(3.5) DETERMINANTS

Definition:

For any square matrix A, the **determinant** of A is a real number denoted by **det(A)** or $|A|$. If A is a square matrix of order n, then det(A) is called a **determinant of order *n*.**

1) The determinant of a 1 x 1 matrix A=[a] is the number **a** itself. det(A)=a.
2) The determinant of the second-order square matrix $A = \begin{matrix} a11 & a12 \\ a21 & a22 \end{matrix}$, is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example:

Find: $\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix}$

Solution:

det(A) =(-1)(14) – 2(-3) = 4+6=10

3) **Third-Order Determinants: the determinant** of 3×3 matrix can be obtained by:

$$\begin{vmatrix} \boldsymbol{a1} & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

Note that each 2 x 2 matrix can be obtained by deleting, in the original matrix, the row and column containing its coefficient.

$$a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If we think of the sign $(-1)^{i+j}$ as being located in position (i,j) of an $n\times n$ matrix, then the signs form a checkerboard pattern that has a (+) in the (1,1) position. The patterns for n=3 and n=4 are as follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Example: Find:

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$$

Solution:

We can choose any row or column to expand. We will choose the first row because it has zero: ☺

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 2\left[(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}\right] + (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + 0$$

$$= (2)(1)[(1)(-1) - (-3)(2)] + (-2)(-1)[(-3)(-1) - (1)(2)]$$

$$= (2)(5) + (2)(1) = 12$$

Example: Evaluate:

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{3+2}(0)\begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix}$$

$$+ (-1)^{3+3}(0)\begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} + (-1)^{3+4}(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$=(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} +(3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$= (3)\left(2\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}\right) + (3)\left((-2)\begin{vmatrix} -4 & 1 \\ 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix}\right)$$

$$= (3)(2)(3+6) - (3)(2)(-9+8) + (3)(-2)(8-2) + (3)(2)(-2+6)$$

$$= 54 + 6 - 36 + 24 = 48.$$

H.W: Evaluate $\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix}$

Theorem: (Determinant properties)

(1) If a matrix B results from a matrix A by interchanging two rows (columns) of A, then $B =-|A|$.

(2) If two rows (column) of A are equal, then $|A| =0$.

(3) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

(4) If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of the new determinant is k times of the given determinant.

(5) Let A and B be two matrix, then det(AB) = det(A)*det(B) or $|AB| = |A|\,|B|$

(6) If A is a nonsingular matrix, then the determinant of Inverse of matrix can be defined as $|A^{-1}| = \frac{1}{|A|}$.

(7) The determinant of a matrix and its transpose are equal; that is, $|A| = |A^T|$.

(8) If a matrix $A=[a_{ij}]_{n\times n}$ is upper (lower) triangular, then $A=a_{11}\cdot a_{22}\cdot\cdots\cdot a_{nn}$ = Product of the elements on the main diagonal.

(9) In a determinant each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as sum of two determinants of same order. For example,

$$\begin{vmatrix} a & b & \alpha + x \\ c & d & \beta + y \\ e & f & \gamma + z \end{vmatrix} = \begin{vmatrix} a & b & \alpha \\ c & d & \beta \\ e & f & \gamma \end{vmatrix} + \begin{vmatrix} a & b & x \\ c & d & y \\ e & f & z \end{vmatrix}$$

Theorem: A matrix is invertible if and only if its determinant is not zero. Then a matrix is singular if its determinant is zero.

Theorem: If a **2** x **2** matrix A is invertible then we can obtain the inverse of matrix by using the determinant of A as follows: (i) interchanging the elements on the main diagonal, (ii) taking the negative **of** the other elements, and (iii) dividing each element by the determinant of the original matrix.

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example:

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

|A|=2 ×5 - 3 ×4= -2

$$A^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}$$

H.W.

1. **Find the determinant of each matrix:**

(i) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & -5 \\ 0 & 2 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Determine those values of k for which $\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 0.$

3. **Find the inverse of each matrix:** (i) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}$

4. Express the determinant of a 3 x 3 matrix as a linear combination of determinants of order two with coefficients from ii) the first column, (ii) the second column, (iii) the third column.

4. Systems of linear equations

Definition: A **system of linear equations** (or **linear system**) is a collection of one or more linear equations involving the same set of variables.

For example, A linear equation in two unknowns x and y is of the form

$ax + by = c$ where a, b, c are real numbers.

Example:

Consider the equation

$$2x + y = 4$$

If we substitute $x = -2$ in the equation, we obtain

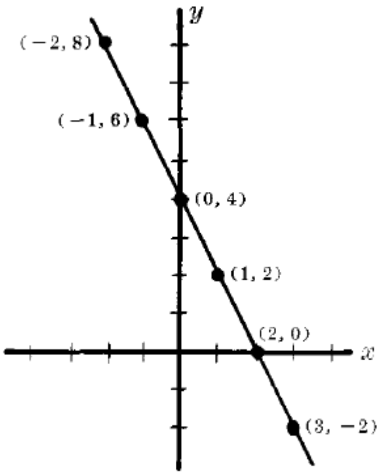
$$2 \cdot (-2) + y = 4 \quad \text{or} \quad -4 + y = 4 \quad \text{or} \quad y = 8$$

Hence $(-2, 8)$ is a solution. If we substitute $x = 3$ in the equation, we obtain

$$2 \cdot 3 + y = 4 \quad \text{or} \quad 6 + y = 4 \quad \text{or} \quad y = -2$$

Hence $(3, -2)$ is a solution. The table on the right lists six possible values for x and the *corresponding* values for y, i.e. six solutions of the equation.

x	y
−2	8
−1	6
0	4
1	2
2	0
3	−2



Graph of $2x + y = 4$

Example:

The equation $6x_1-3x_2+4x_3=-13$ is linear equation of three variables. $x_1=2, {}_2=3$, $x_3=-4$ is a solution to the linear equation $6\cdot2-3\cdot3+4\cdot(-4)=-13$. This is not the only solution to the given linear equation, since $x_1=3$, $x_2=1$, $x_3=-7$ is another solution.

Example:

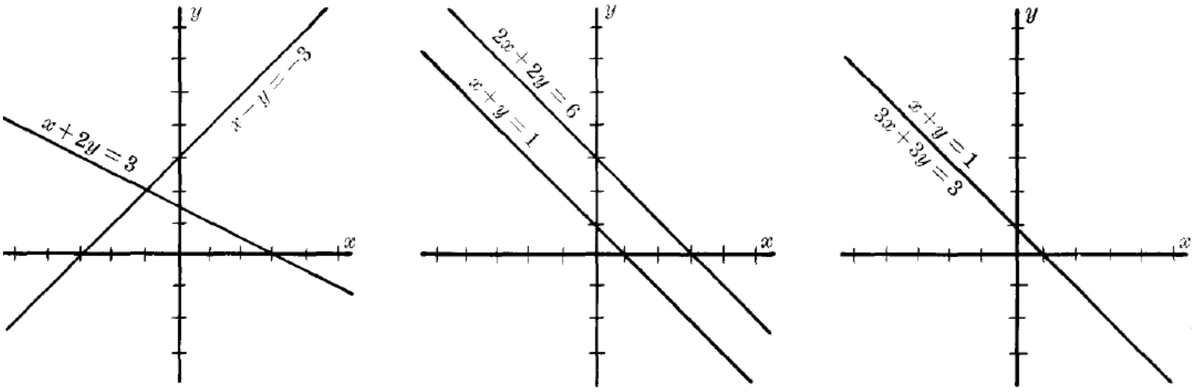
Now consider systems of two linear equations in two unknowns x and y:

(a) (b) (c)

$$\begin{aligned} x - y &= -3 \\ x + 2y &= 3 \end{aligned} \qquad \begin{aligned} x + y &= 1 \\ 2x + 2y &= 6 \end{aligned} \qquad \begin{aligned} x + y &= 1 \\ 3x + 3y &= 3 \end{aligned}$$

we study before how to solve these systems, by :

1) graph 2) substitution 3) elimination



(a) The system has exactly one solution.

(b) The system has no solutionns.

(c)The system has an infinite number of solutions.

In **general a system of *m* equations** in n unknowns $x_1, x_2, x_3, \dots, x_n$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \text{.......................(1)}$$

The numbers a_{ij} are called the **coefficients** of x_j and b_i is called the **constant term** for each i. A solution to a linear system is a sequence of ***n*** numbers $s_1, _2, \cdots, s_n$, which has the property that each equation in the system is satisfied when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in the system.

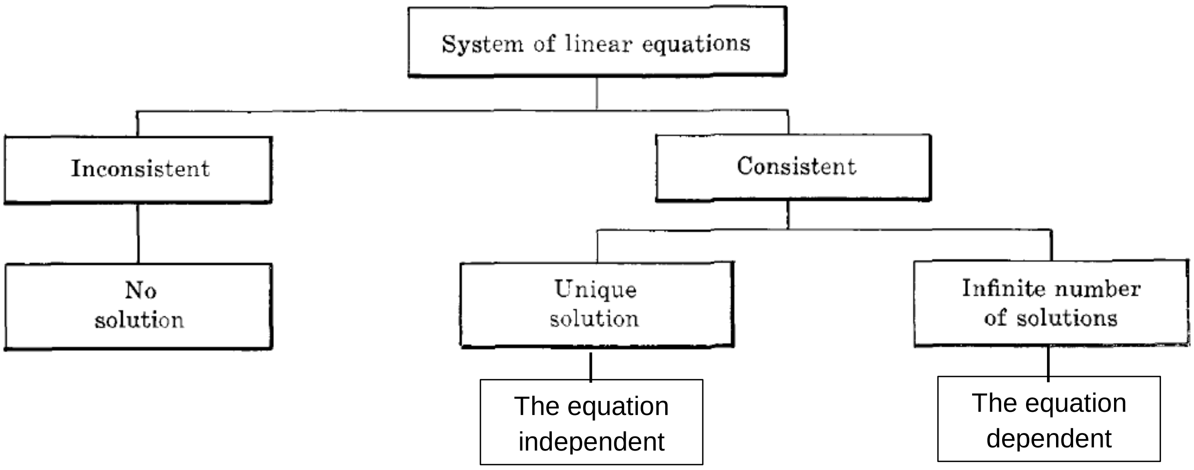
Definition:

(1) A system which each constant term is zero called **homogenous system**.

The solution $x_1=x_2=\cdots x_n=0$ to the homogenous system is called **trivial solution.** A solution $x_1=s_1,{}_2=s_2,\cdots x_n=s_n$ to a homogenous system in which not all the $s_i=0$ is called **nontrivial solution**.

(2) If a system of equations has at least one solution, it is said to be **consistent**.
(3) If it has no solution, it is said to be **inconsistent**.

(4) If a consistent system of equations has exactly one solution, the equations of the system are said to be **independent**. If it has an infinite number of solutions, the equations are called **dependent**.



(4.1) Represent linear system in augmented matrix

We can represent the above system of linear equations (1) by:

$$AX=B$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad X=\begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad B=\begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Where A is the **coefficient matrix** of the linear system (1).

The **augmented matrix** can be obtained by: [A|B] as follows:

$$C=\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]_{m\times(n+1)}$$

Example:

(1) Express the following system as AX = B and find the augmented matrix.

$$2x + 3y - 4z = 5$$
$$3x + 4y - 5z = 6$$
$$5x - 6z = 7$$

Solution:

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

The augmented matrix is: $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & 6 \\ 5 & 0 & -6 & 7 \end{array}\right]$

(2) The matrix $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{array}\right]$ is the augmented matrix of the following linear system

$$x + 3y - z = 0$$
$$2x \quad + z = 1$$
$$3x + y \quad = 1$$

Theorem: (1) A homogenous system of m equations in n unknown always has a nontrivial solution if $m<n$.

(2) If A is an $n\times n$ matrix, then the homogenous system $AX=0$ has a nontrivial solution ⟺ A is singular ⟺ $|A| = 0$.

(4.2) Solution of linear system by using the Gauss-Jordan method:

Definition:

The two linear systems are equivalent if and only if they have the same solution set.

The following row operations give the equivalent linear systems.

Row Operations:

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.

3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

Gauss-Jordan method: this method reduces the system into a series of equivalent systems by employing the row operations. This row reduction continues until the system is expressed in what is called the reduced row echelon form as follows:

1. Write the augmented matrix.

2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.

3. Use a row operation to make the entry in a11 = 1.

4. Use row operations to make all other entries as zeros in column one, i.e a21=0, a23=0,…

5. Interchange rows if necessary to obtain a nonzero number in a22. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

Example:

Use Gauss-Jordan method to solve the following linear system:

$$x + 3y = 7$$

$$3x + 4y = 11$$

Solution:

The augmented matrix for the system is as follows

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 3 & 4 & | & 11 \end{bmatrix}$$

To make the position of 3 in the second row = 0, we multiply the first row by -3, and add to the second row

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right]$$

To make the position of -5 =1, we divide the second row by -5, we get

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

Finally, to make the position of 3 = 0, we multiply the second row by - 3 and add to the first row, and we get

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \left[\begin{array}{c} x = 1 \\ y = 2 \end{array}\right]$$

Example: Use Gauss-Jordan method to solve the following linear system:

$$2x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_3 = 7$$
$$x_1 - 3x_2 + 2x_3 = 0$$

Solution: Write the augmented matrix and follow the steps indicated at the right to produce a reduced form.

Need a 1 here.

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

Step 1: Choose the leftmost nonzero column and get a 1 at the top.

Need 0's here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} (-3)R_1 + R_2 \rightarrow R_2 \\ (-2)R_1 + R_3 \rightarrow R_3 \end{array}$$

Step 2: Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Need a 1 here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad 0.1R_2 \rightarrow R_2$$

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top (shaded) row.