(5.4) Horner's Method (Synthetic Division)

To divide the polynomial $P(x)$ by $x - r$:

Step 1. Arrange the coefficients of $P(x)$ in order of descending powers of x. Write 0 as the coefficient for each missing power.

Step 2. After writing the divisor in the form $x - r$, use r to generate the second and third rows of numbers as follows. Bring down the first coefficient of the dividend and multiply it by r; then add the product to the second coefficient of the dividend. Multiply this sum by r, and add the product to the third coefficient of the dividend. Repeat the process until a product is added to the constant term of $P(x)$.

Step 3. The last number to the right in the third row of numbers is the remainder. The other numbers in the third row are the coefficients of the quotient, which is of degree   less than $P(x)$.

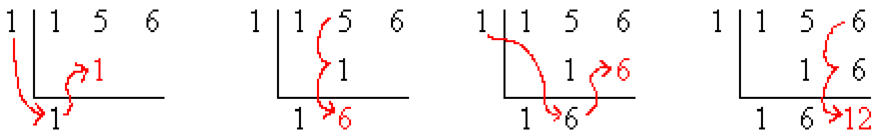
Example:

Divide $x^2 + 5x + 6$ by $x - 1$ by using synthetic division?

Solution:

First write the coefficients of the dividend and the *negative* of the constant term of the divisor in the format shown below at the left. Bring down the 1 as indicated next on the right, multiply by 1, and record the product 1. Add 5 and 1, bringing down their sum 6.

Repeat the process until the coefficients of the quotient and the remainder are obtained.



Example: Use synthetic division to divide

$$P(x) = 4x^5 - 30x^3 - 50x - 2 \text{ by } (x + 3).$$

Find the quotient and remainder. Write the conclusion in the form of Division Algorithm Theorem.

Solution: Because $x + 3 = x - (-3)$, we have $r = -3$ then the synthetic division is:

	4	0	−30	0	−50	−2
-3		−12	36	−18	54	−12
	4	−12	6	−18	4	−14

The quotient is $4x^4 - 12x^3 + 6x^2 - 18x + 4$

The remainder -14

The form of division algorithm is :

$$4x^5 - 30x^3 - 50x - 2 = (x+3)(4x^4 - 12x^3 + 6x^2 - 18x + 4) - 14$$

Theorem: (Remainder Theorem)

If R is the remainder after dividing the polynomial (x) by $(- r)$, then $P(r) = R$.

Example:

If $P(x) = 4x^4 + 10x^3 + 19x + 5$, find $P(-3)$ by

(i) Using the remainder theorem and synthetic division.

(ii) Evaluating (-3) directly.

Solution:

(i) Use synthetic division to divide (x) by $x - (-3)$.

	4	10	0	19	5	
		−12	6	−18	−3	
−3	4	−2	6	1	2	$= R = P(-3)$

(ii) $P(-3) = 4(-3)^4 + 10(-3)^3 + 19(-3) + 5 = 2.$

Theorem: (Factor Theorem)

If r is a zero of the polynomial $P(x)$, then $(x - r)$ is a factor of (x). Conversely, if $(x - r)$ is a factor of (x), then r is a zero of (x).

Proof:

The remainder theorem shows that the division algorithm equation,

$P(x) = (x - r)Q(x) + R$

can be written in the form where R is replaced by $P(r)$:

$P(x) = (x - r)Q(x) + P(r)$

Therefore, x- r is a factor of $P(x)$ if and only if $(r) = 0$, that is, if and only if

r is a zero of the polynomial (x).

Example:

Use the factor theorem to show that $(x+1)$ is a factor of $P(x) = x^{25} + 1$ but is not a factor of $Q(x) = x^{25} - 1$

Solution:

Since x+1=x-(-1) , $P(-1) = (-1)^{25} + 1 = -1 + 1 = 0$

Then x+1 is a factor of $x^{25} + 1$.

$$Q(-1) = (-1)^{25} - 1 = -1 - 1 = -2$$

And x+1 is not a factor of $x^{25} - 1$

Example:

Find the remainder when $f(x) = 2x^4 - 3x^3 + 7x$ is divided by $(x+2)$.

By using: (i) Remainder Theorem (ii) Long division (iii) synthetic division

Solution:

(i) Remainder Theorem. Since $(x - r) = (x + 2)$, it follows that $r = -2$. Thus,

$$R = f(-2) = 32 + 24 - 14 = 42.$$

(ii) Long Division.

$$2x^3 - 7x^2 + 14x - 21$$

$$(x+2)|\ 2x^4 - 3x^3 + 0x^2 + 7x + 0$$

$$\mp 2x^4 \mp 4x^3$$

$$-7x^3 + 0x^2 + 7x + 0$$

$$\pm 7x^3 \pm 14x^2$$

$$14x^2 + 7x + 0$$

$$\mp 14x^2 \mp 28x$$

$$-21x + 0$$

$$\pm 21x \pm 42$$

$$+42$$

Then the remainder is 42 and the quotient (x) is

$$q(x) = 2x^3 - 7x^2 + 14x - 21.$$

(iii) Synthetic substitution.

$$\begin{array}{r|rrrrr} & 2 & -3 & +0 & +7 & +0 \\ & & -4 & +14 & -28 & +42 \\ \hline -2 & 2 & -7 & +14 & -21 & +42 = R = f(-2) \end{array}$$

Example:

Determine whether $(x-2)$ is a factor of $f(x) = x^6 - 64$.

(i) Factor theorem. $a = 2,\ f(2) = 2^6 - 64 = 64 - 64 = 0$.
Thus, $x - 2$ is a factor of $f(x)$.

(ii) Synthetic division.

$$\begin{array}{r|rrrrrrr} & 1 & +0 & +0 & +0 & +0 & +0 & -64 \\ & & +2 & +4 & +8 & +16 & +32 & +64 \\ \hline 2 & 1 & +2 & +4 & +8 & +16 & +32 & +0 = R = f(2) \end{array}$$

Hence, $x - 2$ is a factor of $f(x) = x^6 - 64$ and the second factor is

$x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$. That is,

$f(x) = x^6 - 64 = (x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32)$.

Example:

Determine whether $(x-1)$ is a factor of $f\ (x) = x^\wedge 3 + 7x^\wedge 2 - 3x - 4$

Solution:

Synthetic division gives

$$\begin{array}{r|rrrr} & 1 & +7 & -3 & -4 \\ & & +1 & +8 & +5 \\ \hline 1 & 1 & +8 & +5 & +1 = R = f(1) \end{array}$$

Since $(1) = 1 \neq 0$, then $x - 1$ is not a factor of the given polynomial.

Fundamental Theorem of algebra:
If P(x) is a polynomial of degree n then P(x) will have exactly n zeroes, some of which may repeat.

Example:
$f(x) = x^2 - 1 = (x+1)(x-1)$ has two real roots 1 and -1.

Def:
If a is a root of f(x), and if $(x-a)^m$ is a factor of f(x), then we say that the root has multiplicity m.

Example:
Find the root of (i) $f(x) = 5(x-2)^3(x+1)$ (ii) $f(x) = x^3 - 2x^2 - 9x + 18$

(i) The roots are : 2 has multiplicity 3 and -1

(ii) $f(x) =$

Example
Find a polynomial function of degree 3 having the roots : 1,3i, -3i

Sol:

$$f(x) = a_0(x - c_1)(x - c_2)(x - c_3)$$

Let $a_0 = 1$

$$f(x) = (x-1)(x-3i)(x-3i)$$

$$f(x) = (x-1)(x^2+9) = x^3 - x^2 + 9x - 9$$

Example:
Given that x=2 is a zero of $P(x) = x^3 + 2x^2 - 5x - 6$ find the other two zeroes.

Sol
We need to do synthetic division to get quotient

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$$P(x) = (x-2)(x^2+4x+3)$$

So, this means that,

$$Q(x) = x^2 + 4x + 3$$

$$Q(x) = x^2 + 4x + 3 = (x+3)(x+1) \qquad \Rightarrow \qquad x = -3,\ x = -1$$

So the roots are: -3, -1, 2

H.W.:

List the multiplicities of the zeroes of each of the following polynomials

(a) $P(x) = x^2 + 2x - 15$

(b) $P(x) = x^2 - 14x + 49$

(c) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

(d) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

(e) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x+4)$

Example

Find a polynomial function having the roots : {0,3,3,-2}

Sol:

H.W.:

. Which of the following functions are polynomial functions?

(a) $f(x) = 4x^2 + 2$ (b) $f(x) = 3x^3 - 2x + \sqrt{x}$ (c) $f(x) = 12 - 4x^5 + 3x^2$

(d) $f(x) = \sin x + 1$ (e) $f(x) = 3x^{12} - 2/x$ (f) $f(x) = 3x^{11} - 2x^{12}$

Write down a polynomial function with roots:

(a) $1, 2, 3, 4$ (b) $2, -4$ (c) $12, -1, -6$

Write down the roots and identify their multiplicity for each of the following functions:

(a) $f(x) = (x-2)^3(x+4)^4$ (b) $f(x) = (x-1)(x+2)^2(x-4)^3$