$$\sim\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \qquad \begin{array}{l} 3R_2 + R_1 \to R_1 \\ \\ (-4)R_2 + R_3 \to R_3 \end{array}$$

Need 0's here.

Step 4: Repeat step 2 with the *entire matrix.*

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \qquad (-5)R_3 \to R_3$$

Need a 1 here.

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top two (shaded) rows.

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} 0.1R_3 + R_1 \to R_1 \\ 0.7R_3 + R_2 \to R_2 \end{array}$$

Need 0's here.

Step 4: Repeat step 2 with the *entire matrix.*

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The matrix is now in reduced form, and we can proceed to solve the corresponding reduced system.

The system which has the above augmented matrix is

$$\begin{aligned} x_1 + 0 + 0 &= 2 \\ 0 + x_2 + 0 &= 0 \\ 0 + 0 \ + x_1 &= -1 \end{aligned}$$

Therefore, $S.S. = \{(2,0,-1)\}$.

Example: Use Gauss-Jordan method to solve the following linear system:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \xrightarrow[\text{(To get 1 in upper left corner)}]{0.5R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right]$$

$$\xrightarrow[2R_1 + R_3 \to R_3]{(-4)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{0.2R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2 + R_3 \to R_3]{(-0.5)R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

We stop the Gauss–Jordan elimination, even though the matrix is not in reduced form, since the last row produces a contradiction. The system is inconsistent and has no solution.

Example: Use Gauss-Jordan method to solve the following linear system:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right]$$

$$\xrightarrow[2R_1 + R_3 \leftrightarrow R_3]{(-2)R_1 + R_2 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in reduced form. Write the corresponding reduced system and solve.

$x_1 + 0 + x_3 = -3 \quad \Rightarrow x_1 = -x_3 - 3$

$0 + x_2 - 2x_3 = 4 \quad \Rightarrow x_2 = 2x_3 + 4$

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions.

$$\begin{aligned} x_3 &= t \\ x_2 &= 2t + 4 \\ x_1 &= -t - 3 \end{aligned}$$

Where $t \in R$. Therefore, $S.S. = \{(-t - 3, 2t + 4, t) | t \in R\}$.

Example: Use Gauss-Jordan method to solve the following linear system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\ x_1 + 3x_2 + 7x_3 + 3x_5 &= -2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 2 & 4 & 8 & 3 & -4 & 2 \\ 1 & 3 & 7 & 0 & 3 & -2 \end{array}\right] \xrightarrow[(-1)R_1 + R_3 \leftrightarrow R_3]{(-2)R_1 + R_2 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 4 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{(-2)R_2 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & -9 & 7 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \leftrightarrow R_2]{(-3)R_3 + R_1 \leftrightarrow R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -3 & 7 \\ 0 & 1 & 3 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

This matrix is in reduce row echelon form. Write the corresponding reduced system and solve.

$$\begin{aligned} x_1 - 2x_3 - 3x_5 &= 7 \\ x_2 + 3x_3 + 2x_5 &= -3 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

Solve for the leftmost variables x_1, x_2, and x_4 in terms of the remaining variables x_3 and x_5:

$$\begin{aligned} x_1 &= 2x_3 + 3x_5 + 7 \\ x_2 &= -3x_3 - 2x_5 - 3 \\ x_4 &= 2x_5 \end{aligned}$$

If we let $x_3 = s$and $x_5 = t$, then for any real numbers s and t,

$x_1 = 2s + 3t + 7$

$x_2 = -3s - 2t - 3$

$x_3 = s$

$x_4 = 2t$

$x_5 = t$

$S.S. = \{(2s + 3t + 7, -3s - 2t - 3, s, 2t, t) | s, t \in R\}$.

H.W.

$$\begin{aligned} 2x + 6y - z &= 4 \\ 3x - 2y - z &= 1 \\ 5x + 9y - 2z &= 12 \end{aligned} \qquad \begin{aligned} x + 2y - 4z &= -4 \\ 5x - 3y - 7z &= 6 \\ 3x - 2y + 3z &= 11 \end{aligned}$$

(4.3) Solving Linear System By Cramer's Rule (determinant)

This method can be used only for square matrix and computationally inefficient for n>4.

Let $AX = B$ be an $n \times n$ linear system, where $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then the Cramer's rule is as follows:

If $|A| \neq 0$, then

$$x_i = \frac{|A_i|}{|A|}, i = 1,2,\cdots,n$$

where A_i is the matrix obtained from A by replacing the ith column by B.

If $n = 3$ then Cramer's rule as follows:

Given the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= k_1 \\ a_{21}x + a_{22}y + a_{23}z &= k_2 \\ a_{31}x + a_{32}y + a_{33}z &= k_3 \end{aligned} \qquad \text{with} \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

Example:

Solve using Cramer's rule:

$$\begin{aligned} x + y \quad &= 2 \\ 3y - z &= -4 \\ x \quad + z &= 3 \end{aligned}$$

Solution:

$$|A| = D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

$$x = \frac{\begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix}}{2} = \frac{7}{2} \qquad y = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix}}{2} = -\frac{3}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 1 & 0 & 3 \end{vmatrix}}{2} = -\frac{1}{2}$$

H.W.

Solve using determinants:

1) $\begin{aligned} 2x - 3y &= 7 \\ 3x + 5y &= 1 \end{aligned}$ $\qquad \begin{aligned} 2x - 4y &= 7 \\ 3x - 6y &= 5 \end{aligned}$

2) $\begin{aligned} 2x + y - z &= 3 \\ x + y + z &= 1 \\ x - 2y - 3z &= 4 \end{aligned}$

(4.4) Solving Linear System Using Inverses

Using reduced row echelon form to find the inverse of matrix

Steps for finding the inverse of a matrix of dimension $n \times n$:

STEP 1: Form the augmented matrix $A|I_n$.

STEP 2: Using row operations, write $A|I_n$ in reduced row echelon form.

STEP 3: If the resulting matrix is of the form $I_n|B$ that is, if the identity matrix appears on the left side of the bar, then B is the inverse of A. Otherwise, A has no inverse.

Example:

Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution:

STEP 1 Since A is of dimension 3×3, use the identity matrix I_3. The matrix $[A|I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

STEP 2 Proceed to obtain the reduced row echelon form of this matrix:

Use $\begin{array}{l} R_2 = -2r_1 + r_2 \\ R_3 = -1r_1 + r_3 \end{array}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$

Use $R_2 = -1r_2$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$

Use $\begin{array}{l} R_1 = -1r_2 + r_1 \\ R_3 = -1r_2 + r_3 \end{array}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & -4 & -3 & 1 & 1 \end{array}\right]$

Use $R_3 = -\frac{1}{4}r_3$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

Use $\begin{array}{l} R_1 = 2r_3 + r_1 \\ R_2 = -4r_3 + r_2 \end{array}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

The matrix $[A|I_3]$ is in reduce row echelon form.

STEP 3: Since the identity matrix I_3 appears on the left side, the matrix appearing on the right is the inverse. That is,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

H.W. Show that if A has inverse or not.

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}.$$

Solving Linear System AX=B Using Inverses

$AX = B$	A has an inverse A^{-1}.
$A^{-1}(AX) = A^{-1}B$	Multiply both sides by A^{-1}.
$(A^{-1}A)X = A^{-1}B$	Apply the Associative Property on the left side.
$I_nX = A^{-1}B$	Apply the Inverse Property: $A^{-1}A = I_n$.
$X = A^{-1}B$	Apply the Identity Property: $I_nX = X$.

Example:

Solve the system of equations:

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + y \qquad &= 2 \\ x + 2y + 2z &= 3 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

the solution X of the system is

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

Therefore, $S.S. = \{(0,2,\frac{-1}{2})\}$.