Calculus I

First Semester

Lecturer 4

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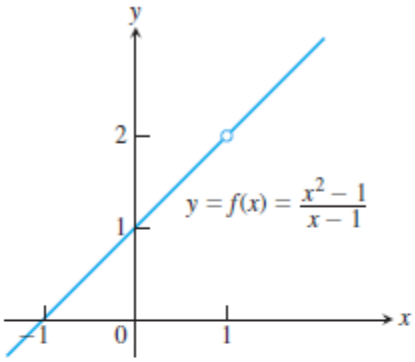
Chapter Two: Limits and Continuity.

2.1 Limits of Function Values

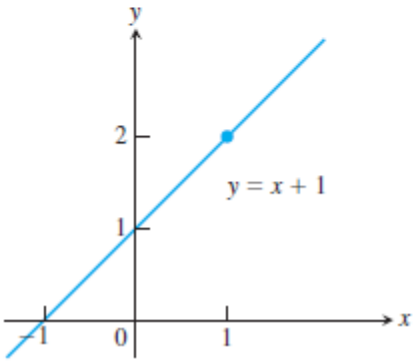
In this section we hall study the function's $y = f(x)$ behavior near a particular point c, but not at c. For instance, how does the behave of the function $f(x) = \frac{x^2-1}{x-1}$ near $x = 1$.



The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for } x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point (1, 2) removed. This removed point is shown as a "hole". Even though $f(1)$ is not defined, it is clear that we can make the value of f(x) as close as we want to 2 by choosing x close enough to 1.



x		$f(x) = \frac{x^2-1}{x-1}$
0.9		1.9
0.99		1.99
0.999		1.999
0.999999		1.999999
	1.1	2.1
	1.01	2.01
	1.001	2.001
	1.000001	2.000001

suppose $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the limit L as x approaches c, and write

$$\lim_{x\to c} f(x) = L,$$

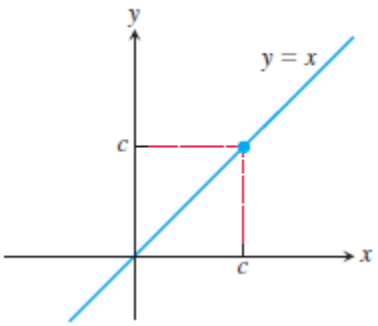
which is read "the limit of $f(x)$ as x approaches c is L." For instance, in Example we would say that $f(x) = \frac{x^2-1}{x-1}$ approaches the limit 2 as x approaches 1, and write

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Example:

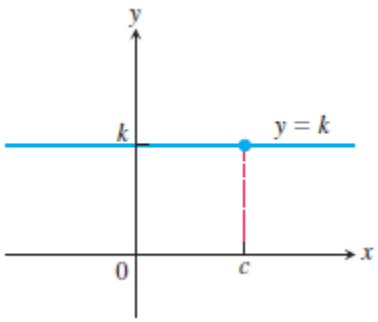
a) If f is the identity function $f(x) = x$, then for any value of c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$



b) If f is the constant function $f(x) = k$ (function with the constant value k), then for any value of c

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$



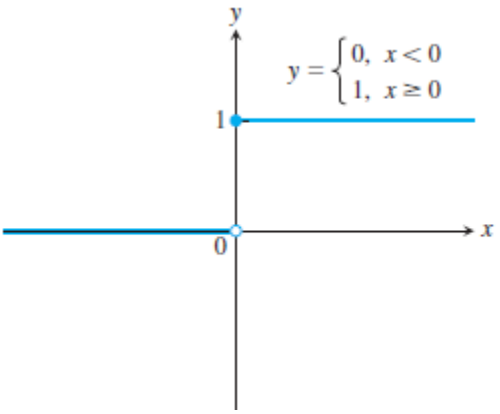
Example:

Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

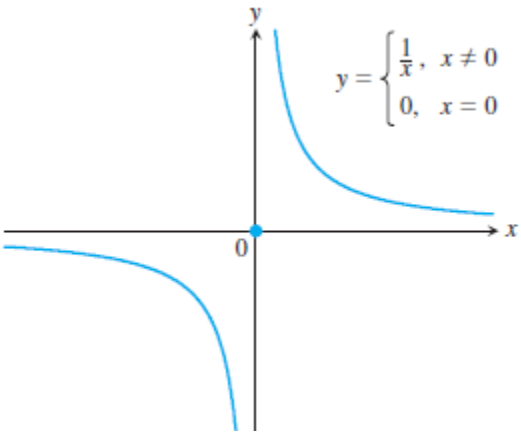
a) $U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$, **b)** $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, **c)** $f(x) = \begin{cases} 0 & x \leq 0 \\ \sin\frac{1}{x} & x \geq 0 \end{cases}$.

Solution:

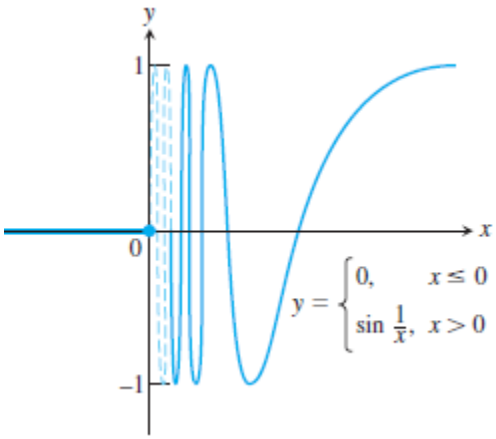
a) It jumps: The unit step function $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no single value L approached by $U(x)$ as $x \to 0$.



b) It grows too "large" to have a limit: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to any fixed real number. We say the function is not bounded.

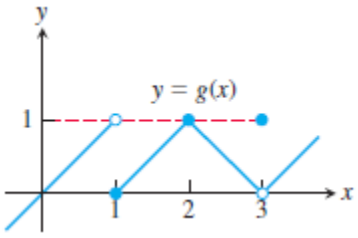


c) It oscillates too much to have a limit: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between +1 and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.



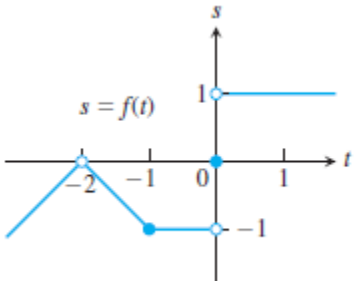
Exercises:

1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.



a) $\lim_{x \to 1} g(x)$, **b)** $\lim_{x \to 2} g(x)$, **c)** $\lim_{x \to 3} g(x)$, **d)** $\lim_{x \to 2.5} g(x)$.

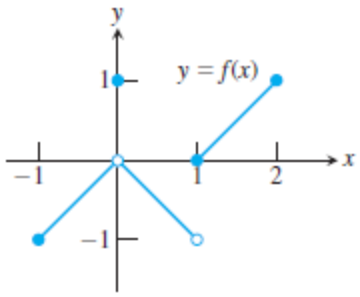
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.



a) $\lim_{t\to -2} f(t)$ **b)** $\lim_{t\to -1} f(t)$

c) $\lim_{t\to 0} f(t)$ **d)** $\lim_{t\to 0.5} f(t)$

3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



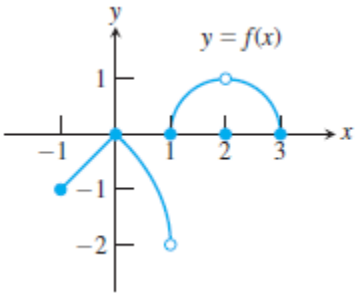
a) $\lim_{x\to 0} f(x)$ exists. **b)** $\lim_{x\to 0} f(x) = 0$

c) $\lim_{x\to 0} f(x) = 1$ **d)** $\lim_{x\to 1} f(x) = 1$

e) $\lim_{x\to 1} f(x) = 0$ **f)** $\lim_{x\to c} f(x)$ exists at every point c in (-1,1).

g) $\lim_{x\to 1} f(x)$ does not exist.

4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to 2} f(x)$ does not exist. **b)** $\lim_{x\to 2} f(x) = 2$.

c) $\lim_{x\to 1} f(x)$ does not exist. **d)** $\lim_{x\to c} f(x)$ exists at every point c in (-1,1).

e) $\lim_{x\to c} f(x)$ exists at every point c in (1,3).

5. Find the limits in following

a) $\lim_{x\to -3} (x^2 - 13)$

b) $\lim_{x\to 2} (-x^2 + 5x - 2)$

c) $\lim_{t\to 6} 8(t-5)(t-7)$

d) $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

e) $\lim_{x\to 2} \frac{2x+5}{11-x^3}$

f) $\lim_{s\to 2/3} (8 - 3s)(2s - 1)$

g) $\lim_{s\to -1/2} 4x(3x+4)^2$

h) $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$

i) $\lim_{y\to -3} (5-y)^{4/3}$

j) $\lim_{z\to 4} \sqrt{z^2 - 10}$

k) $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$

l) $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$

m) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

n) $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

o) $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$

p) $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

q) $\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$

r) $\lim_{y\to 0}\frac{y^3+8y^2}{3y^4-16y^2}$

s) $\lim_{x\to -2}\frac{-2x-4}{x^3+2x^2}$

t) $\lim_{x\to 1}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

u) $\lim_{x\to 1}\frac{x^{-1}-1}{x-1}$

v) $\lim_{u\to 2}\frac{v^3-8}{v^4-16}$

w) $\lim_{u\to 1}\frac{u^4-1}{u^3-1}$

x) $\lim_{x\to 4}\frac{4x-x^2}{2-\sqrt{x}}$

y) $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

z) $\lim_{x\to \pi/4}\sin^2 x$

aa) $\lim_{x\to 0}(2\sin x-1)$

bb) $\lim_{x\to \pi/3}\tan x$

cc) $\lim_{x\to 0}\sec x$

dd) $\lim_{x\to 0}(x^2-1)(2-\cos x)$

ee) $\lim_{x\to 0}\frac{1+x+\sin x}{3\cos x}$

ff) $\lim_{x\to 0}\sqrt{7+\sec^2 x}$

gg) $\lim_{x\to -\pi}\sqrt{x+4}\cos(x+\pi)$

2.2 The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

Theorem (Limit Laws):

If L, M, c, and k are real numbers and $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$ then

1. **Sum Rule:** $\lim_{x\to c}(f(x)+g(x))=L+M$
2. **Difference Rule:** $\lim_{x\to c}(f(x)-g(x))=L-M$
3. **Constant Multiple Rule:** $\lim_{x\to c}(k\cdot f(x))=k\cdot L$
4. **Product Rule:** $\lim_{x\to c}(f(x)\cdot g(x))=L.M$
5. **Quotient Rule:** $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}, \quad M\neq 0.$
6. **Power Rule:** $\lim_{x\to c}[f(x)]^n=L^n$, **n a positive integer.**
7. **Root Rule:** $\lim_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{L}=L^{1/n}$, **n a positive integer.**

(If *n* is even, we assume that $\lim_{x\to c} f(x)=L>0.$)

Example:

Use the fundamental rules of limits to find the following limits.

a) $\lim_{x\to c}(x^3+4x^2-3)$ **b)** $\lim_{x\to c}\frac{x^4-x^2-1}{x^2+5}$ **c)** $\lim_{x\to -2}\sqrt{4x^2-3}$

Solution:

a) $\lim_{x\to c}(x^3+4x^2-3) = \lim_{x\to c}x^3+\lim_{x\to c}4x^2-\lim_{x\to c}3$ — **Sum and Difference Rules**

$= c^3+4c^2-3$ — **Power and Multiple Rules**

b) $\lim_{x\to c}\frac{x^4-x^2-1}{x^2+5} = \frac{\lim_{x\to c}(x^4-x^2-1)}{\lim_{x\to c}(x^2+5)}$ — **Quotient Rule**

$= \frac{\lim_{x\to c}x^4-\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5}$ — **Sum and Difference Rules**

$= \frac{c^4-c^2-1}{c^2+5}$ — **Power or Product Rule**

c) $\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)}$ — **Root Rule with $n = 2$**

$= \sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$ — **Difference Rule**

$= \sqrt{4(-2)^2-3}$ — **Product and Multiple Rules**

$= \sqrt{16-3}$

$= \sqrt{13}$

Theorem (Limits of Polynomials):

If $P(x) = a_n x^n + a_{n-1}x^n + \cdots + a_0$, then

$$\lim_{x\to c}P(x) = P(x) = a_n c^n + a_{n-1}c^n + \cdots + a_0.$$

Theorem (Limits of Rational Functions):

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example:

$$\lim_{x\to c}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

Example:

Evaluate $\lim_{x\to c}\frac{\sqrt{x^2+100}-10}{x^2}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \lim_{x\to 0}\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$ **Common factor x^2**

$$= \lim_{x\to 0}\frac{1}{\left(\sqrt{x^2+100}+10\right)}$$ **Cancel x^2 for $x \neq 0$.**

$$= \frac{1}{\left(\sqrt{0^2+100}+10\right)}$$ **Denominator not 0 at $x = 0$; substitute**

$$= \frac{1}{20} = 0.05.$$

Theorem (Limits of Rational Functions):

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

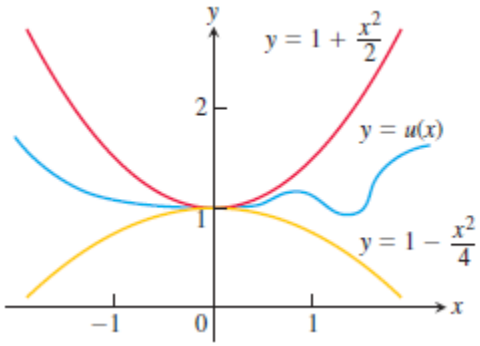
Then $\lim_{x\to c} f(x) = L$.

Example:

Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.



Solution:

Since $\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1$ and $\lim_{x\to 0}\left(1 - \frac{x^2}{2}\right) = 1$,by the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$.

Example:

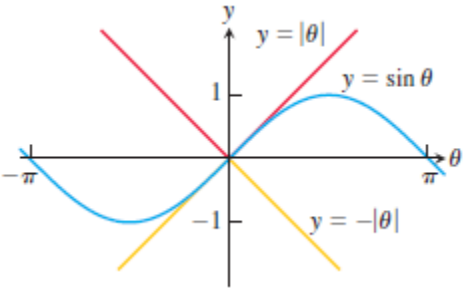
The Sandwich Theorem helps us establish several important limit rules:

a) $\lim_{\theta \to 0} \sin\theta = 0$ **b)** $\lim_{\theta \to 0} \cos\theta = 1$

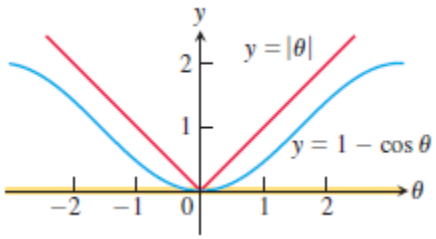
c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution:

a) Since $-|\theta| \le sin\,\theta \le |\theta|$, for all θ and since $\lim_{\theta \to 0} -|\theta| \lim_{\theta \to 0} |\theta| = 0$, we have $\lim_{\theta \to 0} \sin\theta = 0$.



b) Since $-|\theta| \le 1 - \cos\theta \le |\theta|$, for all θ and since $\lim_{\theta \to 0} -|\theta| \lim_{\theta \to 0} |\theta| = 0$, we have $\lim_{\theta \to 0} 1 - \cos\theta = 0$ or $\lim_{\theta \to 0} \cos\theta = 1$.



c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ 0 and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$.

Theorem:

If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Exercises:

1. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x \to 0} f(x)$.

2. If $2 - x^2 \le g(x) \le 2$ for all x, find $\lim_{x \to 0} g(x)$.

3. It can be shown that the inequalities $1 - \frac{x^2}{6} \le \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to zero. What, if anything, does this tell you about $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$? Give reasons for your answer.

4. Suppose that the inequalities $\frac{1}{2} - \frac{x^2}{24} \leq \frac{1-\cos x}{x^2} < \frac{1}{2}$ hold for values of x close to zero. What, if anything, does this tell you about $\lim_{x\to 0} \frac{1-\cos x}{x^2}$? Give reasons for your answer.

5. Find the limits in following

a) $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$

b) $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

d) $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

e) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

f) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

g) $\lim_{x\to -3} \frac{-2-\sqrt{x^2-5}}{x+3}$

2.3 The Precise Definition of a Limit

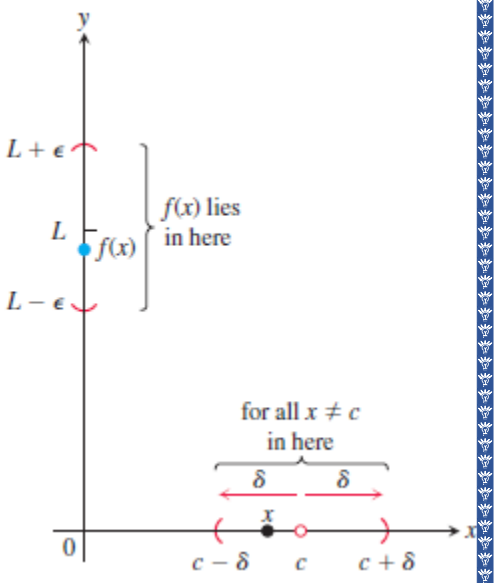
Definition:

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



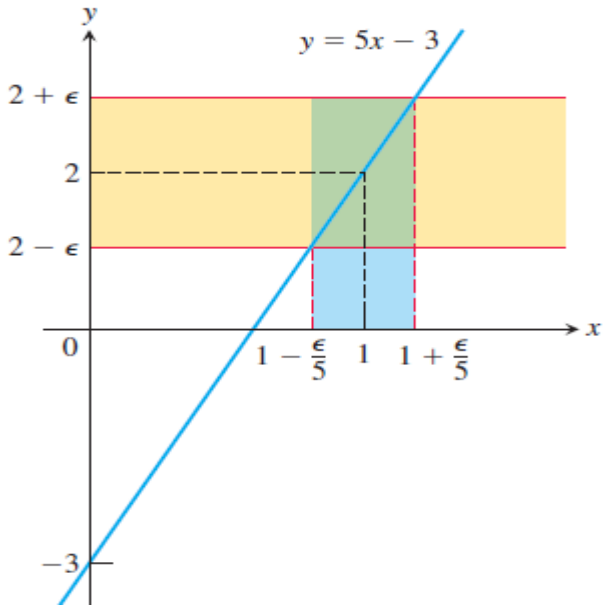
Examples: Testing the Definition

Example:

Show that $\lim_{x\to 1}(5x - 3) = 2$.

Solution:

Set c = 1, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of c = 1, that is, whenever $0 < |x - 1| < \delta$, it is true that $f(x)$ is within distance ϵ of $L = 2$, so $|f(x) - 2| < \epsilon$.



We find δ by working backward from the ϵ -inequality:

$$|(5X - 3) - 2| = |5x - 5| = 5|x - 1| < \epsilon \Rightarrow |x - 1| < \epsilon/5.$$

Thus, we can take $\delta = \epsilon/5$. If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5X - 3) - 2| = |5x - 5| = 5|x - 1| < 5\frac{\epsilon}{5} = \epsilon.$$

Which proves that $\lim_{x \to 1}(5x - 3) = 2$.

Example:

Prove the following results

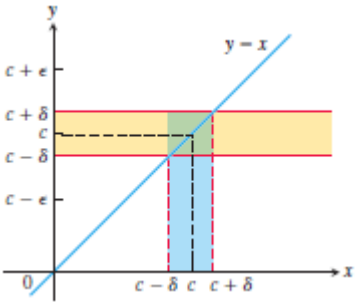
a) $\lim_{x \to c} x = c$

b) $\lim_{x \to c} k = k$. (k constant).

Solution:

a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

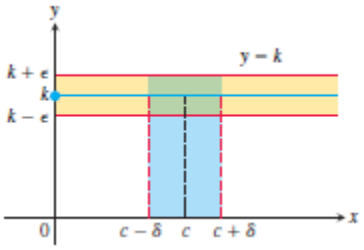
$$0 < |x - c| < \delta \Rightarrow |x - c| < \epsilon.$$



The implication will hold if δ equals ϵ or any smaller positive number. This proves that $\lim_{x \to c} x = c$.

b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |k - k| < \epsilon.$$

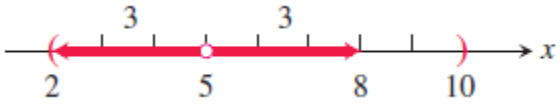


Since k - k = 0, we can use any positive number for δ and the implication will hold. This proves that $\lim_{x \to c} k = k$.

Example:

For the limit $\lim_{x\to 5}\sqrt{x-1}=2$, find a $\delta>0$ that works for $\epsilon=1$. That is, find a $\delta>0$ such that for all x

$$0<|x-5|<\delta \Rightarrow \left|\sqrt{x-1}-2\right|<1.$$



Solution:

We organize the search into two steps.

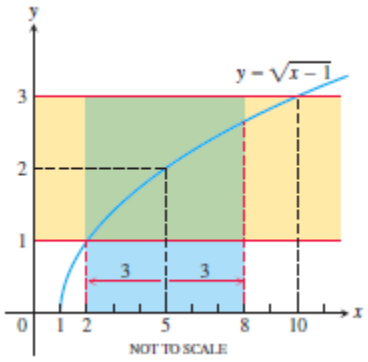
1. Solve the inequality $\left|\sqrt{x-1}-2\right|<1$ to find an interval containing $x=5$ on which the inequality holds for all $x\neq 5$.

$$\left|\sqrt{x-1}-2\right|<1$$
$$-1<\sqrt{x-1}-2<1$$
$$1<\sqrt{x-1}<3$$
$$1<x-1<9$$
$$2<x<10.$$

The inequality holds for all x in the open interval (2, 10), so it holds for all $x\neq 5$ in this interval as well.

2. Find a value of $\delta>0$ to place the centered interval $5-\delta<x<5+\delta$ (centered at $x=5$) inside the interval (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3. If we take $\delta=3$ or any smaller positive number, then the inequality $0<|x-5|<\delta$ will automatically place x between 2 and 10 to make $\left|\sqrt{x-1}-2\right|<1$:

$$0<|x-5|<\delta \Rightarrow \left|\sqrt{x-1}-2\right|<1.$$



Example:

Prove that $\lim_{x\to 2} f(x)=4$ if $f(x)=\begin{cases}x^2 & x\neq 2\\ 1 & x=2\end{cases}$.

Solution:

Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$

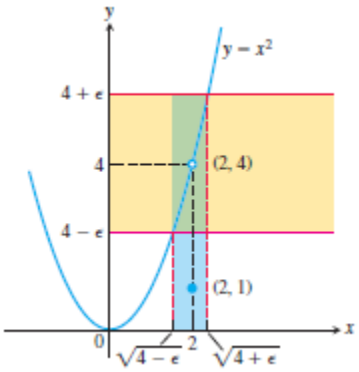
1. Solve the inequality $|f(x) - 4| < \epsilon$ to find an interval containing $x = 2$ on which the inequality holds for all $x \neq 2$.

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2 - 4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$
$$4 - \epsilon < x^2 < 4 + \epsilon$$
$$\sqrt{4 - \epsilon} < |x| < \sqrt{4 + \epsilon} \quad \{\text{Assumes } \epsilon < 4\}$$
$$\sqrt{4 - \epsilon} < x < \sqrt{4 + \epsilon}$$

An open interval about $x = 2$ that solves the inequality

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.



2. Find a value of $\delta > 0$ that places the centered interval $(2 - \delta, 2 + \delta)$ inside the interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$. Take δ to be the distance from $x = 2$ to the nearer endpoint of $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$. In other words, take $\delta = min\{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\}$, the minimum (the smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$

If $\epsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $(0, \sqrt{4 + \epsilon})$. In other words, take $\delta = min\{2, \sqrt{4 + \epsilon} - 2\}$.

Exercises:

1. In following sketch, the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - c| < \delta \Rightarrow a < x < b$.

a) $a = 1, b = 7, c = 5$ **b)** $a = 1, b = 7, c = 2$

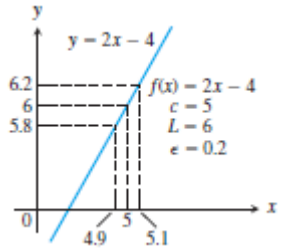
c) $a = -7/2, b = -1/2, c = -3$ **d)** $a = -7/2, b = -1/2, c = -3/2$

e) $a = 4/9, b = 4/7, c = 1/2$

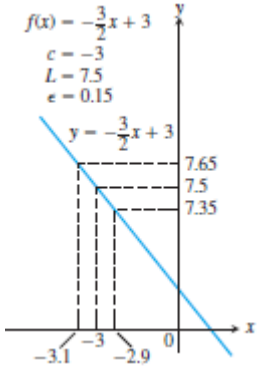
2. In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

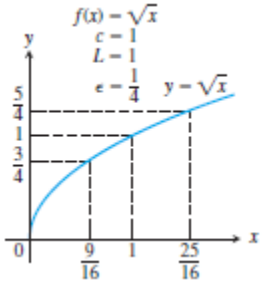
a)



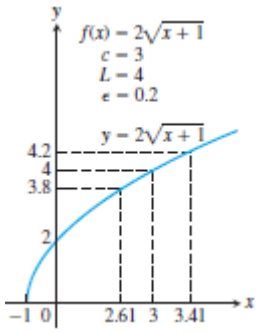
b)



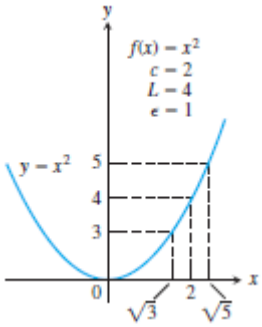
c)



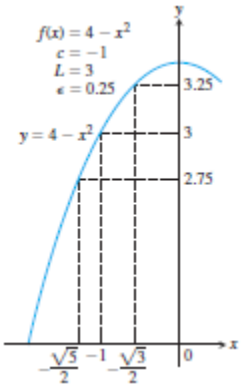
d)



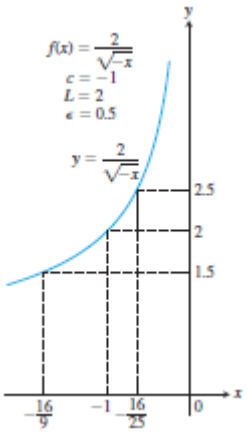
e)



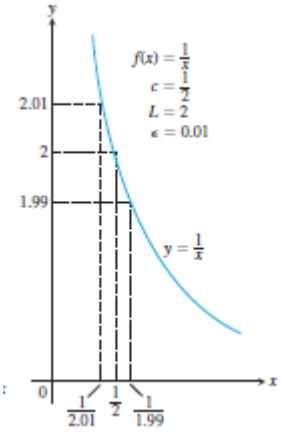
f)



g)



h)



3. Each of following gives a function $f(x)$ and numbers L, c, and $\epsilon > 0$. In each case, find an open interval about c on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \epsilon$.

a) $f(x) = x + 1, L = 5, c = 4, \epsilon = 0.01$

b) $f(x) = 2x - 2, L = -6, c = -2, \epsilon = 0.02$

c) $f(x) = \sqrt{x+1}, L = 1, c = 0, \epsilon = 0.1$

d) $f(x) = \sqrt{x}, L = 1/2, c = 1/4, \epsilon = 0.1$

e) $f(x) = \sqrt{19 - x}, L = 3, c = 10, \epsilon = 1$

f) $f(x) = \sqrt{x - 7}, L = 4, c = 23, \epsilon = 1$

g) $f(x) = 1/x, L = 1/4, c = 4, \epsilon = 0.05$

h) $f(x) = x^2 + 1, L = 3, c = \sqrt{3}, \epsilon = 0.1$

i) $f(x) = x^2, L = 4, c = -2, \epsilon = 0.5$

j) $f(x) = 1/x, L = -1, c = -1, \epsilon = 0.1$

4. Each of following gives a function $f(x)$, a point c, and a positive number ϵ. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

a) $f(x) = 3 - 2x, c = 3, \epsilon = 0.02$

b) $f(x) = -3x - 2, c = -1, \epsilon = 0.03$

c) $f(x) = \frac{x^2-4}{x-2}, c = 2, \epsilon = 0.05$

d) $f(x) = \frac{x^2+6x+5}{x+5}, c = -5, \epsilon = 0.05$

e) $f(x) = \sqrt{1 - 5x}, c = -3, \epsilon = 0.5$

f) $f(x) = 4/x, c = 4, \epsilon = 0.4$

5. Prove the limit statements of following

a) $\lim_{x \to 4}(9 - x) = 5$

b) $\lim_{x \to 3}(3x - 7) = 2$

c) $\lim_{x \to 9} \sqrt{x - 5} = 2$

d) $\lim_{x \to 0} \sqrt{4 - x} = 2$

e) $\lim_{x \to 1} f(x) = 1 \; if \; f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$

f) $\lim_{x \to -2} f(x) = 4 \; if \; f(x) = \begin{cases} x^2 & x \neq -2 \\ 1 & x = -2 \end{cases}$

g) $\lim_{x \to 1} (1/x) = 1$

h) $\lim_{x \to \sqrt{3}} (1/x^2) = 1/3$

i) $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$

j) $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

k) $\lim_{x \to 1} f(x) = 2 \; if \; f(x) = \begin{cases} 4 - 2x & x < 1 \\ 6x - 4 & x \geq 1 \end{cases}$

l) $\lim_{x \to 0} f(x) = 0 \; if \; f(x) = \begin{cases} 2x & x < 0 \\ x/2 & x \geq 0 \end{cases}$

m) $\lim_{x \to 0} x \sin 1/x = 0$

n) $\lim_{x \to 0} x^2 \sin 1/x = 0$