V_t = P - [(P - S)x\ (A/F,\ i\%, N\ )x(F/A, i\%, t\ )}\ \dots\dots\dots\dots\dots \textbf{(41)}$$

<u>Example 57</u>

A new asset is purchased for $ 120,000 and is estimated to have a life of 10 years and a salvage value of $ 20,000 at the end of that time. What will be the annual depreciation cost up at the end of the sixth year, and the book value at the end of the sixth year? Assume the interest rate of 3%.

<u>Solution</u>

$D_1 = (P - S) X\ Factor\ (\frac{i}{(1+i)^{L}-1})$= 100,000 x 0.08723= 8,723

$B.V_1 = P - D_1 = 120{,}000 - 8{,}723 = 111{,}277$

$D_t = (P - S)x(A/F, i\%,\ N)x(F/A, i\%, t)$

$D_t = D_{1\,x}\ ((1+i)^{t-1})$

$B.V_t = B.V_{t-1} - D_t$

$B.V_t = P - [(P - S)x\ (A/F,\ i\%, N\ )x(F/A, i\%, t\ )$

n	1	2	3	4	5	6	7	8	9	10
Dep.	8.723	8.98469	9.2542	9.5318	9.8178	10.1123	10.4157	10.7281	11.05	11.3815
B.V	111.277	102.292	92.76811	83.2363	73.4185	63.3062	52.8905	42.1624	31.1124	19.7309

<u>Example 58</u>

Determine the Depreciation by different methods also determine the Book Value for each year for machine as following:-

Purchased (initial cost) P = $ 8,000

Salvage Value S = $ 1,000

Rate interest = %10 The useful life = 10 years

Solution

1- The straight line Method

d = $\frac{P-S}{n} = \frac{8,000-1,000}{10}$ = 700 and $D_n = 1\ x\ d$ and $B.V_n = P - t\ D_t$

n	1	2	3	4	5	6	7	8	9	10
Dep.	700	700	700	700	700	700	700	700	700	700
B.V	7,300	6,600	5,900	5,200	4,500	3,800	3,100	2,400	1,700	1,000

2- Sum of years (Integers) Method

S.O.Y = 10+9+8+7+6+5+4+3+2+1= 55

$or \quad S.O.Y = \frac{N\ (N+1)}{2} = \frac{10\ (11)}{2} = 55$

$Dt_1 = \frac{10}{55}\ x\ (P-S)$ = 0.1818 X 7,000 = 1,272. 6

B.V_1 = 8,000 -1,272.6= 6,727.4

$D.t_2 = \frac{9}{55}\ x\ (P-S)$ = 0.1636 X 7,000 = 1,145.45

$B.V_{\ 2} = 6{,}727.4 - 1{,}145.45 = 5{,}581.95$

$D_t = \frac{2\ x\ (N-t+1)}{N(N+1)}\ x(P-S) \quad or \quad D_t = \frac{N-t+1}{S.O.Y}\ x\ (P-S)$

And $\quad B.V_t = P - [(P-S)x\frac{2tN-t^2+t}{N\ (N+1)}]$

n	1	2	3	4	5	6	7	8	9	10
Dep.	1,272.6	1,144	1,018.18	890.9	763.6	636.36	509.09	381.81	254.54	127.27
B.V	6,727.4	5,581.95	4,563.77	3,672.87	2,909.27	2,272.90	1763.80	1381.98	1127.43	1,000.27

3- Declining Balance Method

D = 1- $\sqrt[10]{\frac{1}{8}}$ = 0.188 = 18.8%

D_1 =8,000 x 0.188 = 1,500

$B.V_1 = 8{,}000 - 1{,}500 = 6{,}500$

$D_2 = 6{,}500\ X0.188 = 1{,}222$

$B.V_2 = 6{,}500 - 1{,}222 = 6{,}500$

$D_t = B.V_{t-1}xk$

$B.V_t = B.V_{t-1} - D_t$

n	1	2	3	4	5	6	7	8	9	10
Dep.	1,500	1,222	992.26	805.71	654.24	531.24	431.36	350.27	284.41	230.94
B.V	6,500	5,278	4,285.73	3,480.01	2,825.76	2,294.51	1863.14	1,512.86	1228.44	997.49

4- Sinking Fund Method

$D = (P - S)X\ Factor\ (\frac{i}{(1+i)^N - 1})$

$\mathrm{D_1 = (P - S)X\ Factor = (8{,}000 - 1{,}000)x0.06274 = 439.18}$

$B.V_1 = P - D_1$

$B.V_1 = 8{,}000 - 439.18 = 7560.82$

$439.18\ x\ Factor\ [((1+i)^{t=1}) = 1.1]$

439.18 X 1.1 = 483.098

483.098X1.1= 531.399

$D_t = (P - S)x(A/F\ , i\%,\ N)x(F/A\ , i\ \%, t)$

$D_t = D_{1\ x}\ ((1+i)^{t-1})$

$B.V_t = B.V_{t-1} - D_t$

$B.V_t = P - [(P - S)x\ (A/F,\ i\%, N\ )x(F/A\ , i\%, t\ )$

n	1	2	3	4	5	6	7	8	9	10
Dep.	439.18	483.09	531.399	584.53	642.983	707.28	778.00	855.80	941.38	1035.52
B.V	7,560.82	7077.72	6546.32	5961.78	5318.79	4611.50	3833.5	2977.7	2036.32	1000.8

<u>**Example 59**</u>

A contractor deposited amount of money in a bank for (8) eight years with rate interest of 15%. At the end of the eighth year, the amount was withdrawn and bought a construction machine which has a useful life 10 years and the salvage value of is 1 (MID) Million Iraqi Dinars and the book value at the end of the sixth year is 4.5 (MID) Million Iraqi Dinars in accordance with the sinking fund depreciation. How much was the original amount which deposited in the bank?

Solution

$\sum P = (P - S) = (P - 1.0)$

$B.V_6 = P - \sum_6^1 D$

$D_1 = (P - 1.0)x \left(\frac{i}{(1+i)^N - 1}\right) = (P - 1.0)x \left(\frac{0.1}{(1+0.1)^{10} - 1}\right)$

$= (P - 1.0)x(0.04925)$

$\sum_6^1 D = (P - 1.0)x \left(\frac{i}{(1+i)^N - 1}\right) x \left(\frac{(1+i)^t - 1}{i}\right)$

$= (P - 1.0)x0.04925x8.753 = 0.431\, P - 0.431$

$B.V_6 = P - (0.43P - 0.43) = 4.5$

$0.57\, P = 4.07$

$\therefore P_1 = 7.14$

$P_0 = F \left(\frac{1}{(1+i)^n}\right) = 7.14\, x \left(\frac{1}{(1+0.1)^8}\right) = 7.14\, x\, 0.3269 = 2.33\, MID$

another method with S.O.Y.D

$S.O.Y.D = 1 + 2 + 3 + \cdots \ldots \ldots \ldots + 10 = 55$

$\sum_6^1 D = \frac{10+9+8+7+6+5}{55} x\, (P - 1.0)$

$B.V_6 = 0.82\ x\, (P - 1.0) = 0.82\, P - 0.82 = 4.5$

$\therefore P_1 = \frac{3.68}{0.18} = 20.4$

$\therefore P_0 = 20.4x\, 0.3269 = 6.66\, MID$

ENGINEERING ALERNATIVES: DATA, METHODS OF COMPARISON AND DECISION MAKING

Many engineering problems related to business and projects lead to a variety of alternatives solution, and this requires for selection the optimum alternative among these choices.

In its economic dimension, the study is conducted for projects or engineering works that prove their usefulness in the other deportation, especially in the technical dimension.

The purpose of economic-engineering studies is usually to reduce costs or maximize profits, which imposes two problem situations:

Either the cost alternatives are either alone or they are concerned with both costs and benefits (revenues and savings). Therefore, the criterion for decision making may be to select the alternative with the lowest equivalent costs or the alternative that maximizes the net benefits.

There are many different methods and techniques for evaluation and comparisons between the alternatives for projects and engineering works, each of them has a features and constraints. It can be listed as following:-

- Present Worth Method
- Equivalent Uniform Annual Worth Method
- Rate of Return Method
- Capitalized Cost Method
- Benefit / Cost Ratio Method
- Service Life Method
- Breakeven Cost points Method
- Approximate Hoskold Method
- Minimum Cost Points Method
- Service Life When i= 0

First: The Present Worth Method

In this method all the cash flow (revenues and expenditures) will transfer to their equivalents at the beginnings of their assumed ages. It will calculate their costs for each alternative alone. Then the comparison will be considered for all these equivalents together in the same time for selecting the best one accordance with criterions.

There are three types (kinds) of application of this method as following:-

1- Equal – Lived Alternatives
2- Different – Lived Alternatives
3- Assumed to Last Forever

1- Present Worth Comparison of Equal – Lived Alternatives

Example 60

There is an order for purchasing a new equipment for a project and there is two equipment for purchasing this project as following: -

	Equipment 1	Equipment 2
Purchase price	$ 25,000	$ 28,000
Annual Maintenance	$ 1,800	$ 1,200
Salvage Value	$ 4,500	$ 4,500
Useful life	7 years	7 years
Rate interest	8%	8%

Which is the economical equipment for selecting to the project?

Solution

1- Costs of the Present worth for the 1st equipment

Costs	$ Cost	P.W. F	Net Amount of present Worth
1- Purchase cost	-25,000	1.0	-25000
2- Annual Maintenance 7 years	-1,800	5.206	-9370.8
3- Salvage Value	+4,500	0.5835	+ 2625.75
Total			**- $ 31,745.1**

2- Costs of the Present worth for the 2nd equipment

Costs	$ Cost	P.W. F	Net Amount of present Worth
1-Purchase cost	-28,000	1.0	-28000
2- Annual Maintenance 7 years	-1,200	5.206	-6247.2
3- Salvage Value	+4,500	0.5835	2625.75
Total			- **$ 31, 621.5**

The decision: alternative B should be selected

Example 61

There is a suggestion for three alternatives for solving the technical problem in a project during the 6 years, which is the best alternative for selecting from following with rate interest 15%?

Alternative A		
Initial cost $ 90,000	Annual Operating cost $ 10,000	Salvage Value $ 20,000

Alternative B		
Initial cost $ 110,000	Annual Operating cost $ 8,000 starting from the end of 2nd year, there is additional cost $ 20,000 from the beginning of the 4th year.	There is no Salvage Value

Alternative C		
It needs a $ 30,000 at the beginning of 2 years	Annual Maintenance cost $ 2,000	There is no Salvage Value

Solution

Alternative A		
Initial cost $ 90,000	Annual Operating cost $ 10,000	Salvage Value $ 20,000
Present Worth = - $\$ 90{,}000\ x\ 1.0 - 10{,}000x\ (P/A, 15\%, 6) + 20{,}000\ x\ (P/F, 15\%, 6)$		

Present Worth = - 90,000 – 10,000 x (3.784) + 20,000 x (0.4324)= **- $ 119,192**

Alternative B		
Initial cost $ 110,000	Annual Operating cost $ 8,000 starting from the end of 2^{nd} year, there is additional cost $ 20,000 from the beginning of the 4^{th} year.	There is no Salvage Value
Present Worth = -$ 110,000 x 1.0- 20,000x (P⁄ F,15%,3) – 8,000 x (P /A , 15%, 5) x (P/ F, 15%,1)		
Present Worth = -$ 110,000 x 1.0- 20,000x (0.6575) – 8,000 x (3.352)x (0.8696) = **- $ 146,469.19**		

Alternative C		
It needs a $ 30,000 in the beginning of each 2 years.	Annual Maintenance cost $ 2,000	There is no Salvage Value
Present Worth = - $ 30,000 x 1.0-30,000 x (P/F,15%,2)- 30,000 x (P/F,15%,4)- 2,000x (P/A,15%,6)		
Present Worth = -$ 30,000 x -30,000 x (0.7561)- 30,000 x (0.5718)- 2,000x (3.784**)= - $ 77,405**		

By comparing the equivalent of the Present Worth Values for each alternative, the third alternative (C) would be the least cost.

2-Present Worth Comparison of Different – Lived Alternatives

It is necessary to achieve the uniform conditions for the compared alternatives which have different ages. It is not permissible to compare alternatives unless they have the same age. The best and common way is to find the Least Common Multiple (LCM), for these ages .

It means to repeat the age of alternative by many times equal to (LCM), this needs to repeat the cash flows for each repeated time.

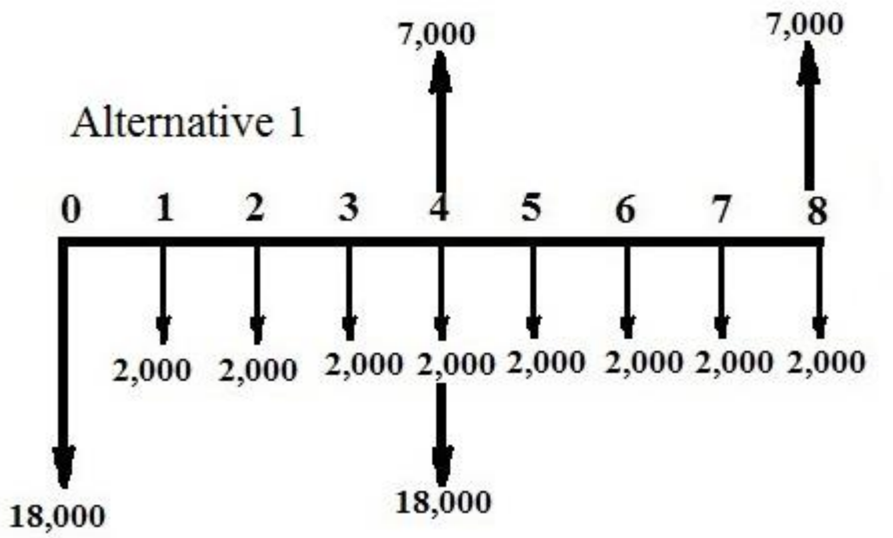
Example 62

There is two alternatives for manufacturing of products as following:-

	Alternative 1	Alternative 2
Initial Cost	$ 18,000	$ 60,000
Annual Operating	$ 2,000	$ 3,000
Salvage Value	$ 7,000	$ 30,000
Useful life	4 years	8 years
Rate interest	15%	15%

Solution

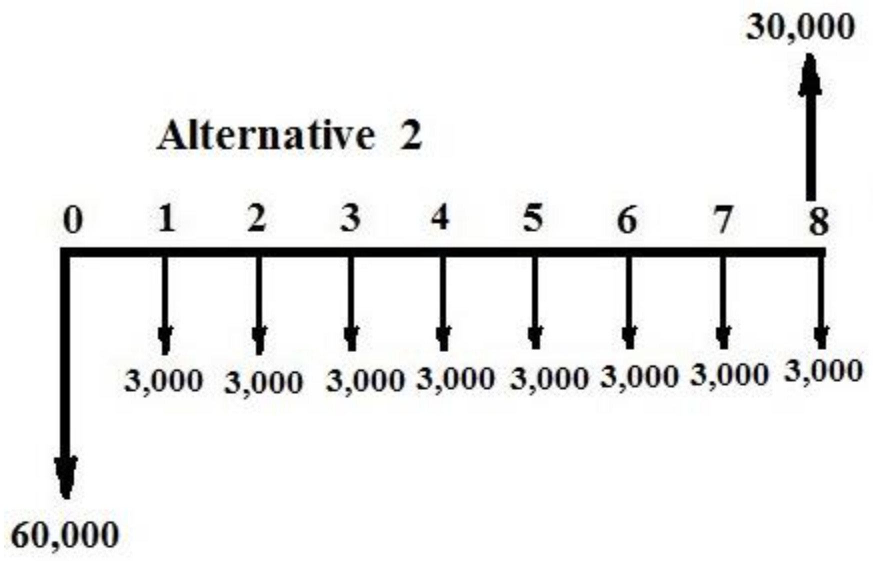
At the beginning we have to find the Least Common Multiple (LCM) for those two alternatives which = 8



The Cash flow diagram above represents that the age of this alternative (Alternative 1) was repeated two times as well as the cash flows have repeated with the same manner.

$P.W_{\,Alternative\ 1} =$
$-18{,}000 - 2{,}000\,x(\,P/A, 15\%, 8) - 18{,}000x\,(P/F, 15\%, 4) +$
$7{,}000\,x(P/F, 15\%, 4) + 7{,}000\,x(P/F, 15\%, 8) = -\$\,\mathbf{27{,}711.5}$

<u>For the Alternative 2</u>



The Cash flow diagram above represents that the age of this alternative (Alternative 2) still without any change.

$P.W_{\ Alternative\ 2} =$
$-60{,}000 - 3{,}000\ x(\ P/A, 15\%, 8) + 30{,}000x\ (P/F, 15\%, 8) =$
$-\$\ \mathbf{58{,}758}$

Alternative 1 is the best one

<u>Example 63</u>

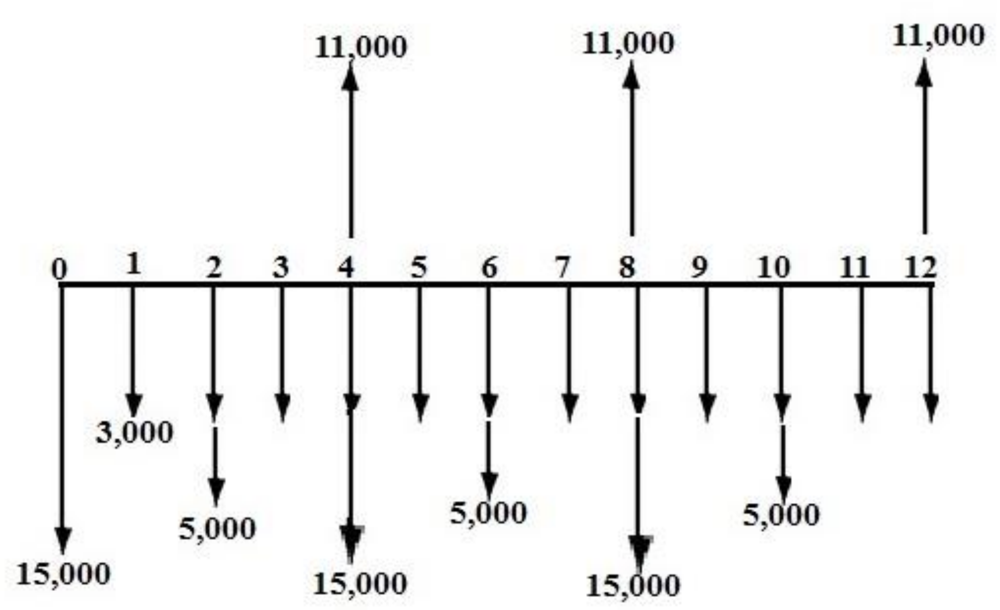
There is a plan for avoiding the flood was submitted by one of the government agencies in the implementing this project. The plan has two options (alternatives), as following information and the economic evaluation is the standard in decision-making and choose the best alternative?

	Alternative A	Alternative B
Initial Cost	\$ 15,000	\$ 45,000
Maintenance Cost (started from the end of 3rd year)	0.0	\$ 3,000
Annual Operating Cost	\$ 3,000	\$ 7,000
Extra Cost, (end of 2nd year)	\$ 5,000	\$ 18,000
Salvage Value	\$ 11,000	\$ 22,000
Useful life	4 years	6 years
Rate interest	i= 15%	i=15%

At the beginning we have to find the Least Common Multiple (LCM) for those two alternatives which = 12

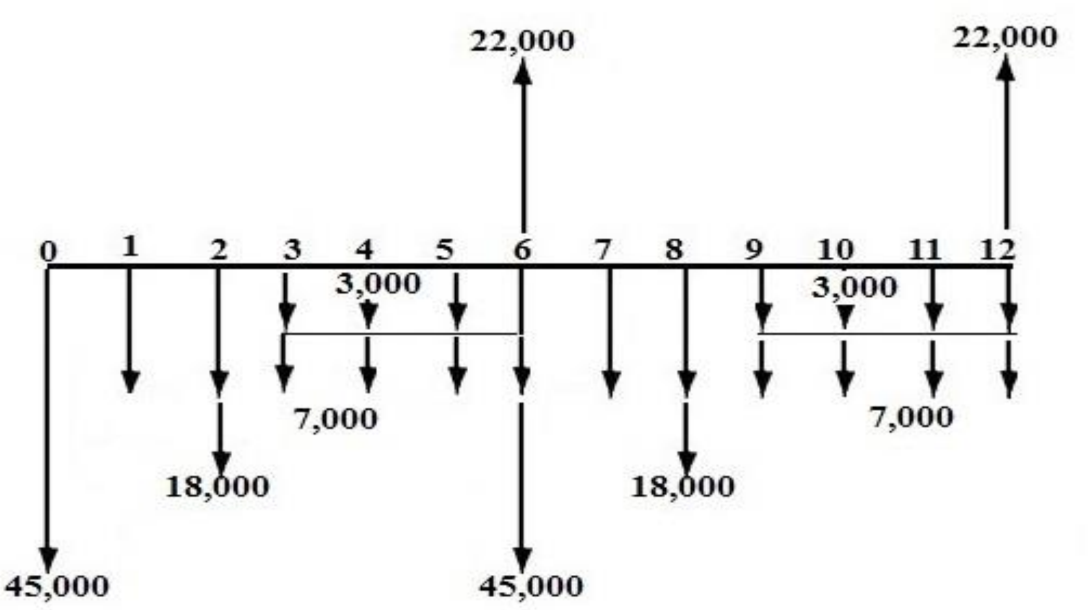
Solution

$P.W_{\ Alternative\ A}\ =$



$-15{,}000 - 5{,}000x(P/F, 15\%, 2) - 4{,}000\ x(P/F, 15\%, 4)$
$-5{,}000x(P/F, 15\%, 6) - 4{,}000\ x(P/F, 15\%, 8)$
$-5{,}000x(P/F, 15\%, 10) - 3{,}000\ x(P/A, 15\%, 12)$
$+11{,}000x(P/F, 15\%, 12) = -\$\ 39{,}979.9$

$P.W_{\ Alternative\ B}\ =$



$-45{,}000 - 18{,}000x(P/F, 15\%, 2) - 18{,}000\ x(P/F, 15\%, 8)$

$-23{,}000x(P/F, 15\%, 6) - 7{,}000x(P/A, 15\%, 12)$

$-3{,}000x(P/A, 15\%, 4)x(P/F, 15\%, 2)$

$-3{,}000\ x(P/A, 15\%, 4)\ x(P/F, 15\%, 8)$

$+22{,}000x(P/F, 15\%, 12) = -\$\ 80{,}909.29$

The best alternative is A

Second: The Equivalent uniform Annual Worth Method (EUAW)

In the case of using (EUAW) methods of alternatives of different ages there will be no uniformity of ages with (LCM) where the same equivalent values will be given as in the original ages of alternatives.

This feature makes the method (EUAW) more common and useful.

Example 64

The new project needs a construction equipment (Grader) for leveling the project site. There is two machines are described in following table.

	Machine A	Machine B
Initial Cost	$ 25,000	$ 60,000
Annual Maintenance Cost	$ 3,000	$ 4,500
Salvage Value	$ 6,000	$ 12,000
Useful life	4 years	8 years
Rate interest	i= 9%	i= 9%

$EUAW_{\ Machine\ A} =$

$-25{,}000\ (A/P\ , 9\%, 4) - 3{,}000 + 6{,}000\ x(A/F, 9\%, 4)$=
-7,885-3,000+1,292.82 = - $ 9,592.18

$EUAW_{\ Machine\ B} =$
$-60{,}000\ (A/P\ , 9\%, 8) - 4{,}500 + 12{,}000\ x(A/F, 9\%, 8)$=
-11,246.4-4,500+ 1,049.28= - $ 14,694.12
Machine A is the Best One

Rate of Return Analysis: One Project

The most commonly quoted measure of economic worth for a project or alternative is its rate of return (ROR). Whether it is an engineering project with cash flow estimates or an investment in a stock or bond, the rate of return is a well-accepted way of determining if the project or investment is economically acceptable. Compared to the PW or AW value, the ROR is a generically different type of measure of worth, as is discussed in this chapter. Correct procedures to calculate a rate of return using a PW or AW relation are explained here, as are some cautions necessary when the ROR technique is applied to a single project's cash flows.

The ROR is known by other names such as the *internal rate of return* (IROR), which is the technically correct term, and *return on investment* (ROI). We will discuss the computation of ROI in the latter part of this chapter.

In some cases, more than one ROR value may satisfy the PW or AW equation. This chapter describes how to recognize this possibility and an approach to find the **multiple values.**

Alternatively, one reliable ROR value can be obtained by using additional information established separately from the project cash flows. Two of the techniques are covered: the modified ROR technique and the ROIC (return on invested capital) technique.

Only one alternative is considered here; Chapter 8 applies these same principles to multiple alternatives. Finally, the rate of return for a bond investment is discussed.

From the perspective of someone who has borrowed money, the interest rate is applied to the *unpaid balance* so that the total loan amount and interest are paid in full exactly with the last loan payment. From the perspective of a lender of money, there is an *unrecovered balance* at each time period. The interest rate is the return on this unrecovered balance so that the total amount lent and the interest are recovered exactly with the last receipt. *Rate of return* describes both of these perspectives.

Rate of return (ROR) is the rate paid on the **unpaid balance of borrowed money,** or the rate earned on the **unrecovered balance of an investment,** so that the final payment or receipt brings the **balance to exactly zero** with interest considered.

The rate of return is expressed as a percent per period, for example, i = 10% per year. It is stated as a positive percentage; the fact that interest paid on a loan is actually a negative rate of return from the borrower's perspective is not considered. The numerical value of i can range from _100% to infinity, that is, _100% $\leq i \leq \infty$. In terms of an investment, a return of i = 100% means the entire amount is lost.

The definition above does not state that the rate of return is on the initial amount of the investment; rather it is on the **unrecovered balance,** which changes each time period. The following Example illustrates this difference.

Example 65

To get started in a new telecommuting position with AB Hammond Engineers, Jane took out a $1000 loan at i = 10% per year for 4 years to buy home office equipment. From the lender's perspective, the investment in this young engineer is expected to produce an equivalent net cash flow of $315.47 for each of 4 years.

A = $1000 (A / P, 10%, 4) = $315.47

This represents a 10% per year rate of return on the unrecovered balance. Compute the amount of the unrecovered investment for each of the 4 years using (a) the rate of return on the unrecovered balance (the correct basis) and (b) the return on the initial $1000 investment. (c) Explain why the entire initial $1000 amount is not recovered by the final payment in part (b).

Solution

a- Table below shows the unrecovered balance at the end of each year in column 6 using the 10% rate on the *unrecovered balance at the beginning of the year.* After 4 years the total $1000 is recovered, and the balance in column 6 is exactly zero.

TABLE	Unrecovered Balances Using a Rate of Return of 10% on the Unrecovered Balance				
1	**2**	**3**	**4**	**5**	**6**
		= 0.10 x 2		= 4 - 3	= 2+5
Year	Beginning Unrecovered Balance	Interest on Unrecovered Balance	Cash Flow	Recovered Amount	Ending Unrecovered Balance
0	------	--------	$_1000.00	—	$ - 1000.00
1	$ - 1000.00	$100.00	+315.47	$215.47	-784.53
2	-784.53	78.45	+315.47	237.02	- 547.51
3	- 547.51	54.75	+315.47	260.72	- 286.79
4	-286.79	28.68	+315.47	286.79	0
		$ 261.88		$ 1000.00	

TABLE	Unrecovered Balances Using a Rate of Return of 10% on the Unrecovered Balance				
1	**2**	**3**	**4**	**5**	**6**
		= 0.10 x 2		= 4 - 3	= 2+5
Year	Beginning Unrecovered Balance	Interest on Unrecovered Balance	Cash Flow	Recovered Amount	Ending Unrecovered Balance
0	------	--------	$_1000.00	—	$ - 1000.00
1	$ - 1000.00	$100.00	+315.47	$215.47	-784.53
2	-784.53	$100.00	+315.47	$215.47	- 569.06
3	- 569.06	$100.00	+315.47	$215.47	-353.59
4	-353.59	$100.00	+315.47	$215.47	-138.12
		$ 400.00		$ 861.88	

b- Table 7–2 shows the unrecovered balance if the 10% return is always figured on the *initial $1000.* Column 6 in year 4 shows a remaining unrecovered amount of $138.12, because only $861.88 is recovered in the 4 years (column 5).
c- As shown in column 3, a total of $400 in interest must be earned if the 10% return each year is based on the initial amount of $1000. However, only $261.88 in interest must be earned if a 10% return on the unrecovered balance is used. There is more of the annual cash flow available to reduce the remaining loan when the rate is applied to the unrecovered balance as in part (**a**) and first Table. The Figure below illustrates the correct interpretation of rate of return in first Table.

Rate of Return Calculation Using a PW or AW Relation

The ROR value is determined in a generically different way compared to the PW or AW value for a series of cash flows. For a moment, consider only the present worth relation for a cash flow series. Using the MARR, which is established independent of any particular project's cash flows, a mathematical relation determines the PW value in actual monetary units, say, dollars or euros.
For the ROR values calculated in this and later sections, only the cash flows themselves are used to determine an interest rate that balances the present worth relation. Therefore, ROR may be considered a relative measure, while PW and AW are absolute measures. Since the resulting interest rate depends only on the cash flows themselves, the correct term is internal rate of return (IROR); however, the term ROR is used interchangeably. Another definition of rate of return is based on our previous interpretations of PW and AW.

The *rate of return* is the interest rate that makes the present worth or annual worth of a cash flow series exactly equal to 0.

To determine the rate of return, develop the ROR equation using either a PW or AW relation, set it equal to 0, and solve for the interest rate. Alternatively, the present worth of cash outflows (costs and disbursements) PW O may be equated to the present worth of cash inflows (revenues and savings) PW I. That is, solve for *i* using either of the relations

$$\mathbf{0 = PW} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(42)}$$

Or

$$\boldsymbol{PW_O = PW_I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(43)}$$

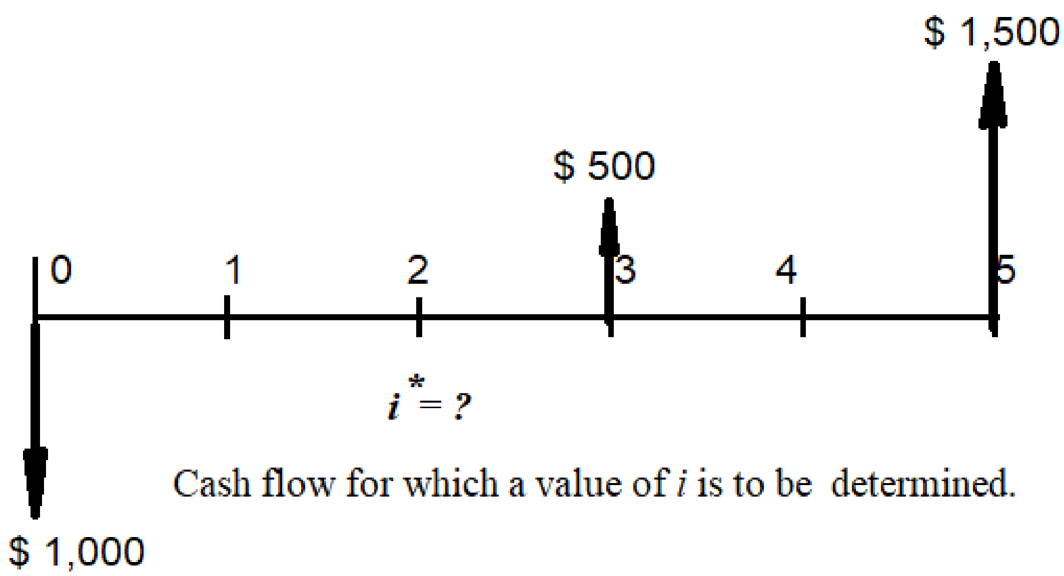
The annual worth approach utilizes the AW values in the same fashion to solve for *i* .

$$\mathbf{0 = AW} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(44)}$$

Or

$$\boldsymbol{PA_O = PA_I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(45)}$$

The *i* value that makes these equations numerically correct is called *i* *. It is the root of the ROR relation. To determine if the investment project's cash flow series is viable, compare *i* * with the established MARR.



Cash flow for which a value of *i* is to be determined.

Project Evaluation If $i^* \geq$ MARR, accept the project as economically viable.
If $i^* \leq$ MARR, the project is not economically viable.

The purpose of engineering economy calculations is *equivalence* in PW or AW terms for a stated *i* = 0%. In rate of return calculations, the objective is to *find the interest rate i** at which the cash flows are equivalent.. For example, if you deposit $1000 now and are promised payments of $500 three years from now and $1500 five years from now, the rate of return relation using PW factors and Equation 42 is

$$1{,}000 = 500\ (P/F, i^*, 3) + 1{,}500(P/F, i^*, 5)$$

The value of *i* * that makes the equality correct is to be determined (see Figure above). If the $1000 is moved to the right side of Equation above, we have the form 0 = PW.

$0 = -1{,}000 + 500\ (P/F, i^*, 3) + 1{,}500(P/F, i^*, 5)$

The equation is solved for *i* * = 16.9% by hand using trial and error or using a spreadsheet function.

The rate of return will always be greater than zero if the total amount of cash inflow is greater than the total amount of outflow, when the time value of money is considered. Using *i* * =16.9%, a graph similar to Figure above can be constructed. It will show that the unrecovered balances each year, starting with $-1000 in year 1, are exactly recovered by the $500 and $1500 receipts in years 3 and 5.

It should be evident that rate of return relations are merely a rearrangement of a present worth equation. That is, if the above interest rate is known to be 16.9%, and it is used to find the present worth of $500 three years from now and $1500 five years from now, the PW relation is

$$PW = 500\ (P/F, 16.9\%, 3) + 1{,}500(P/F, 16.9\%, 5) = \$\ 1{,}000$$

This illustrates that rate of return and present worth equations are set up in exactly the same fashion.

The only differences are what is given and what is sought.

There are several ways to determine *i* * once the PW relation is established: solution via trial and error by hand, using a programmable calculator, and solution by spreadsheet function. The spreadsheet is faster; the first helps in understanding how ROR computations work. We summarize two methods here.

i* Using Trial and Error The general procedure of using a PW-based equation is as follows:

1. Draw a cash flow diagram.

2. Set up the rate of return equation in the form of Equation 42 or 43 .

3. Select values of *i* by trial and error until the equation is balanced.

When the trial-and-error method is applied to determine *i* *, it is advantageous in step 3 to get fairly close to the correct answer on the first trial. If the cash flows are combined in such a manner that the income and disbursements can be represented by a *single factor* such as *P*/*F* or *P*/*A*, it is possible to look up the interest rate (in the tables) corresponding to the value of that factor for *n* years. The problem, then, is to combine the cash flows into the format of only one of the factors.

This may be done through the following procedure:

1. Convert all *disbursements* into either single amounts (P or *F)* or uniform amounts (A*)* by neglecting the time value of money. For example, if it is desired to convert an *A* to an *F* value, simply multiply the *A* by the number of years *n.* The scheme selected for movement of cash flows should be the one that minimizes the error caused by neglecting the time value of money. That is, if most of the cash flow is an *A* and a small amount is an *F*, convert the *F* to an *A* rather than the other way around.

2. Convert all *receipts* to either single or uniform values.

3. Having combined the disbursements and receipts so that a *P*/*F*, *P*/*A*, or *A*/*F* format applies, use the interest tables to find the approximate interest rate at which the *P*/*F* , *P*/*A* , or *A*/*F* value is satisfied. The rate obtained is a good estimate for the first trial.

It is important to recognize that this first-trial rate is only an *estimate* of the actual rate of return, because the time value of money is neglected. The procedure is illustrated in next Example **(66).**

<u>i* by Spreadsheet</u> The fastest way to determine an *i* * value when there is a series of equal cash flows (*A* series) is to apply the RATE function. This is a powerful one-cell function, where it is acceptable to have a separate *P* value in year 0 and a separate *F* value in year *n* . The format is

$$\boldsymbol{RATE = (n, A, P, F)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boldsymbol{(46)}$$

When cash flows vary from year to year (period to period), the best way to find *i* * is to enter the net cash flows into contiguous cells (including any $0 amounts) and apply the IRR function in any cell. The format is

$$\boldsymbol{= \ IRR\ (first\ \_cell{:}\ last\_cell, guses)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boldsymbol{(46)}$$

Where “guess” is the *i* value at which the function starts searching for *i* *.

The PW-based procedure for sensitivity analysis and a graphical estimation of the *i* * value is as follows:

1. Draw the cash flow diagram.
2. Set up the ROR relation in the form of Equation 42 , PW = 0.
3. Enter the cash flows onto the spreadsheet in contiguous cells.
4. Develop the IRR function to display *i* *.

5. Use the NPV function to develop a PW graph (PW versus i values). This graphically shows the i * value at which PW= 0.

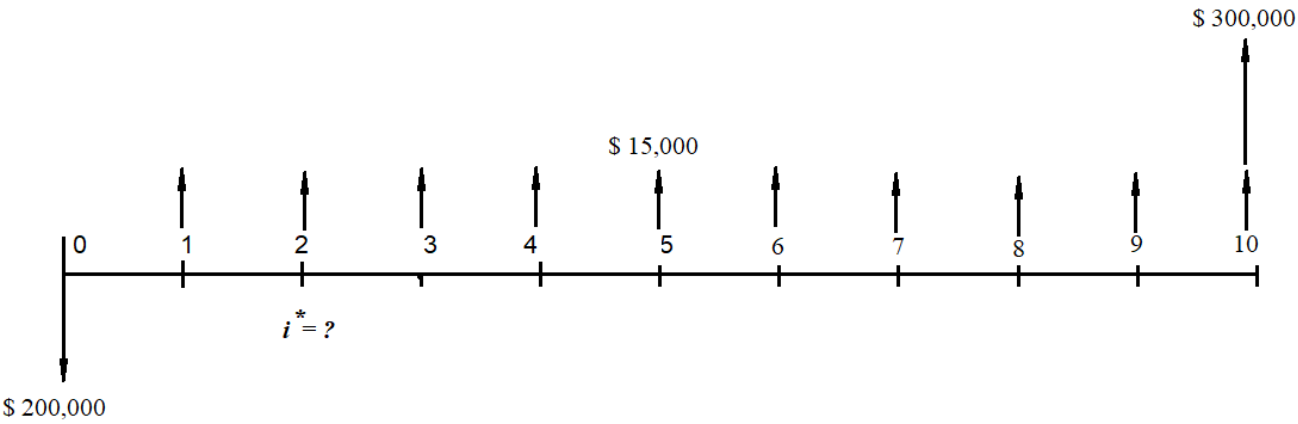
Example 90

Applications of green, lean manufacturing techniques coupled with value stream mapping can make large financial differences over future years while placing greater emphasis on environmental factors. Engineers with Monarch Paints have recommended to management an investment of \$200,000 now in novel methods that will reduce the amount of wastewater, packaging materials, and other solid waste in their consumer paint manufacturing facility. Estimated savings are \$15,000 per year for each of the next 10 years and an additional savings of \$300,000 at the end of 10 years in facility and equipment upgrade costs. Determine the rate of return using hand and spreadsheet solutions.

Solution by Hand

Use the trial-and-error procedure based on a PW equation.
1. Figure below shows the cash flow diagram.
2. Use Equation **42** format for the ROR equation.

$$0 = -200{,}000 + 1{,}5000\ (P/A, i^*, 10) + 300{,}00(P/F, i^*, 10) \text{……….. } \mathbf{(47)}$$



3. Use the estimation procedure to determine i for the first trial. All income will be regarded as a single F in year 10 so that the P/F factor can be used. The P/F factor is selected because most of the cash flow (\$300,000) already fits this factor and errors created by neglecting the time value of the remaining money will be minimized. Only for the first estimate of i , define P = \$200,000, n = 10, and F = 10(15,000) + 300,000 = \$450,000.
Now we can state that

$$200{,}000 = 450{,}000\ (P/F, i^*, 10)$$

$$(P/F, i, 10) = 0.444$$

The roughly estimated *i* is between 8% and 9%. Use 9% as the first trial because this approximate rate for the *P*/*F* factor will be lower than the true value when the time value of money is considered.

$$0 = -200{,}000 + 15{,}000\ (P/A, 9\%, 10) + 300{,}000(P/F, 9\%, 10)$$
$$0 < \$\ 22{,}986$$

The result is positive, indicating that the return is more than 9%. Try *i* = 11%.

$$0 = -200{,}000 + 15{,}000\ (P/A, 11\%, 10) + 300{,}000(P/F, 11\%, 10)$$
$$0 > \$ - 6{,}002$$

Since the interest rate of 11% is too high, linearly interpolate between 9% and 11%.

$$r^* = 9.00 + \frac{22{,}986 - 0}{22{,}986 - (-6{,}002)}\ (2.0)$$
$$= 9.00 + 1.58 = 10.58\%$$

Solution by Spreadsheet

The fastest way to find *i* * is to use the RATE function (Equation 46). The entry = RATE (10, 15000,-200,000, 300000) displays *i* * = 10.55% per year. It is equally correct to use the IRR function. Figure below, column B, shows the cash flows and = IRR(B2:B12) function to obtain *i* *.

For a complete spreadsheet analysis, use the procedure outlined above.

1. Figure above shows cash flows.
2. Equation [47] is the ROR relation.
3. Figure below shows the net cash flows in column B.
4. The IRR function in cell B14 displays *i* * = 10.55%.
5. To graphically observe *i* * = 10.55%, column D displays the PW graph for different *i* values. The NPV function is used repeatedly to calculate PW for the *xy* scatter chart.

Year	Cash Flow	Trial i%	PW Value, $
0	-200,000		
1	15,000	5%	99,999.9999999999
2	15,000	6%	77,919.7388457558
3	15,000	7%	57,858.5107544042
4	15,000	8%	39,609.2674095269
5	15,000	9%	22,988.1075860918
6	15,000	10%	7,831.4934144296
7	15,000	11%	-6,006.1762005418
8	15,000	12%	-18,654.6835966282
IRR	10.5 %		

Example 91

Find the Rate of Return ROR (i^{*}) for machine which has the purchase cost= $\$ \, 5x10^3$, the maintenance cost $= \$ \, 0.2 \, x \, 10^3 \, per \, year$ while the income for operating of this machine is $= \$ \, 1.5x10^3$ and the salvage value (S.V) = $\$ \, 1.0x \, 10^3$. The useful life for this machine =8 $years$.

Solution

To determine the (R.O.R), assume the $i = 10\%$ and find the total present values for cash flows

Year	Cost \$ $(x \, 10^3)$	Income \$ $(x \, 10^3)$	Net cash flow	P.W.F	P.W. of cash flow
0	-5.0	-------	-5.0	1.0	-5
1	-0.2	+ 1.5	+ 1.3	0.90909	1.181817
2	-0.2	+ 1.5	+ 1.3	0.82644	1.074372
3	-0.2	+ 1.5	+ 1.3	0.75131	0.976703
4	-0.2	+ 1.5	+ 1.3	0.68301	0.887913
5	-0.2	+ 1.5	+ 1.3	0.62092	0.807196
6	-0.2	+ 1.5	+ 1.3	0.56447	0.733811
7	-0.2	+ 1.5	+ 1.3	0.51315	0.667095
8	-0.2	+ 1.5 +1.0	+ 2.3	0.4665	1.07295
					$\sum +$ 2.401857

For the positive sign we need to add the value of (i), let to be = 15%

Year	Net cash flow	P.W.F	P.W. of cash flow
0	-5.0	1.0	-5
1-7	+ 1.3	4.1608	5.40904
2	+ 2.3	0.32690	0.75187
			$\sum +$ 1.16091

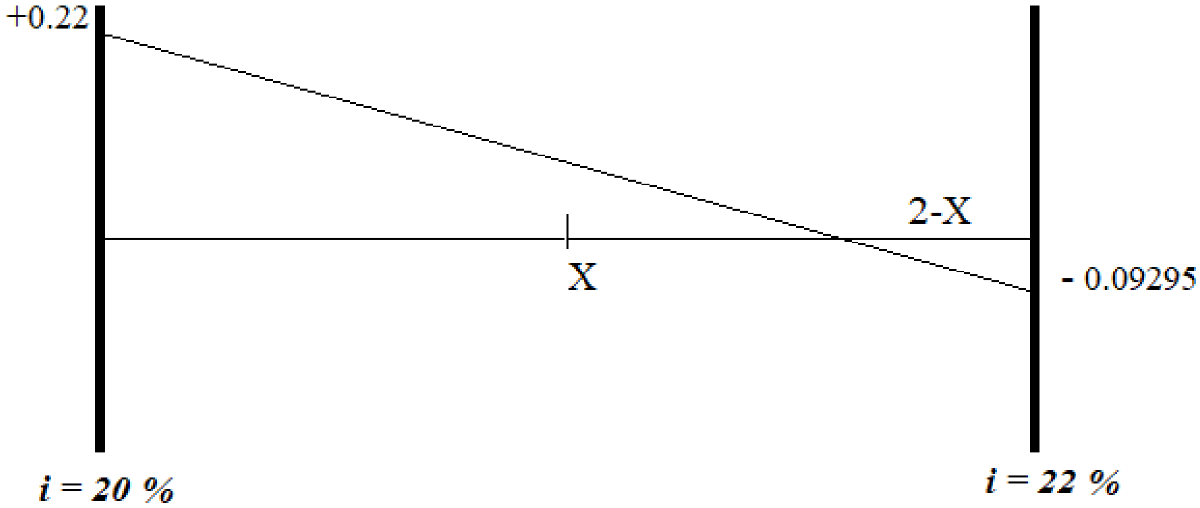
For the positive sign we need to add the value of (i), let to be = 20%

Year	Net cash flow	P.W.F	P.W. of cash flow
0	-5.0	1.0	-5.00
1-7	+ 1.3	3.6045	4.68585
2	+ 2.3	0.23256	0.534888
			$\sum +$ 0.220738

For the positive sign we need to add the value of (i), let to be = 22%

Year	Net cash flow	P.W.F	P.W. of cash flow
-5	-5.0	1.0	-5
5.40904	+ 1.3	3.4155	4.44015
0.75187	+ 2.3	0.203	0.4669
			$\sum -0.09295$

For those results to find the (R.O.R $= i^*$) we will make the proportion between those values as shown in figure below:-



$$\frac{0.22}{x} = \frac{0.09295}{2-x}$$

$$x = 1.375$$

$$R.O.R\ (i)^* = 20\ \% + 1.375 = 21.375\ \%$$

Rate of Return Analysis: Multiple Alternatives

This Section presents the methods by which two or more alternatives can be evaluated using a rate of return (ROR) comparison based on the methods of the previous section. The ROR evaluation, correctly performed, will result in the same selection as the PW and AW analyses, but the computational procedure is considerably different for ROR evaluations. The ROR analysis evaluates the increments between two alternatives in pairwise comparisons. As the cash flow series becomes more complex, spreadsheet functions help speed computations.

Why Incremental Analysis Is Necessary

When two or more mutually exclusive alternatives are evaluated, engineering economy can identify the one alternative that is the best economically. As we have learned, the PW and AW techniques can be used to do so, and are the recommended methods. Now the procedure using ROR to identify the best is presented.

Let's assume that a company uses a MARR of 16% per year that the company has $90,000 available for investment, and that two alternatives (A and B) are being evaluated. Alternative A requires an investment of $50,000 and has an internal rate of return iA * of 35% per year. Alternative B requires $85,000 and has an iB * of 29% per year. Intuitively we may conclude that the better alternative is the one that has the larger return, A in this case. However, this is not necessarily so. While A has the higher projected return, its initial investment ($50,000) is much less than the total money available ($90,000). What happens to the investment capital that is left over? It is generally assumed that excess funds will be invested at the company's MARR, as we learned in previous section. Using this assumption, it is possible to determine the consequences of the two alternative investments. If alternative A is selected, $50,000 will return 35% per year. The $40,000 left over will be invested at the MARR of 16% per year. The rate of return on the total capital available, then, will be the weighted average. Thus, if alternative A is selected,

$$\textbf{Overall } ROR_A = \frac{50,000(0.35) + 40,000(0.16)}{90,000} = 26.6\%$$

If alternative B is selected, $85,000 will be invested at 29% per year, and the remaining $5,000 will earn 16% per year. Now the weighted average is

$$\textbf{Overall } ROR_B = \frac{85,000(0.29) + 5,000(0.16)}{90,000} = 28.3\%$$

These calculations show that even though the i * for alternative A is higher, alternative B presents the better overall ROR for the $90,000. If either a PW or AW comparison is conducted using the MARR of 16% per year as i , alternative B will be chosen.

This simple example illustrates a major fact about the rate of return method for ranking and comparing alternatives:

Under some circumstances, project ROR values do not provide the same ranking of alternatives as do PW and AW analyses. This situation does not occur if we conduct an **incremental ROR analysis** (discussed below).

When **independent projects** are evaluated, no incremental analysis is necessary between projects. Each project is evaluated separately from others, and more than one can be selected.
Therefore, the only comparison is with the do-nothing alternative for each project. The project ROR can be used to accept or reject each one.

Calculation of Incremental Cash Flows for ROR Analysis

To conduct an incremental ROR analysis, it is necessary to calculate the **incremental cash flow** series over the lives of the alternatives. Based upon the equivalence relations (PW and AW), ROR evaluation makes the equal-service assumption.

Table	Format for Incremental Cash Flow Tabulation		
year	Cash Flow		Incremental Cash Flow
	Alternative A (1)	Alternative B (2)	(3) = (2) - (1)
0			
1			
2			
3			

Equal-service requirement	The incremental ROR method requires that the equal-service requirement be met. Therefore, the **LCM (least common multiple) of lives for each pairwise comparison** must be used. All the assumptions of equal service present for PW analysis are necessary for the incremental ROR method.

A format for hand or spreadsheet solutions is helpful (above). Equal-life alternatives have *n* years of incremental cash flows, while unequal-life alternatives require the LCM of lives for analysis. At the end of each life cycle, the salvage value and initial investment for the next cycle must be included for the LCM case.
When a **study period** is established, only this number of years is used for the evaluation. All incremental cash flows outside the period are neglected. As we learned earlier, using a study period, especially one shorter than the life of either alternative, can change the economic decision from that rendered when the full lives are considered. Only for the purpose of simplification, use the convention that between two alternatives, the one with the *larger initial investment* will be regarded as *alternative B.* Then, for each year in Table above,

$$\textbf{Incremental cash flow} = \textbf{cash flow}_B - \textbf{cash flow}_A \ldots\ldots\ldots\ldots\ldots\ldots..(48)$$

The initial investment and annual cash flows for each alternative (excluding the salvage value):
Revenue alternative, where there are both negative and positive cash flows

Cost alternative, where all cash flow estimates are negative.
In either case, Equation [48] is used to determine the incremental cash fl ow series with the sign of each cash flow carefully determined.

Example 92

A tool and die company in Hanover is considering the purchase of a drill press with fuzzy-logic software to improve accuracy and reduce tool wear. The company has the opportunity to buy a slightly used machine for $15,000 or a new one for $21,000. Because the new machine is a more sophisticated model, its operating cost is expected to be $7000 per year, while the used machine is expected to require $8200 per year. Each machine is expected to have a 25-year life with a 5% salvage value. Tabulate the incremental cash flow.

Solution

Incremental cash flow is tabulated in Table 8–2. The subtraction performed is (new – used) since the new machine has a larger initial cost. The salvage values in year 25 are separated from ordinary cash flow for clarity. When disbursements are the same for a number of consecutive years, *for hand solution only* , it saves time to make a single cash flow listing, as is done for years 1 to 25. However, remember that several years were combined when performing the analysis. This approach *cannot be used for spreadsheets*, when the IRR or NPV function is used, as each year must be entered separately.

Table	Format for Incremental Cash Flow Tabulation		
year	Cash Flow		Incremental Cash Flow
	Used Press	New Press	(New- Used)
0	$ - 15,000	$-21,000	$ - 6,000
1-25	-8,200	-7,000	+1,200
25	+750	+1,050	+300

Example 93

A sole-source vendor can supply a new industrial park with large transformers suitable for underground utilities and vault-type installation. Type A has an initial cost of $70,000 and a life of 8 years. Type B has an initial cost of $95,000 and a life expectancy of 12 years.
The annual operating cost for type A is expected to be $9000, while the AOC for type B is expected to be $7000. If the salvage values are $5000 and $10,000 for type A and type B, respectively, tabulate the incremental cash flow using their LCM for hand and spreadsheet

Solution

Solution by Hand

The LCM of 8 and 12 is 24 years. In the incremental cash flow tabulation for 24 years
(Table 8–3), note that the reinvestment and salvage values are shown in years 8 and 16 for type A and in year 12 for type B.

Solution by Spreadsheet

Figure below shows the incremental cash flows for the LCM of 24 years. As in the hand tabulation, reinvestment is made in the last year of each intermediate life cycle. The incremental values in column D are the result of subtractions of column B from C.
Note that the final row includes a summation check. The total incremental cash flow should agree in both the column D total and the subtraction C29 _ B29. Also note that the incremental values change signs three times, indicating the possibility of multiple *i** values, per Descartes' rule of signs. This possible dilemma is discussed later in the chapter.

Table	Format for Incremental Cash Flow Tabulation		
year	Cash Flow		Incremental Cash Flow
	Type A	Type B	(B - A)
0	$-70,000	$ -95,000	$-25,000
1-7	-9,000	–7,000	-2,000
8	-70,000		
	-9,000	–7,000	+ 67,000
	+5,000		
9-11	-9,000	–7,000	+ 2,000
12		–95,000	
	-9,000	–7,000	-83,000
		+10,000	
13–15	-9,000	–7,000	+ 2,000
16	-70,000		
	-9,000	–7,000	+ 67,000
	+5,000		
17–23	- 9,000	–7,000	-2,000
24	-9,000	–7,000	+ 2,000
	+5,000	+10,000	+ 7,000
	- $ 411,000	$ - 338,000	$+ 73,000

The table below illustrates the Spreadsheet computation of incremental cash flows for unequal-life alternatives, Example above.

Year	Type A	Type B	Incremental cash flow (B-A)
0	-70,000	-95,000	-25,000
1	-9,000	-7,000	2,000
2	-9,000	-7,000	2,000
3	-9,000	-7,000	2,000
4	-9,000	-7,000	2,000
5	-9,000	-7,000	2,000
6	-9,000	-7,000	2,000
7	-9,000	-7,000	2,000
8	-74,000	-7,000	67000
9	-9,000	-7,000	2,000
10	-9,000	-7,000	2,000
11	-9,000	-7,000	2,000
12	-9,000	-92,000	-83000
13	-9,000	-7,000	2,000
14	-9,000	-7,000	-83,000
15	-9,000	-7,000	2,000
16	-74,000	-7,000	67000
17	-9,000	-7,000	2,000
18	-9,000	-7,000	2,000
19	-9,000	-7,000	2,000
20	-9,000	-7,000	2000
21	-9,000	-7,000	2,000
22	-9,000	-7,000	2,000
23	-9,000	-7,000	2000
24	-4,000	3,000	7,000
	-411,000	-338,000	73000

Starting new life cycle for A
= initial cost + AOC + Salvage
= - 70,000 - 9,000 + 5,000

= -74,000

Interpretation of Rate of Return on the Extra Investment

The incremental cash flows in year 0 of Tables 8–2 and 8–3 reflect the *extra investment* or *cost* required if the alternative with the larger first cost is selected. This is important in an incremental ROR analysis in order to determine the ROR earned on the extra funds expended for the larger investment alternative. If the incremental cash flows of the larger investment don't justify it, we must select the cheaper one. In Example 8.1 the new drill press requires an extra investment of \$6000 (Table 8–2). If the new machine is purchased, there will be a "savings" of \$1200 per year for 25 years, plus an extra \$300 in year 25. The decision to buy the used or new machine can be made on the basis of the profitability of investing the extra \$6000 in the new machine. If the equivalent worth of the savings is greater than the equivalent worth of the extra investment at the MARR, the extra investment should be made (i.e., the larger first-cost proposal should be accepted). On the other hand, if the extra investment is not justified by the savings, select the lower-investment proposal.

It is important to recognize that the rationale for making the selection decision is the same as if only *one alternative* were under consideration, that alternative being the one represented by the incremental cash flow series. When viewed in this manner, it is obvious that unless this investment yields a rate of return equal to or greater than the MARR, the extra investment should not be made. As further clarification of this extra investment rationale, consider the following: The rate of return attainable through the incremental cash flow is an alternative to investing at the MARR. Section 8.1 states that any excess funds not invested in the alternative are assumed to be invested at the MARR. The conclusion is clear:

ME alternative selection	If the rate of return available through the incremental cash flow equals or exceeds the MARR, the alternative associated with the extra investment should be selected.

Not only must the return on the extra investment meet or exceed the MARR, but also the return on the investment that is common to both alternatives must meet or exceed the MARR. Accordingly, prior to performing an incremental ROR analysis, it is advisable to determine the internal rate of return i^* for each alternative. This can be done only for *revenue alternatives*, because cost alternatives have only cost (negative) cash flows and no i^* can be determined. The guideline is as follows:

For multiple revenue alternatives, calculate the internal rate of return i^* for each alternative, and eliminate all alternatives that have an $i^* <$ MARR. Compare the remaining alternatives incrementally.

As an illustration, if the MARR = 15% and two alternatives have i^* values of 12% and 21%, the 12% alternative can be eliminated from further consideration. With only two alternatives, it is obvious that the second one is selected. If both alternatives have $i^* <$ MARR, no alternative is justified and the do-nothing alternative is the best economically. When three or more alternatives are evaluated, it is usually worthwhile, but not required, to calculate i^* for each alternative for preliminary screening. Alternatives that cannot meet the MARR may be eliminated from further evaluation using this option. This option is especially useful when performing the analysis by spreadsheet. The IRR function applied to each alternative's cash flow estimates can quickly indicate unacceptable alternatives, as demonstrated in Section 8.6.

When **independent projects** are evaluated, there is no comparison on the extra investment.

The ROR value is used to accept all projects with $i^* \geq$ MARR, assuming there is no budget limitation. For example, assume MARR = 10%, and three independent projects are available with ROR values of

$i^*_A = 12\%$ $\quad i^*_B = 9\%$ $\quad i^*_C = 23\%$

Projects A and C are selected, but B is not because $i^*_B <$ MARR.