4. Not all water tanks are shaped like cylinders. Suppose a tank has cross-sectional area $A(h)$ at height h. Then the volume of water up to height h is $V = \int_0^h A(u)\,du$ and so the Fundamental Theorem of Calculus gives $dV/dh = A(h)$. It follows that

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\,\frac{dh}{dt}$$

and so Torricelli's Law becomes

$$A(h)\,\frac{dh}{dt} = -a\sqrt{2gh}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we take $g = 10\ \text{m/s}^2$, show that h satisfies the differential equation

$$(4h - h^2)\,\frac{dh}{dt} = -0.0001\sqrt{20h}$$

(b) How long will it take for the water to drain completely?

9.4 Models for Population Growth

In Section 9.1 we developed two differential equations that describe population growth. In this section we further investigate these equations and use the techniques of Section 9.3 to obtain explicit models for a population.

The Law of Natural Growth

One of the models for population growth that we considered in Section 9.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

1
$$\frac{dP}{dt} = kP$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

Because Equation 1 is a separable differential equation, we can solve it by the methods of Section 9.3:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C} = e^C e^{kt}$$

$$P = Ae^{kt}$$

where A ($= \pm e^C$ or 0) is an arbitrary constant. To see the significance of the constant A, we observe that

$$P(0) = Ae^{k \cdot 0} = A$$

Therefore A is the initial value of the function.

Examples and exercises on the use of (2) are given in Section 3.8.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Another way of writing Equation 1 is

$$\frac{dP/dt}{P} = k$$

which says that the *relative growth rate* (the growth rate divided by the population size; see Section 3.8) is constant. Then (2) says that a population with constant relative growth rate must grow exponentially.

We can account for emigration (or "harvesting") from a population by modifying Equation 1: if the rate of emigration is a constant m, then the rate of change of the population is modeled by the differential equation

3 $$\frac{dP}{dt} = kP - m$$

See Exercise 17 for the solution and consequences of Equation 3.

The Logistic Model

As we discussed in Section 9.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{dP/dt}{P} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**, which we first saw in Section 9.1:

4 $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Notice from Equation 4 that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population P lies between 0 and M, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

EXAMPLE 1 Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and here we are interested only in what happens after $t = 0$.

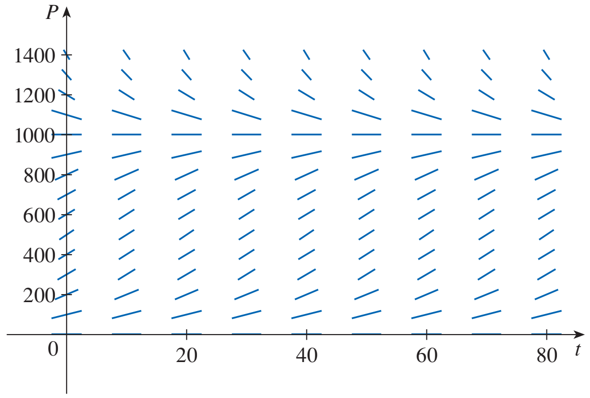


FIGURE 1
Direction field for the logistic equation in Example 1

The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$. Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing. The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$. In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500. (See Exercise 13.)

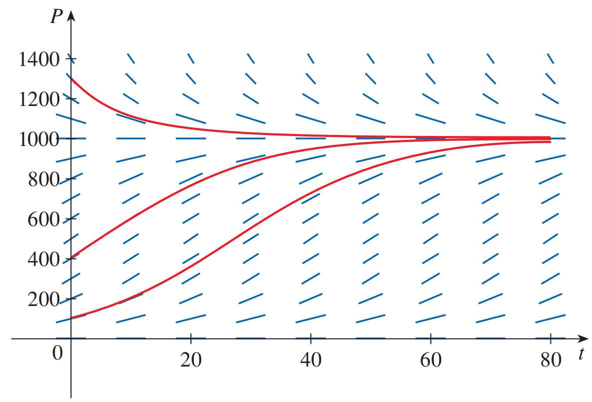


FIGURE 2
Solution curves for the logistic equation in Example 1

■

The logistic equation (4) is separable and so we can solve it explicitly using the method of Section 9.3. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

5 $$\int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions (see Section 7.4), we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k\,dt$$

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

6 $$\frac{M-P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$. Solving Equation 6 for P, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so $$P = \frac{M}{1 + Ae^{-kt}}$$

We find the value of A by putting $t = 0$ in Equation 6. If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

7 $$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}$$

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

EXAMPLE 2 Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

SOLUTION The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$. So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \qquad \text{where } A = \frac{1000 - 100}{100} = 9$$

Thus $$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and $t = 80$ are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6 \qquad P(80) = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Compare the solution curve in Figure 3 with the lowest solution curve we drew from the direction field in Figure 2.

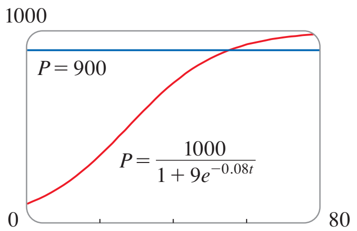


FIGURE 3

Solving this equation for t, we get

$$1 + 9e^{-0.08t} = \tfrac{10}{9}$$

$$e^{-0.08t} = \tfrac{1}{81}$$

$$-0.08t = \ln \tfrac{1}{81} = -\ln 81$$

$$t = \frac{\ln 81}{0.08} \approx 54.9$$

So the population reaches 900 when t is approximately 55. As a check on our work, we graph the population curve in Figure 3 and observe that it intersects the line $P = 900$ at $t \approx 55$. ■

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

EXAMPLE 3 Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit for each model.

SOLUTION Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt} = 2e^{0.7944t}$$

Gause used the same value of k for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$). The equation

$$\left.\frac{1}{P_0}\frac{dP}{dt}\right|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

Then the solution of the logistic equation, given in Equation 7, is

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}$$

where $$A = \frac{M - P_0}{P_0} = \frac{64 - 2}{2} = 31$$

So $$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	...										

We observe from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model. For $t \geqslant 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.

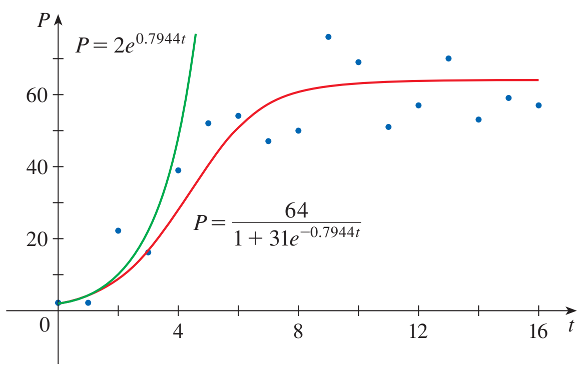


FIGURE 4
The exponential and logistic models for the *Paramecium* data

Many countries that formerly experienced exponential growth are now finding that their rates of population growth are declining and the logistic model provides a better model. The table in the margin shows midyear values of the population of Japan, in thousands, from 1960 to 2015. Figure 5 shows these data points, using $t = 0$ to represent 1960, together with a shifted logistic function (obtained from a calculator with the ability to fit a logistic function to data points by regression; see Exercise 15). At first the data points appear to be following an exponential curve but overall a logistic function provides a much more accurate model.

Year	Population (thousands)
1960	94,092
1965	98,883
1970	104,345
1975	111,573
1980	116,807
1985	120,754
1990	123,537
1995	125,327
2000	126,776
2005	127,715
2010	127,579
2015	126,920

Source: U.S. Census Bureau / International Programs / International Data Base. Revised Sept. 18, 2018. Version data 18.0822. Code 12.0321.

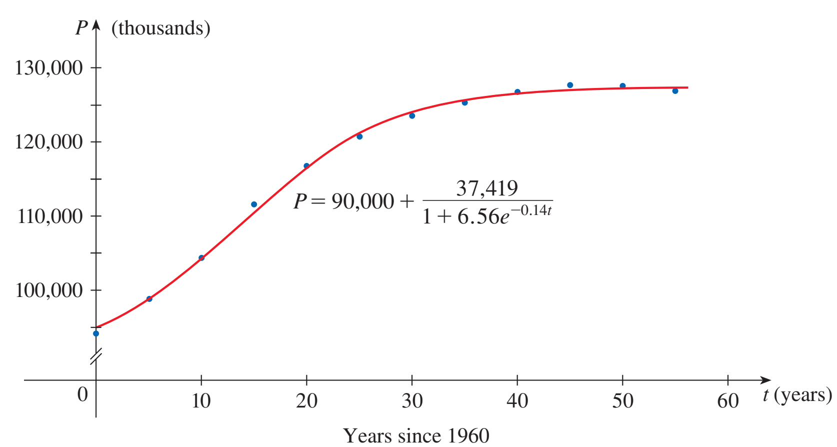


FIGURE 5
Logistic model for the population of Japan

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. In Exercise 22 we look at the Gompertz growth function and in Exercises 23 and 24 we investigate seasonal-growth models.

Two additional models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.) This equation is explored in Exercises 19 and 20.

For some species there is a minimum population level m below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, $1 - m/P$, takes into account the consequences of a sparse population (see Exercise 21).

9.4 Exercises

1–2 A population grows according to the given logistic equation, where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?
(b) Write the solution of the equation.
(c) What is the population after 10 weeks?

1. $\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{1200}\right)$, $P(0) = 60$

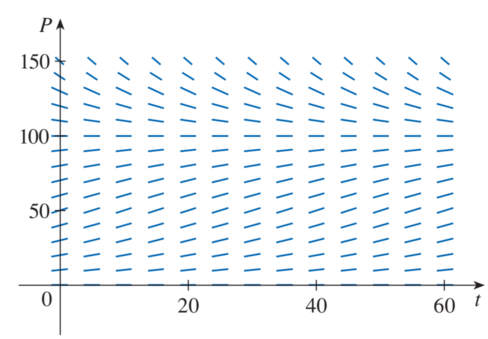
2. $\frac{dP}{dt} = 0.02P - 0.0004P^2$, $P(0) = 40$

3. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?
(b) A direction field for this equation is shown. Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



(c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?
(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

T **4.** Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.

(a) Write the logistic differential equation for these values.
(b) Draw a direction field (either by hand or with a computer). What does it tell you about the solution curves?

(c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?
(d) Program a calculator or computer to use Euler's method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.
(e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).
(f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).

5. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $k = 0.71$ per year.
(a) If $y(0) = 2 \times 10^7$ kg, find the biomass a year later.
(b) How long will it take for the biomass to reach 4×10^7 kg?

6. Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.
(a) What is the carrying capacity?
(b) What is $P'(0)$?
(c) When will the population reach 50% of the carrying capacity?

7. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

8. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.
(b) Use the data to estimate the initial relative growth rate.
(c) Find both an exponential model and a logistic model for these data.
(d) For each model, compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
(e) Use your logistic model to estimate the number of yeast cells after 7 hours.

9. The population of the world was about 6.1 billion in 2000. Birth rates around that time ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 20 billion.
(a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)
(b) Use the logistic model to estimate the world population in the year 2010 and compare with the actual population of 6.9 billion.
(c) Use the logistic model to predict the world population in the years 2100 and 2500.

10. (a) Assume that the carrying capacity for the US population is 800 million. Use it and the fact that the population was 282 million in 2000 to formulate a logistic model for the US population.
(b) Determine the value of k in your model by using the fact that the population in 2010 was 309 million.
(c) Use your model to predict the US population in the years 2100 and 2200.
(d) Use your model to predict the year in which the US population will exceed 500 million.

11. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.
(a) Write a differential equation that is satisfied by y.
(b) Solve the differential equation.
(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

12. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.
(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.
(b) How long will it take for the population to increase to 5000?

13. (a) Show that if P satisfies the logistic equation (4), then

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) Deduce that a population grows fastest when it reaches half its carrying capacity.

14. For a fixed value of M (say $M = 10$), the family of logistic functions given by Equation 7 depends on the initial value P_0 and the proportionality constant k. Graph several members of this family. How does the graph change when P_0 varies? How does it change when k varies?

T 15. A Shifted Logistic Model The table gives the midyear population P of Trinidad and Tobago, in thousands, from 1970 to 2015.

Year	Population (thousands)	Year	Population (thousands)
1970	955	1995	1264
1975	1007	2000	1252
1980	1091	2005	1237
1985	1189	2010	1227
1990	1255	2015	1222

Source: US Census Bureau / International Programs / International Data Base. Revised Sept. 18, 2018. Version data 18.0822. Code 12.0321.

(a) Make a scatter plot of these data. Choose $t = 0$ to correspond to the year 1970.
(b) From the scatter plot, it appears that a logistic model might be appropriate if we first shift the data points downward (so that the initial P-values are closer to 0). Subtract 900 from each value of P. Then use a calculator or computer to obtain a logistic model for the shifted data.
(c) Add 900 to your model from part (b) to obtain a shifted logistic model for the original data. Graph the model with the data points from part (a) and comment on the accuracy of the model.
(d) If the model remains accurate, what do you predict for the future population of Trinidad and Tobago?

T 16. The table gives the number of active Twitter users worldwide, semiannually from 2010 to 2016.

Years since January 1, 2010	Twitter users (millions)	Years since January 1, 2010	Twitter users (millions)
0	30	3.5	232
0.5	49	4.0	255
1.0	68	4.5	284
1.5	101	5.0	302
2.0	138	5.5	307
2.5	167	6.0	310
3.0	204	6.5	317

Source: www.statistica.com/statistics/282087/number-of-monthly-active-twitter-users/. Accessed March 9, 2019.

Use a calculator or computer to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models.

17. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.
(b) What condition on m will lead to an exponential expansion of the population?
(c) What condition on m will result in a constant population? A population decline?
(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

18. Doomsday Equation Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.
(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.
(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.
(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

T 19. Let's modify the logistic differential equation of Example 1 as follows:

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

(a) Suppose $P(t)$ represents a fish population at time t, where t is measured in weeks. Explain the meaning of the final term in the equation (-15).
(b) Draw a direction field for this differential equation.
(c) What are the equilibrium solutions?
(d) Use the direction field to sketch several solution curves. Describe what happens to the fish population for various initial populations.
(e) Solve this differential equation explicitly, either by using partial fractions or with a computer. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).

20. Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.

(a) Draw direction fields for various values of c.

(b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?

(c) Use the differential equation to prove what you discovered graphically in part (b).

(d) What would you recommend for a limit to the weekly catch of this fish population?

21. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.

(b) For the case where $k = 0.08$, $M = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?

(c) Solve the differential equation explicitly, either by using partial fractions or with a computer. Use the initial population P_0.

(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. [*Hint:* Show that the numerator in your expression for $P(t)$ is 0 for some value of t.]

22. The Gompertz Function Another model for a growth function for a limited population is given by the *Gompertz function*, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.

(a) Solve this differential equation.

(b) Compute $\lim_{t\to\infty} P(t)$.

(c) Graph the Gompertz function for $M = 1000$, $P_0 = 100$, and $c = 0.05$, and compare it with the logistic function in Example 2. What are the similarities? What are the differences?

(d) We know from Exercise 13 that the logistic function grows fastest when $P = M/2$. Use the Gompertz differential equation to show that the Gompertz function grows fastest when $P = M/e$.

23. In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.

(a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and ϕ are positive constants.

(b) By graphing the solution for several values of k, r, and ϕ, explain how the values of k, r, and ϕ affect the solution. What can you say about $\lim_{t\to\infty} P(t)$?

24. Suppose we alter the differential equation in Exercise 23 as follows:

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

(a) Solve this differential equation with the help of a table of integrals or a computer.

(b) Graph the solution for several values of k, r, and ϕ. How do the values of k, r, and ϕ affect the solution? What can you say about $\lim_{t\to\infty} P(t)$ in this case?

25. Graphs of logistic functions (Figures 2 and 3) look suspiciously similar to the graph of the hyperbolic tangent function (Figure 3.11.3). Explain the similarity by showing that the logistic function given by Equation 7 can be written as

$$P(t) = \tfrac{1}{2}M\left[1 + \tanh\left(\tfrac{1}{2}k(t - c)\right)\right]$$

where $c = (\ln A)/k$. Thus the logistic function is really just a shifted hyperbolic tangent.

9.5 Linear Equations

In Section 9.3 we learned how to solve separable first-order differential equations. In this section we investigate a method for solving a class of differential equations that are not necessarily separable.