Chapter Four

Qualitative Solutions to Models for Two Interacting Species

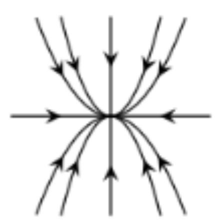
In this chapter we develop some powerful theory, which allows us to predict the dynamics of a system in general terms. It provides the means by which we can establish the phase plane behavior of a system and predict the outcome for any possible parameter combination.

Linear theory

Both the analytic form of the solution and the solution's stability depend on the eigenvalues of the matrix A where the linear system is $\dot{X} = AX$.

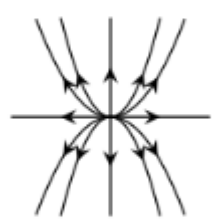
Two distinct real eigenvalues

1) If $\lambda_1 < \lambda_2 < 0$ then the equilibrium point is stable node (sink).



Stable node (sink)

2) If $\lambda_1 > \lambda_2 > 0$ then the equilibrium point is unstable node (source).



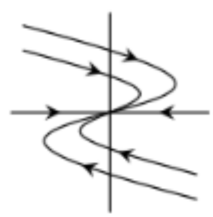
Unstable node (source)

3) If $\lambda_1 < 0 < \lambda_2$ then the equilibrium point is saddle.

Saddle

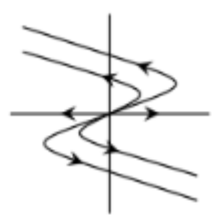
Real eigenvalues with algebraic multiplicity two

1) If $\lambda_1 = \lambda_2 < 0$ then the equilibrium point is degenerate stable node.



Degenerate stable node

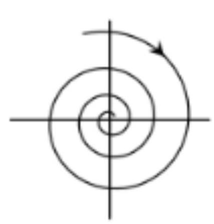
2) If $\lambda_1 = \lambda_2 > 0$ then the equilibrium point is degenerate unstable node.



Degenerate unstable node

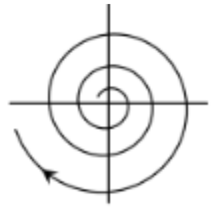
Eigenvalues with nonzero imaginary part

1) If we have $\lambda = \mu \pm iw$ where $\mu < 0$ then the equilibrium point is stable focus (spiral).



Stable focus (spiral)

2) If we have $\lambda = \mu \pm iw$ where $w > 0$ then the equilibrium point is unstable focus (spiral).



Unstable focus (spiral)

3) If $\lambda_{1,2} = \mu \pm iw$ $(\mu = 0)$ then the equilibrium point is center.

Center

Computational shortcuts for two-dimensional system

Although the classification of 2×2 systems in the previous section depended on the eigenvalues of the matrix A, one does not usually need to compute them as there are equivalent conditions that are easier to check. To see this being by writing out the characteristic polynomial of A in terms of its elements

$$\det(A - \lambda I) = \det\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$= \lambda^2 - Tr\,\lambda + \Delta \qquad \ldots (7)$$

where $Tr = Trace\,(A) = a_{11} + a_{22}$ is the trace of the matrix and $\Delta = a_{11}a_{22} - a_{12}a_{21}$ is its determinant. The eigenvalues of A are the roots of this quadratic and so satisfy

$$\lambda = \frac{-Tr \pm \sqrt{Tr^2 - 4\Delta}}{2} \qquad \ldots (8)$$

if we call the two roots λ_1 and λ_2 we can use them to factor the characteristic polynomial and obtain

$$det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \qquad \text{... (9)}$$

Comparing Eq. (7) with the last line of Eq. (9), we obtain

$$Tr = \lambda_1 + \lambda_2 \qquad \text{and} \qquad \Delta = \lambda_1\lambda_2$$

And this, along with Eq.(8), allows us to make the following observations, which are summarised in figure

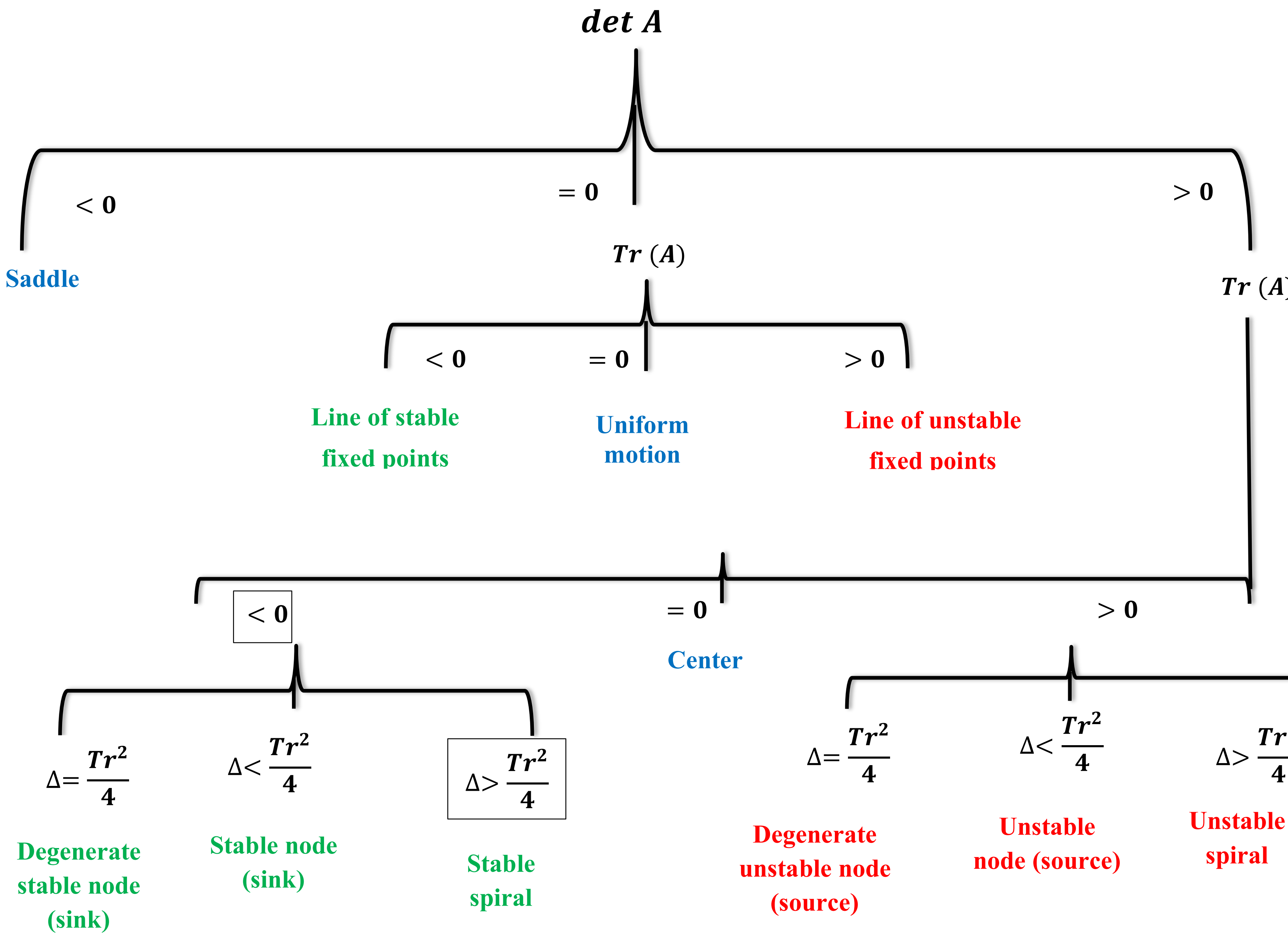
- If $\Delta < 0$ then the eigenvalues are real, nonzeroand have opposite sign: the equilibrium is thus a saddle
- The eigenvalues are distinct real number if $Tr^2 - 4\Delta > 0, (\Delta < \frac{Tr^2}{4})$, but form a complex conjugate pair if $Tr^2 - 4\Delta < 0, (\Delta > \frac{Tr^2}{4})$.
- If $\Delta > 0$ and $Tr < 0$ then the eigenvalues are either a pair of negative real number or a complex conjugate pair with negative real part. In either case, the equilibrium is stable.
- If $\Delta > 0$ and $Tr < 0$ then the eigenvalues are either a pair of postive real number or a complex conjugate pair with postive real part. In either case, the equilibrium is unstable.

The various special cases at the boundaries between the regions described are:

$*$ $\Delta = 0$: then there are one or (if $Tr = 0$) two zero eigenvalues and linear stability analysis is inconclusive and the pattern of solutions will be determined by higher-order terms in the taylor series in Eq.(3).

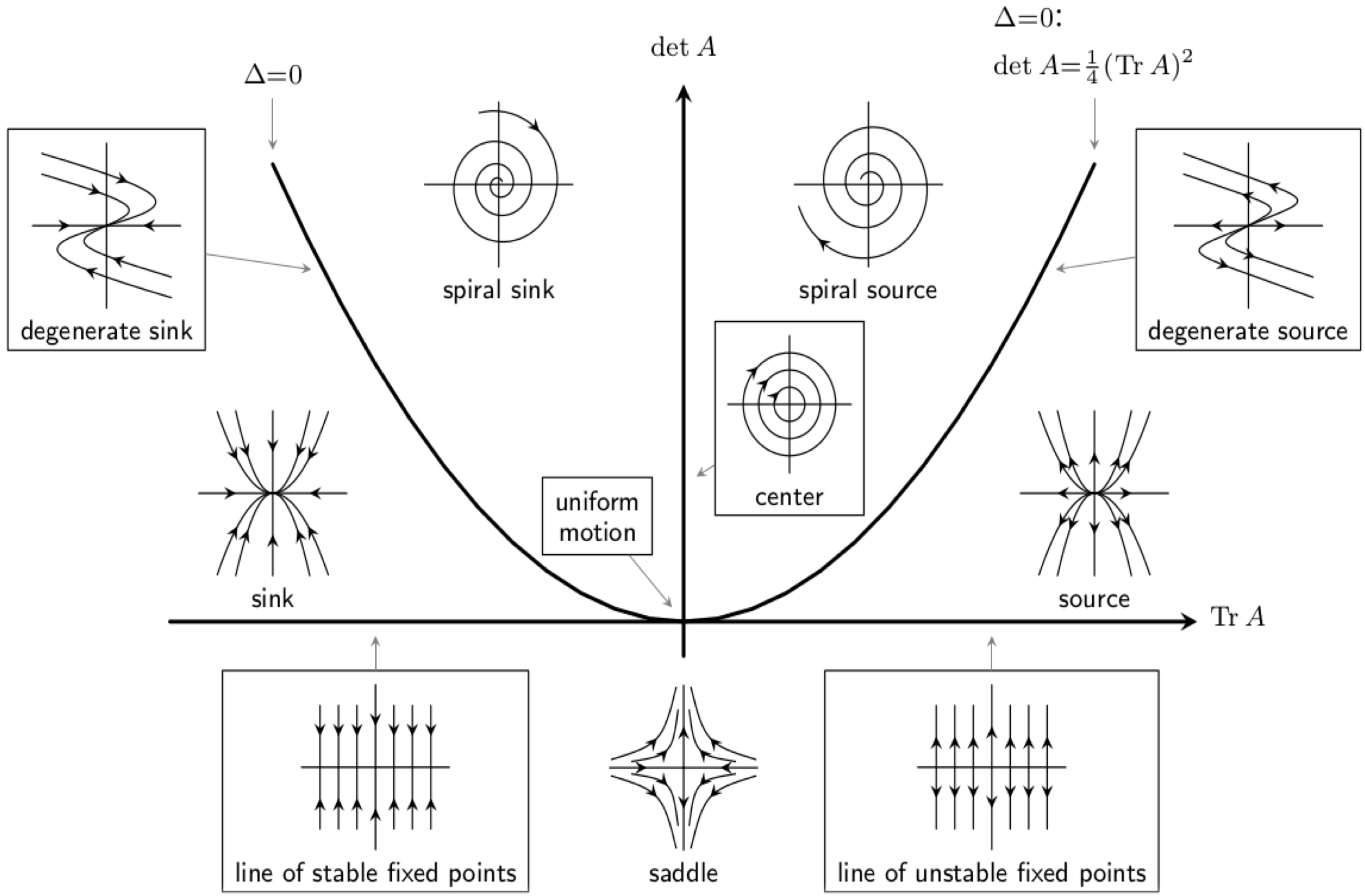
$*$ $\Delta = \left(\frac{Tr}{2}\right)^2$: the characteristic polynomial has a repeated real root and the phase portraits will be degenerate source (in case $Tr > 0$) or degenerate sink (in case $Tr < 0$).

$*$ $\Delta = 0$: and $\Delta > 0$: the eigenvalues are a pure imaginary pair and the phase portrait will be center.



det A
< 0
= 0
> 0
Saddle
Tr (A)
Tr (A)
< 0
= 0
> 0
Line of stable fixed points
Uniform motion
Line of unstable fixed points
< 0
= 0
> 0
Center
$\Delta = \frac{Tr^2}{4}$
$\Delta < \frac{Tr^2}{4}$
$\Delta > \frac{Tr^2}{4}$
Degenerate stable node (sink)
Stable node (sink)
Stable spiral
$\Delta = \frac{Tr^2}{4}$
$\Delta < \frac{Tr^2}{4}$
$\Delta > \frac{Tr^2}{4}$
Degenerate unstable node (source)
Unstable node (source)
Unstable spiral

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane



Examples:

1- Determine the phase portrait of the dynamical system

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Solution:

$$detA = 10 - (-2) = 12 > 0$$

$$Tr(A) = -2 - 5 = -7 < 0$$

$$\frac{[Tr(A)]^2}{4} = \frac{49}{4} = 12.2 > \det(A)$$

$$\therefore \Delta < \frac{Tr^2}{4}$$

∴ The equilibrium point in this system is stable node .

2- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where

$$A = \begin{bmatrix} -2 & -5 \\ 0 & -2 \end{bmatrix}.$$

Solution

$$\Delta = detA = 4 > 0$$

$$Tr(A) = -2 - 2 = -4 < 0$$

$$\frac{[Tr(A)]^2}{4} = \frac{16}{4} = 4 = \Delta$$

∴ The equilibrium point in this system is degenerate stable node.

3- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$$

Solution

$$\Delta = 1 - (-2) = 3 > 0$$

$$Tr(A) = -1 - 1 = -2 < 0$$

$$\frac{[Tr(A)]^2}{4} = \frac{4}{4} = 1 < \Delta$$

∴ The equilibrium point in this system is stable spiral.

4- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 2 & 3 \\ -2 & -2 \end{bmatrix}$

Solution:

$$\Delta = -4 - (-6) = 2 > 0$$

$$Tr(A) = 2 - 2 = 0$$

∴ The equilibrium point in this system is center.

5- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Solution:

$$\Delta = 1 - (-2) = 3 > 0$$
$$Tr(A) = 1 + 1 = 2 > 0$$

$$\frac{[Tr(A)]^2}{4} = \frac{2^2}{4} = 1 < \Delta$$

∴ The equilibrium point in this system is unstable spiral.

6- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

Solution:

$$\Delta = 2 > 0$$
$$Tr(A) = 1 + 2 = 3 > 0$$
$$\frac{[Tr(A)]^2}{4} = \frac{9}{4} = 2.5 > \Delta$$

∴ The equilibrium point in this system is unstable node.

7- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 2 & -5 \\ 0 & 2 \end{bmatrix}$

Solution:

$$\Delta = 4 > 0$$
$$Tr(A) = 4 > 0$$
$$\frac{[Tr(A)]^2}{4} = \frac{16}{4} = 4 = \Delta$$

∴ The equilibrium point in this system is degenerate unstable node.

8- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

Solution:

$$det\ A = -2 < 0$$

∴ The equilibrium point in this system is saddle point.

9- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

Solution:

$$\Delta = 0$$
$$Tr(A) = -1 < 0$$

∴ The equilibrium point in this system is lines of stable fixed points (stable star).

10- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\Delta = 0$$
$$Tr(A) = 1 > 0$$

∴ The equilibrium point in this system is line of unstable fixed points (unstable star).

11- Determine the phase portrait of the dynamical linear system $\dot{X} = AX$, where $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

Solution:

$$\Delta = -4 - (-4) = 0$$
$$Tr(A) = 0$$

∴ The equilibrium point in this system is uniform motion.

Applications of Linear Theory

Example:

Find all equilibrium points associated with the system

$$X' = Y \qquad \text{and} \qquad Y' = -w^2 X$$

and determine their classification(s).

Solution:

In matrix form, the system is

$$X' = AX \text{ where } A = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix}$$

$$Y = 0 \text{ and } -w^2 X = 0 \quad \longmapsto \quad X = 0$$

∴ The only equilibrium point is $(x_e, y_e) = (0,0)$.

To classify this point, we need the characterist eq.

$$\lambda^2 - Tr(A) + det(A) = 0$$

$$\lambda^2 - 0\lambda + w^2 = 0 \quad \longmapsto \quad \lambda^2 + w^2 = 0$$

$$\therefore \quad \lambda = \pm wi$$

Now, with $Tr(A) = 0$ and $det = w^2$

This implies that the equilibrium point (0,0) is a center.

Example:

Classify the equilibrium points for the system

$$\frac{dR}{dt} = -a_1 B, \qquad \frac{dB}{dt} = -a_2 R$$

Solution:

the system can be written in matrix form

$$X' = AX \ , \quad A = \begin{bmatrix} 0 & -a_1 \\ -a_2 & 0 \end{bmatrix}$$

and

$$\left.\begin{matrix} -a_1 B = 0 \longmapsto B = 0 \\ -a_2 R = 0 \longmapsto R = 0 \end{matrix}\right\} \to (R_e, B_e) = (0,0)$$

The characterist equation is

$$\lambda^2 - Tr(A) + det(A) = 0$$

$$\lambda^2 + (-a_1 a_2) = 0$$

with $\det(A) = -a_1 a_2 < 0$

∴ The equilibrium point is a saddle point.

A mathematical Interlude (Non Linear Theory)

Suppose we are considering a model for two interacting species that has the general form

$$\frac{du}{dt} = f(u, v) \qquad \text{and} \qquad \frac{dv}{dt} = g(u, v) \qquad \ldots (1)$$

where f and g are continuous and continuously differetiable. If (u_*, v_*) is an equilibrium of Eq. (1), so that

$$f(u_*, v_*) = 0 = g(u_*, v_*)$$

then we can investigate its stability by defining perturbation $x(t)$ and $y(t)$ such that

$$u(t) = u_* + x(t) \qquad \text{and} \qquad v(t) = v_* + y(t) \qquad \ldots (2)$$

Linearisation near an equilibrium

$$\frac{du}{dt} = \frac{d}{dt}(u_* + x(t)) = \frac{dx}{dt}$$

But then on the other hand

$$\frac{du}{dt} = \frac{dx}{dt} = f\big(u_* + x(t), v_* + y(t)\big)$$

Zero

$$\approx f(u_*, v_*) + x(t)\frac{\partial f}{\partial u}\,|_{u_*,v_*} + y(t)\frac{\partial f}{\partial v}\,|_{u_*,v_*}$$

$$= x(t)\frac{\partial f}{\partial u}\,|_{u_*,v_*} + y(t)\frac{\partial f}{\partial v}\,|_{u_*,v_*} \qquad \ldots (3)$$

In the same manner

$$\frac{dv}{dt} = \frac{dy}{dt} = x(t)\frac{\partial g}{\partial u}\,|_{u_*,v_*} + y(t)\frac{\partial g}{\partial v}\,|_{u_*,v_*} \qquad \ldots (4)$$

It's convenient to combine the results (3) and (4) in matrix form as follows

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix} \qquad \ldots(5)$$

where $A = \begin{bmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\end{bmatrix}_{u_*,v_*} \qquad \ldots(6)$

when derived form an ecological model such as the Lotka-Volterra system the matrix A defined above is called the community matrix. The stability of the equilibrium then depends on the nature of the solutions to the linear system of ODEs (1) and this depends on the eigenvalues of the matrix A

Applications of nonlinear theory

Example:

Find the linearised for model

$$X' = \beta_1 X - c_1 XY$$

$$Y' = -\alpha_2 Y + c_2 XY$$

and hence classify all equilibrium points of the basic predator-prey model.

Solution:

$$\beta_1 X - c_1 XY = 0 \longmapsto X(\beta_1 - c_1 Y) = 0$$

either $X = 0$ or $\beta_1 - c_1 Y = 0 \to Y = \frac{\beta_1}{c_1}$

$$-\alpha_2 Y + c_2 XY = 0 \longmapsto Y(-\alpha_2 + c_2 X)$$

either $Y = 0$ or $-\alpha_2 + c_2 X = 0 \to X = \frac{\alpha_2}{c_2}$

$\therefore$ $(0,0)$ and $(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1})$ are equilibrium points.

$F(X,Y) = \beta_1 X - c_1 XY \qquad\qquad G(X,Y) = -\alpha_2 Y + c_2 XY$

$$\frac{\partial F}{\partial X} = \beta_1 - c_1 Y \qquad \frac{\partial G}{\partial X} = c_2 Y$$

$$\frac{\partial F}{\partial Y} = -c_1 X \qquad \frac{\partial G}{\partial Y} = -\alpha_2 + c_2 X$$

$*$ For $(X_e, Y_e) = (0,0)$

$$\mathrm{J} = \begin{bmatrix} \beta_1 - c_1 Y & -c_1 X \\ c_2 Y & -\alpha_2 + c_2 X \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}$$

with $det(A) = -\beta_1 \alpha_2 < 0 \quad \longmapsto \quad$ (0,0)is saddle point.

$*$ For $(X_e, Y_e) = (\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1})$

$$\mathrm{J} = \begin{bmatrix} \beta_1 - c_1 \frac{\beta_1}{c_1} & -c_1 \frac{\alpha_2}{c_2} \\ c_2 \frac{\beta_1}{c_1} & -\alpha_2 + c_2 \frac{\alpha_2}{c_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{c_1 \alpha_2}{c_2} \\ \frac{\beta_1 c_2}{c_1} & 0 \end{bmatrix}$$

$Tr(A) = 0$ and $det(A) = \frac{c_1 \alpha_2}{c_2} . \frac{\beta_1 c_2}{c_1} = \alpha_2 \beta_1$

$\therefore \left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1} \right)$ is center.

Example:

Classify the equilibrium points of the epidemic model

$$\frac{dS}{dt} = -\beta SI\ ,$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

Solution:

$-\beta SI = 0 \longmapsto$ either $S = 0$ or $I = 0$

$\beta SI - \gamma I = 0 \longmapsto I(\beta S - \gamma) = 0$

$\therefore$ either $I = 0$ or $S = \frac{\gamma}{\beta}$

$\therefore$ (0,0) and $(\frac{\gamma}{\beta}, 0)$ are equilibrium points.

$F(S, I) = -\beta SI$

$$\frac{\partial F}{\partial S} = -\beta I$$

$$\frac{\partial F}{\partial I} = -\beta S$$

$G(S, I) = \beta SI - \gamma I$

$$\frac{\partial G}{\partial S} = \beta I$$

$$\frac{\partial G}{\partial I} = \beta S - \gamma$$

The Jacobian matrix is

$$\mathrm{J} = \begin{bmatrix} \frac{\partial F}{\partial S} & \frac{\partial F}{\partial I} \\ \frac{\partial G}{\partial S} & \frac{\partial G}{\partial I} \end{bmatrix} = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \gamma \end{bmatrix}$$

For $(S_e, I_e) = (0,0)$

$$J = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}$$

$Tr(A) = -\gamma$ and $det(A) = 0$

We have a line of stable fixed points.

For $(S_e, I_e) = (\frac{\gamma}{\beta}, 0)$

$$J = \begin{bmatrix} 0 & -\beta.\frac{\gamma}{\beta} \\ 0 & \beta.\frac{\gamma}{\beta} - \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma \\ 0 & 0 \end{bmatrix}$$

$Tr(A) = 0$ and $det(A) = 0$

We have a uniform motion.