Calculus I

First Semester

Lecturer 3

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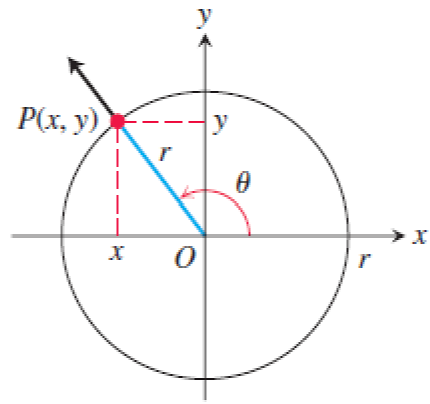
1.3 Trigonometric Functions

1.3.1 The Six Basic Trigonometric Functions

The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



sine: $\sin\theta = \frac{y}{r}$ **cosecant:** $\csc\theta = \frac{r}{y}$

cosine: $\cos\theta = \frac{x}{r}$ **secant:** $\sec\theta = \frac{r}{x}$

tangent: $\tan\theta = \frac{y}{x}$ **cotangent:** $\cot\theta = \frac{x}{y}$

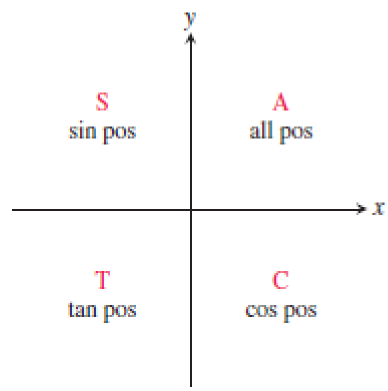
Notice also that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Remark:

1. $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if θ is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which y = 0, namely $\theta = \pm\pi, \pm 2\pi, \cdots$.



2. The CAST rule is useful for remembering when the basic trigonometric functions are positive or negative.

3. The following table shows the Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ.

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Exercises:

1. Copy and complete the following tables of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

a)

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

b)

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

2. In following, one of $sin\ x$, $cos\ x$, and $tan\ x$ is given. Find the other two if x lies in the specified interval.

a) $sin\ x = \frac{3}{5}, x \in [\frac{\pi}{2}, \pi]$

b) $tan\ x = 2, x \in [0, \frac{\pi}{2}]$

c) $cos\ x = \frac{1}{3}, x \in [-\frac{\pi}{2}, 0]$

d) $cos\ x = -\frac{5}{13}, x \in [\frac{\pi}{2}, \pi]$

e) $tan\ x = \frac{1}{2}, x \in [\pi, \frac{3\pi}{2}]$

f) $sin\ x = -\frac{1}{2}, x \in [\pi, \frac{3\pi}{2}]$

1.3.2 Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $sin\,(\theta + 2\pi\,) = sin\,\theta$, $tan\,(\theta + 2\pi) = tan\,\theta$, and so on. Similarly, $cos\,(\theta - 2\pi\,) = cos\,\theta$, $sin\,(\theta - 2\pi\,) = sin\,\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

Definition:

A function $f(x)$ is periodic if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the period of f.

Remark:

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. The tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π.

Periods of Trigonometric Functions

Period π	$tan\,(x + \pi) = tan\,x$ $cot\,(x + \pi) = cot\,x$
Period 2π	$sin\,(x + 2\pi\,) = sin\,x$ $cos\,(x + 2\pi\,) = cos\,x$ $sec\,(x + 2\pi\,) = sec\,x$ $csc\,(x + 2\pi\,) = csc\,x$

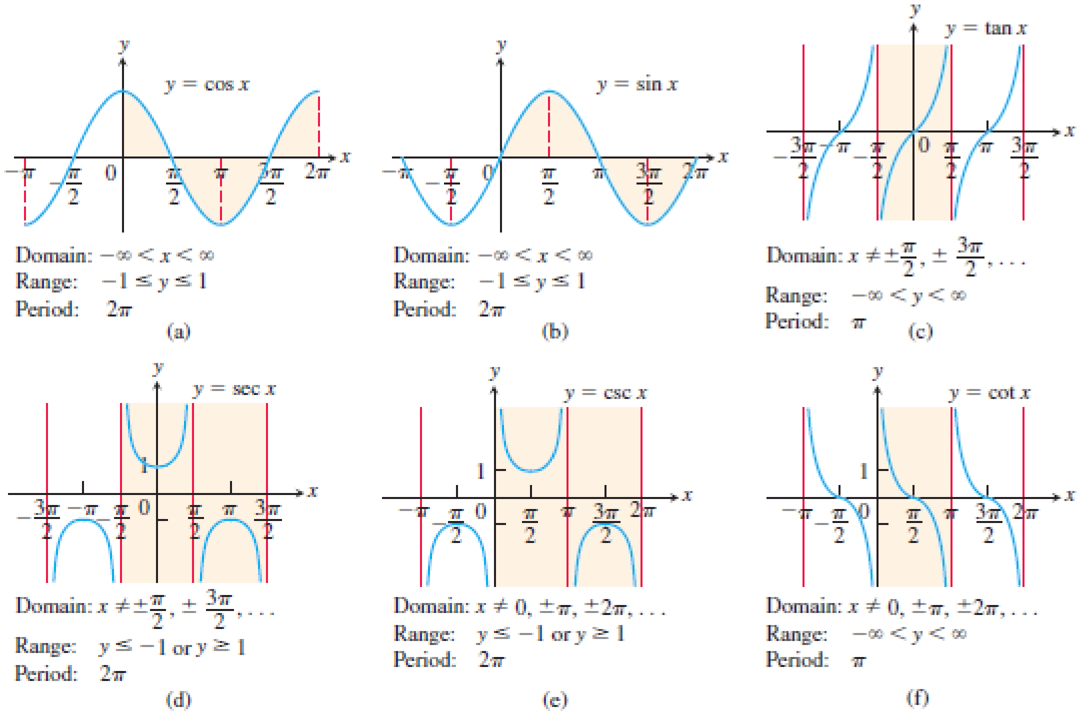
Also, the symmetries of their graphs reveal that the cosine and secant functions are even and the other four functions are odd.

Even
$cos\,(-x\,) = cos\,x$
$sec\,(-x\,) = sec\,x$

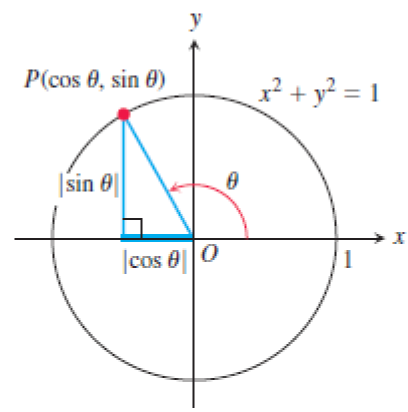
Odd
$sin\,(-x\,) = -sin\,x$
$tan\,(-x\,) = -tan\,x$
$csc\,(-x\,) = -csc\,x$
$cot\,(-x\,) = -cot\,x$

Remark:

The following graphs are graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.



1.3.3 Trigonometric Identities



The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle u that ray OP makes with the positive x-axis. Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \quad (1)$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + tan^2\theta = sec^2\theta \tag{2}$$

$$1 + cot^2\theta = csc^2\theta \tag{3}$$

Remark:

The following formulas hold for all angles A and B.

Addition Formulas

$$\begin{aligned} cos\,(A+B) &= cos\,A\,cos\,B - sin\,A\,sin\,B \\ sin\,(A+B) &= sin\,A\,cos\,B + cos\,A\,sin\,B \end{aligned} \tag{4}$$

Double-Angle Formulas

$$\begin{aligned} cos\,(2A) &= cos^2 A - sin^2 A \\ sin\,(2A) &= 2sin\,A \cdot cos\,A \end{aligned} \tag{5}$$

Half-Angle Formulas

$$\begin{aligned} cos^2 A &= \frac{1+cos\,(2A)}{2} \\ sin^2 A &= \frac{1-cos\,(2A)}{2} \end{aligned} \tag{6}$$

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \le sin\,\theta \le |\theta| \text{ and } -|\theta| \le 1 - \cos\theta \le |\theta|\,.$$

Remark (Transformations of Trigonometric Graphs):

The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions.

Vertical stretch or compression; reflection about $y = d$ if negative

Vertical shift

$$y = af(b(x + c)) + d$$

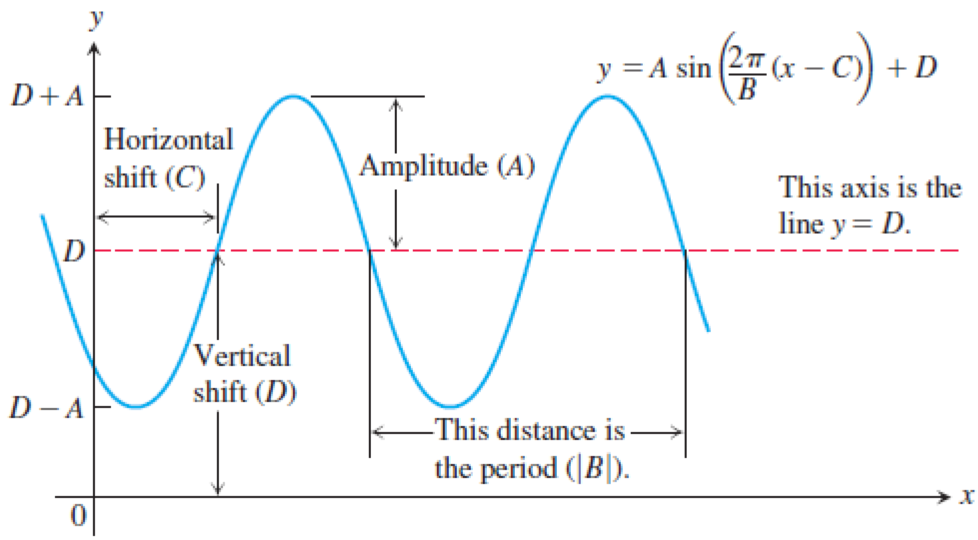
Horizontal stretch or compression; reflection about $x = -c$ if negative

Horizontal shift

The transformation rules applied to the sine function give the ***general sine function*** or ***sinusoid formula***

$$f(x) = A\sin(\tfrac{2\pi}{B}(x - C)) + D.$$

where $|A|$ is the ***amplitude***, B is the ***period***, C is the ***horizontal shift***, and D is the ***vertical shift***. A graphical interpretation of the various terms is given below.



Exercises:

1. Graph the following functions. What is the period of each function?

a) $sin\ 2x$ b) $sin\ (x/2)$ c) $cos\ \pi x$

d) $cos\ \frac{\pi x}{2}$ e) $-sin\frac{\pi x}{3}$ f) $-cos\ 2\pi x$

g) $cos\ \left(x - \frac{\pi}{2}\right)$ h) $sin\ \left(x + \frac{\pi}{6}\right)$ i) $sin\ \left(x - \frac{\pi}{4}\right) + 1$

j) $cos\ \left(x + \frac{2\pi}{3}\right) - 2$

2. Graph $y = cos\ x$ and $y = sec\ x$ together for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$. Comment on the behavior of $sec\ x$ in relation to the signs and values of $cos\ x$.

3. Graph $y = sin\ x$ and $y = csc\ x$ together for $-\pi \leq x \leq 2\pi$. Comment on the behavior of $csc\ x$ in relation to the signs and values of $sin\ x$.

4. Use the addition formulas to derive the following identities

a) $\cos\left(x - \frac{\pi}{2}\right) = sin\ x$ **b)** $\cos\left(x + \frac{\pi}{2}\right) = -sin\ x$

c) $sin\left(x + \frac{\pi}{2}\right) = cos\ x$ **d)** $\mathbf{sin}\left(x - \frac{\pi}{2}\right) = -cos\ x$

5. Express the given quantity in terms of $sin\ x$ and $cos\ x$.

a) $\cos(\pi + x)$ **b)** $sin(2\pi - x)$

c) $sin\left(\frac{3\pi}{2} - x\right)$ **d)** $\cos\left(\frac{3\pi}{2} + x\right)$

6. Evaluate $sin\frac{7\pi}{12}$ as $sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$.

7. Evaluate $cos\frac{11\pi}{12}$ as $sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$.

8. Evaluate $cos\frac{\pi}{12}$.

9. Evaluate $sin\frac{5\pi}{12}$.

10. Find the function values using the Half-Angle Formulas

a) $cos^2\frac{\pi}{8}$ **b)** $cos^2\frac{5\pi}{12}$ **c)** $sin^2\frac{\pi}{12}$ **d)** $sin^2\frac{3\pi}{8}$

11. solve for the angle θ, where $0 \leq \theta \leq 2\pi$.

a) $sin^2\theta = \frac{3}{4}$ **b)** $sin^2\theta = cos^2\theta$

c) $sin2\theta - \cos\theta = 0$ **d)** $cos2\theta - \cos\theta = 0$

1.4 Exponential Functions

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(22P) = 2^3P$. Continuing to double in this fashion leads us to consider the function $f(x) = 2^x$. We call this an exponential function because the variable x appears in the exponent of 2^x. Functions such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function $\boldsymbol{f(x) = a^x}$ is the ***exponential function with base a***.

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically by taking an appropriate number of products, quotients, or roots. If $x = n$ is a positive integer, the number an is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}},$$

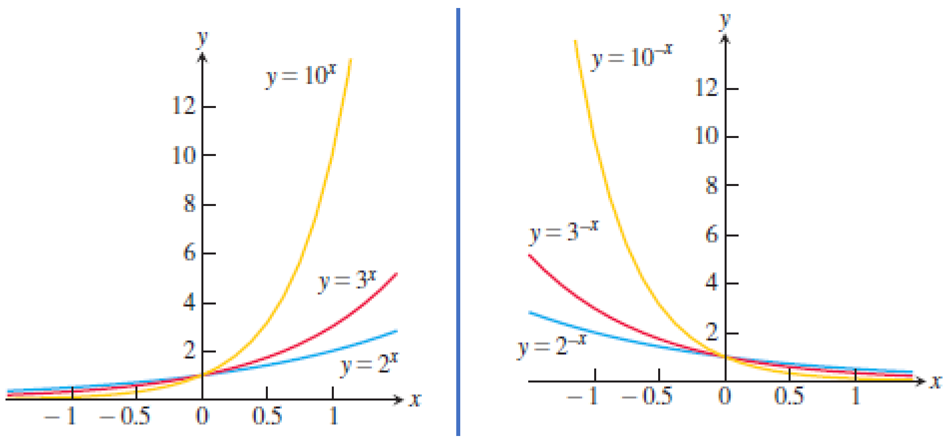
If $x = 0$, then we set $a^0 = 1$, and if $x = -n$ for some positive integer n, then $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$. If $x = 1/n$ for some positive integer n, then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a. If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

When x is irrational, the meaning of a^x is not immediately apparent. The value of a^x can be approximated by raising a to rational numbers that get closer and closer to the irrational number x.



Graphs of exponential functions.

Remark (Rules for Exponents):

If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$

2. $\frac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x \cdot b^x = (a \cdot b)^x$

5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Example:

Use the rules for exponents to simplify some numerical expressions.

1. $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$ **Rule 1**

2. $\frac{(\sqrt{10})^3}{\sqrt{10}} = \left(\sqrt{10}\right)^{3-1} = \left(\sqrt{10}\right)^2 = 10$ **Rule 2**

3. $\left(5^{\sqrt{2}}\right)^{\sqrt{2}} = 5^{\sqrt{2}\cdot\sqrt{2}} = 5^2 = 25$ **Rule 3**

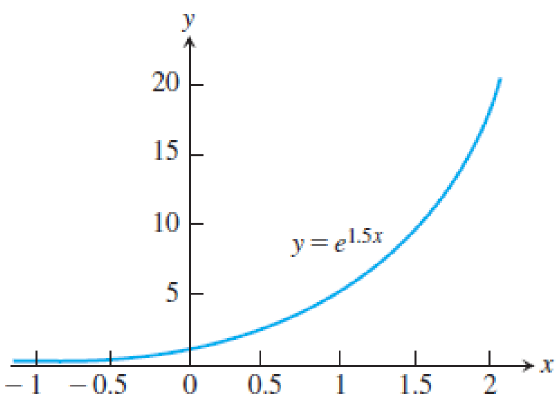
4. $7^\pi \cdot 8^\pi = (56)^\pi$ **Rule 4**

5. $\left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3}$ **Rule 5**

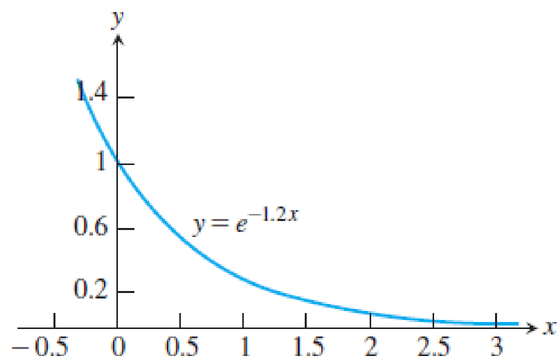
Remark:

The most important exponential function used for modeling natural, physical, and economic phenomena is the ***natural exponential function***, whose base is the special number e. The number e is irrational, and its value to nine decimal places is 2.718281828.

The function $y = y_0\, e^{kx}$, where k is a nonzero constant, is a model for ***exponential growth*** if $k > 0$ and a model for ***exponential decay*** if $k < 0$. Here y_0 is a positive constant that represents the value of the function when $x = 0$.



Graph of exponential growth, k = 1.5 > 0



Graph of exponential decay, k = −1.2 < 0

Exercises:

1. In following, sketch the given curves together in the appropriate coordinate plane, and label each curve with its equation

a) $y = 2^x, y = 4^x, y = 3^{-x}, y = (\frac{1}{5})^x$ **b)** $y = 2^{-t}$ *and* $y = -2^t$

c) $y = 3^x, y = 8^x, y = 2^{-x}, y = (\frac{1}{4})^x$ **d)** $y = 3^{-t}$ *and* $y = -3^t$

e) $y = e^x$ *and* $y = 1/e^x$ **f)** $y = -e^x$ *and* $y = -e^{-x}$

2. In each of following, sketch the shifted exponential curves.

a) $y = 2^x - 1$ *and* $y = 2^{-x} - 1$ **b)** $y = 3^x + 2$ *and* $y = 3^{-x} + 2$

c) $y = 1 - e^x$ *and* $y = 1 - e^{-x}$ **d)** $y = -1 - e^x$ *and* $y = -1 - e^{-x}$

3. Use the laws of exponents to simplify the expressions in following.

a) $16^2 \cdot 16^{-1.75}$ **b)** $9^{1/3} \cdot 9^{-1/6}$ **c)** $\frac{4^{4.2}}{4^{3.7}}$ **d)** $\frac{3^{5/3}}{3^{2/3}}$

e) $\left(25^{1/8}\right)^4$ **f)** $\left(13^{\sqrt{2}}\right)^{\sqrt{2}/2}$ **g)** $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$ **h)** $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$

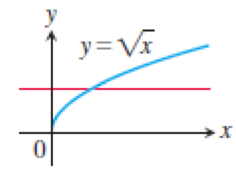
i) $\left(\frac{2}{\sqrt{2}}\right)^4$ **j)** $\left(\frac{\sqrt{6}}{3}\right)^2$

1.5 Inverse Functions and Logarithms

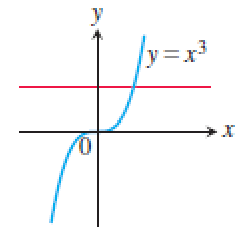
Definition:

A function $f(x)$ is ***one-to-one*** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Example:

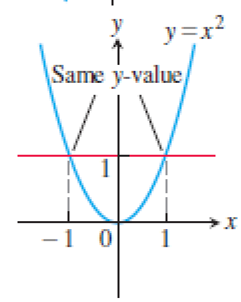


a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $f(x_1) = \sqrt{x_1} \neq f(x_2) = \sqrt{x_2}$ whenever $x_1 \neq x_2$.



b) $f(x) = x^3$ is one-to-one on their domain $(-\infty, \infty)$ because

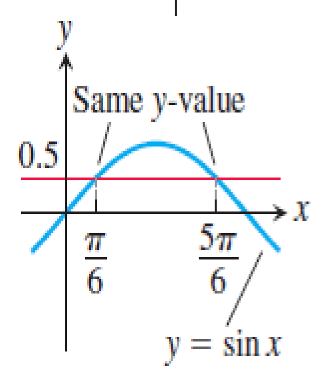
$f(x_1) = x_1^3 \neq f(x_2) = x_2^3$ whenever $x_1 \neq x_2$.



c) $g(x) = x^2$ is not one-to-one on interval $(-\infty, \infty)$ because

$f(-2) = (-2)^2 = 4 = 2^2 = f(2).$

d) $g(x) = \sin x$ is not one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$. The sine function is one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ and therefore gives distinct outputs for distinct inputs in that interval.



The Horizontal Line Test for One-to-One Functions:

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Definition:

Suppose that f is a one-to-one function on a domain D with range R. ***The inverse function*** f^{-1} is defined by

$$f^{-1}(b) = a \quad if \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Example:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

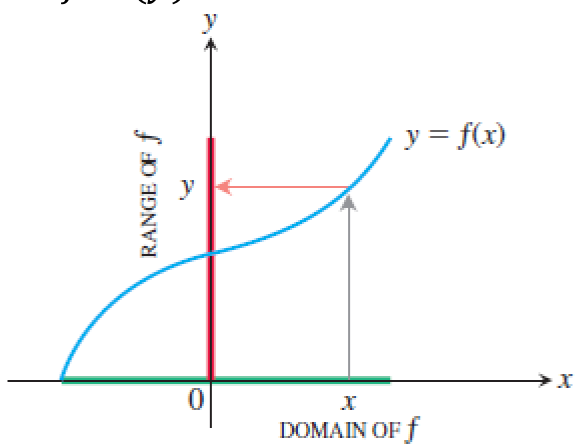
x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f:

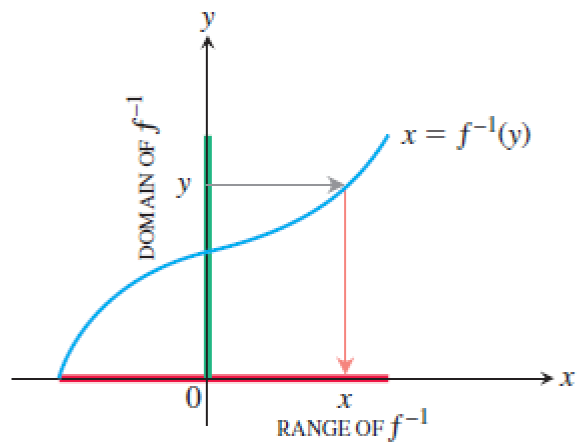
y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Remark:

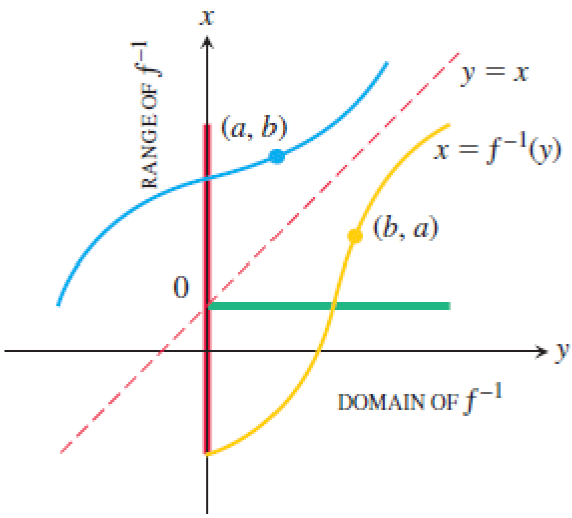
The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the *x*-axis to read the value of $x = f^{-1}(y)$.



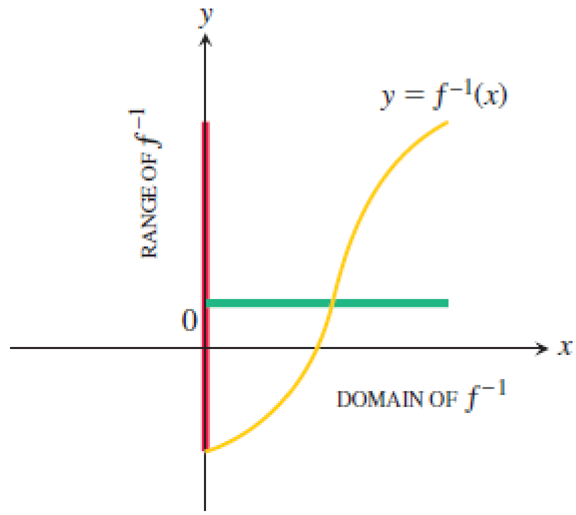
To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

Example:

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution:

1. Solve for x in terms of y:

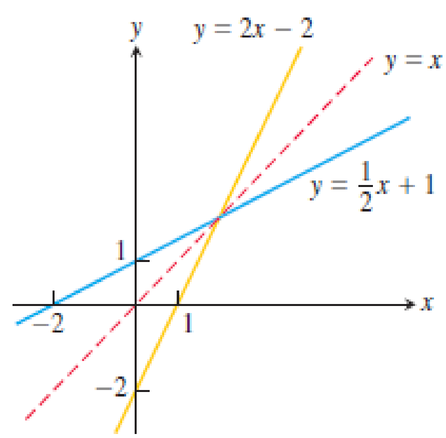
The graph is a straight line satisfying the horizontal line test

$$y = \tfrac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2$$

2. Interchange x and y:

Expresses the function in the usual form where y is the dependent variable.

$$y = 2x - 2$$



The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both compositions give the identity function:

$$f^{-1}(f(x)) = 2\left(\tfrac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

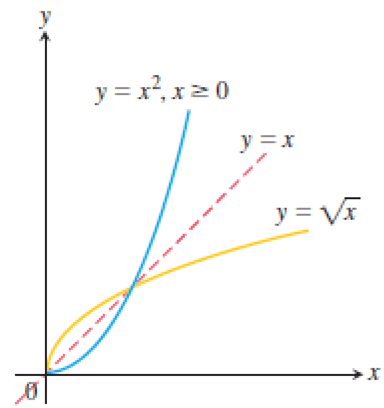
$$f(f^{-1}(x)) = \tfrac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

Example:

Find the inverse of $y = x^2, x \geq 0$, expressed as a function of x.

Solution:

For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y:



$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad \boldsymbol{|x| = x \text{ because } x \geq 0}$$

We then interchange x and y, obtaining $y = \sqrt{x}$.

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$.

Remark:

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, is one-to-one and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, is not one-to-one and therefore has no inverse.

Remark:

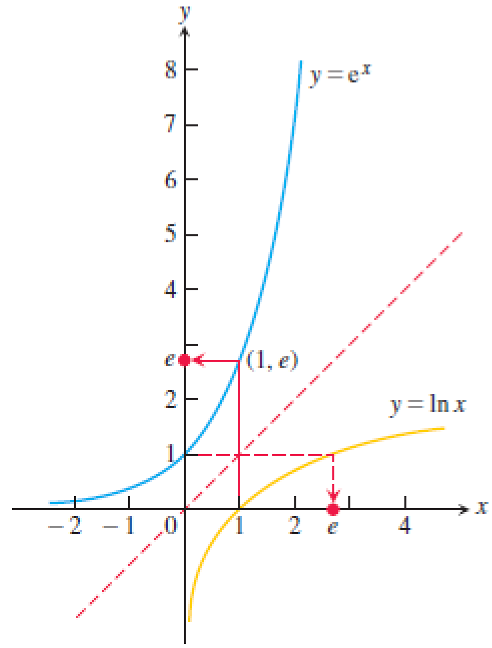
If a is any positive real number other than 1, then the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the logarithm function with base a.

Definition:

The ***logarithm function with base a***, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

Remark:

The graph of $y = a^x, a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$. Logarithms with base e and base 10 are so important in applications that many calculators have special keys for them. They also have their own special notation and names:



$\log_e x$ is written as $\ln x$.

$\log_{10} x$ is written as $\log x$.

The function $y = ln\, x$ is called the ***natural logarithm function***, and $y = log\, x$ is often called the ***common logarithm function***. Since the logarithm is the inverse function of exponentiation, it follows that:

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\ln x = y \Leftrightarrow e^y = x$$

In particular, because $e^1 = e$, we obtain $\ln e = 1$

Theorem (Algebraic Properties of the Natural Logarithm):

For any numbers b > 0 and x > 0, the natural logarithm satisfies the following rules:

1. ***Product Rule:*** $\boldsymbol{ln\, bx = ln\, b + ln\, x}$
2. ***Quotient Rule:*** $\boldsymbol{ln\, \frac{b}{x} = ln\, b - ln\, x}$
3. ***Reciprocal Rule*** $\boldsymbol{ln\, \frac{1}{x} = -\, ln\, x}$ ***Rule 2 with $b = 1$***
4. ***Power Rule:*** $\boldsymbol{ln\, x^r = r\, ln\, x}$

Example:

Use the properties in Theorem Algebraic Properties of the Natural Logarithm to rewrite three expressions.

a) $ln\, 4 + ln \sin x = \ln(4 \sin x)$ **Product Rule**

b) $ln\, \frac{x+1}{2x-3} = ln(x+1) - ln(2x-3)$ **Quotient Rule**

c) $ln\, \frac{1}{8} = -\, ln\, 8$ **Reciprocal Rule**

$= -\, ln\, 2^3 = -3 \ln 2$ **Power Rule**

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

Remark (Inverse Properties for a^x and $\log_a x$):

1. Base $a (a > 0, a \neq 1)$: $a^{log_a x} = x, x > 0$

$$\log_a a^x = x$$

2. Base e: $e^{ln\, x} = x, x > 0$

$$ln\, e^x = x$$

Remark:

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x\ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.For example $2^x = e^{(\ln 2)x} = e^{x\ln 2}$.

Remark (Change-of-Base Formula):

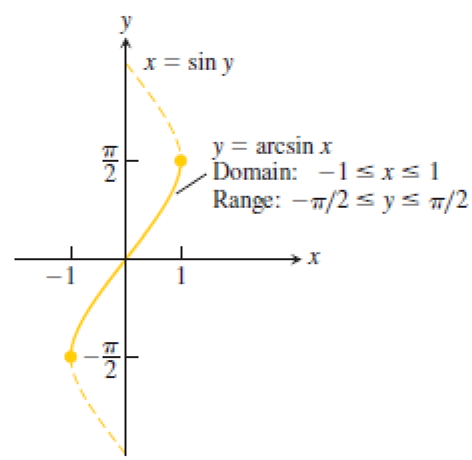
Since $\ln x = \ln(a^{\log_a x}) = (\log_a x)(\ln a)$ then every logarithmic function is a constant multiple of the natural logarithm.

$$\boldsymbol{\log_a x = \frac{\ln x}{\ln a}}, (a > 0, a \neq 1) .$$

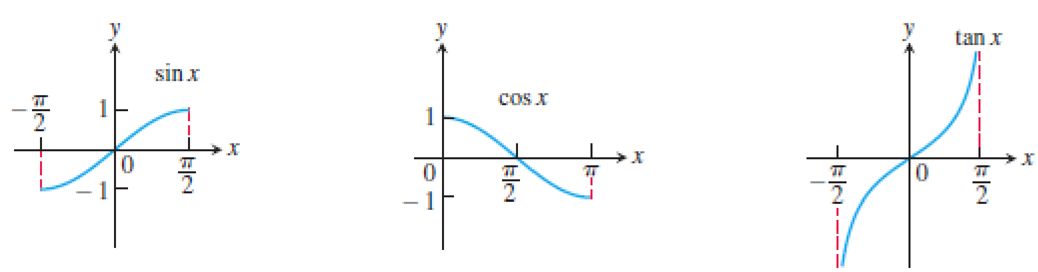
Remark (Inverse Trigonometric Functions):

The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one.

The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse which is called $\arcsin x$. Similar domain restrictions can be applied to all six trigonometric functions.



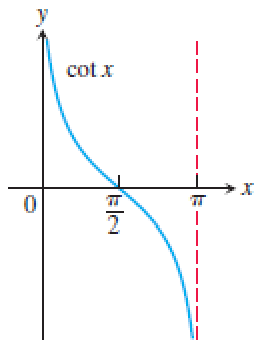
Now the domain restrictions that make the trigonometric functions one-to-one is shown in following:



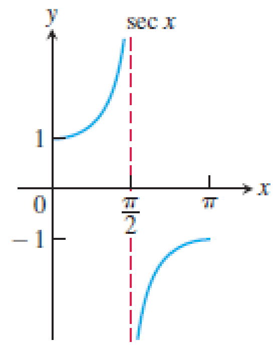
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

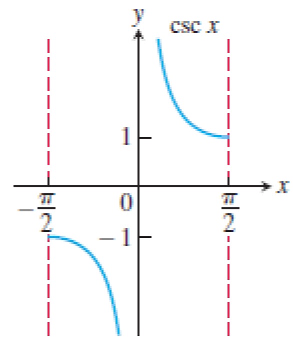
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = sin^{-1}x \ or \ y = \arcsin x\,, \quad y = cos^{-1}x \ or \ y = \arccos \ x$$

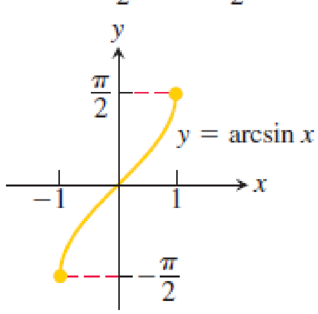
$$y = tan^{-1}x \ or \ y = \arctan x\,, \quad y = cot^{-1}x \ or \ y = \operatorname{arccot} \ x$$

$$y = sec^{-1}x \ or \ y = \operatorname{arcsec} x\,, \quad y = csc^{-1}x \ or \ y = \operatorname{arccsc} \ x$$

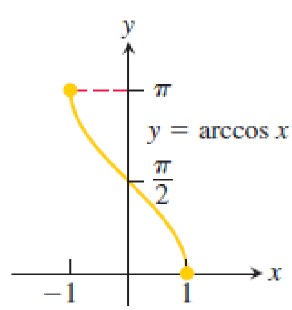
The -1 in the expressions for the inverse means “inverse.” It does not mean reciprocal. For example, the reciprocal of $sin\ x$ is $(sin\ x\,)^{-1} = \frac{1}{sin\,x} = csc\ x$.

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$.

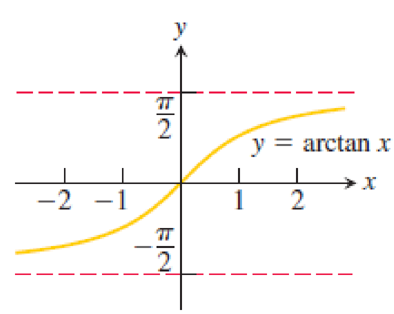
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

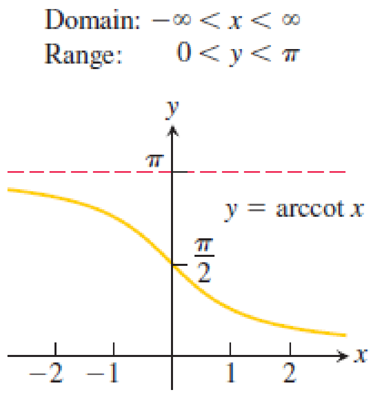


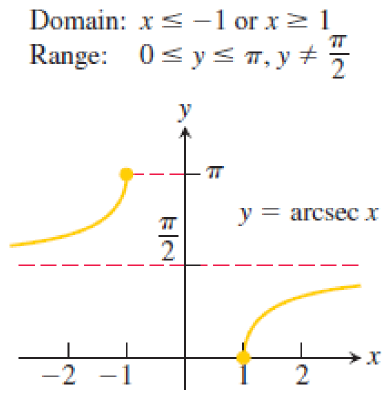
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$

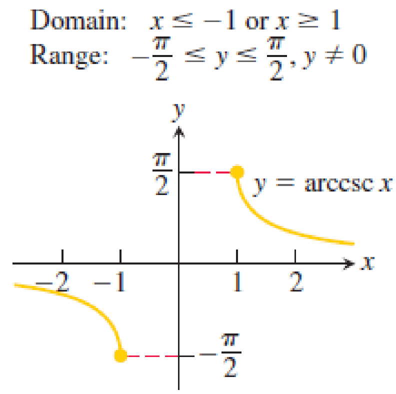


Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$









The graph of $y = arcsin\ x$ is symmetric about the origin. The arcsine is therefore an odd function: $arcsin(-x) = -arcsin\ x$. The graph of $y = arccos\ x$ has no such symmetry.

Definition:

$y = arcsin\ x$ is the number in $[-\pi/2, \pi/2]$for which $sin\ y = x$.

$y = arccos\ x$ is the number in $[\ 0, \pi\ ]$ for which $cos\ y = x$.

Example:

Evaluate **(a)** $arcsin\ (\frac{\sqrt{3}}{2})$ and **(b)** $\arccos(-\frac{1}{2})$.

Solution:

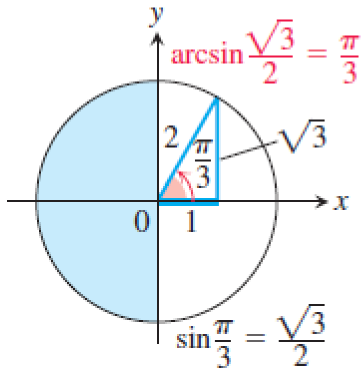
(a) We see that $arcsin\ \left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ because $sin(\ \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ belongs to the range$[-\pi/2, \pi/2]$ of the arcsine function.

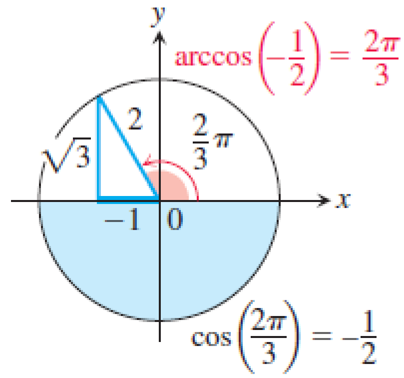
(b) We see that $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$ because $cos\left(\ \frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\frac{2\pi}{3}$ belongs to the range$[\ 0, \pi\ ]$ of the arcsine function.

Remark:

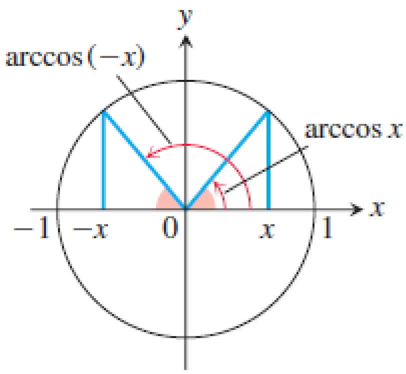
Using the same procedure illustrated in previous example, we can create the following table of common values for the arcsine and arccosine functions.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$





Remark:

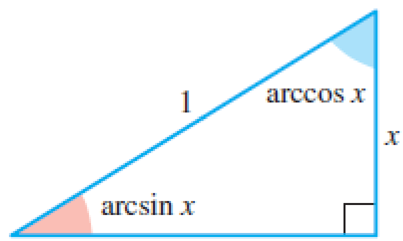


the arccosine of x satisfies the identity

$$\arccos x + \arccos(-x) = \pi,$$

Or

$$\arccos(-x) = \pi - \arccos x.$$

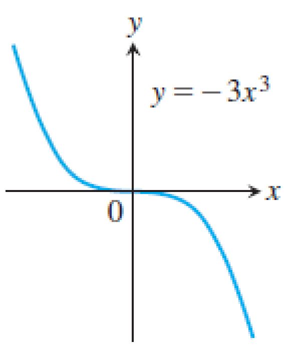


Also, we can see that for $x > 0$, $\arcsin x + \arccos x = \pi/2$.

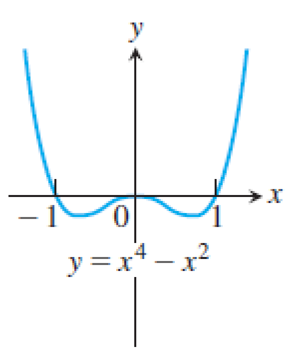
Exercises:

1. Which of the functions graphed in following are one-to-one, and which are not?

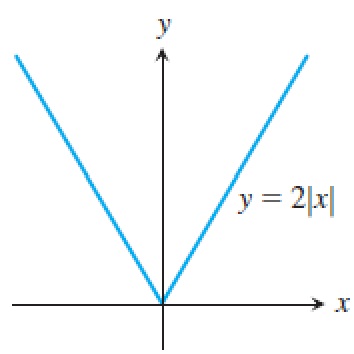
a)



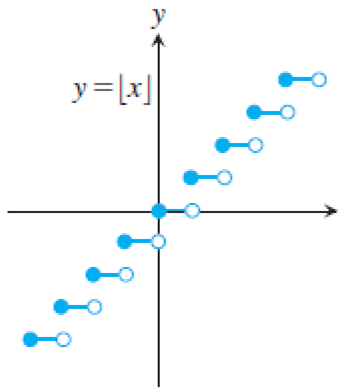
b)



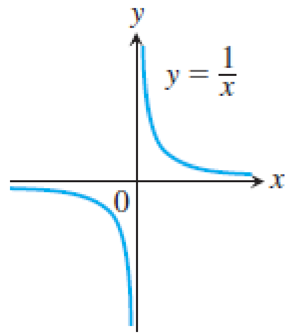
c)



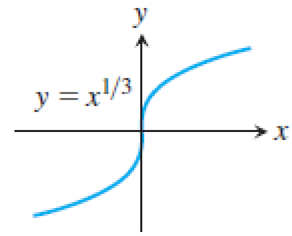
d)



e)



f)



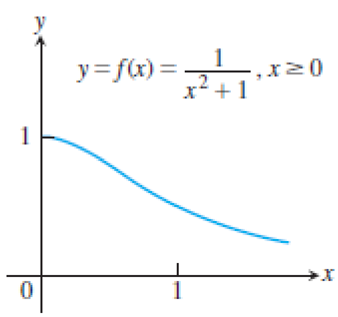
2. In following, determine from its graph whether the function is one-to-one.

a) $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$ **b)** $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$

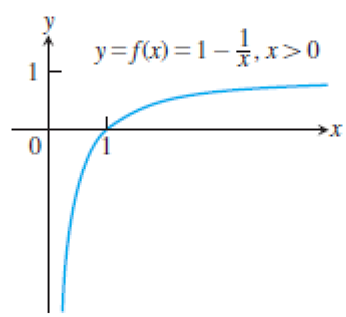
c) $f(x) = \begin{cases} 1 - \frac{x}{2}, & x \leq 0 \\ \frac{x}{x+2}, & x > 0 \end{cases}$ **d)** $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

3. Each of following shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use reflection with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

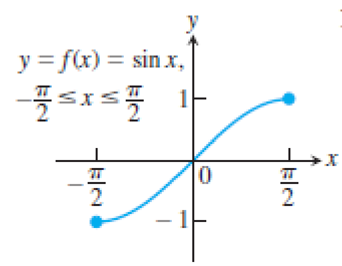
a)



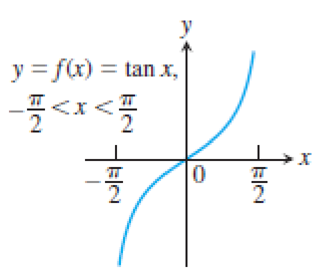
b)



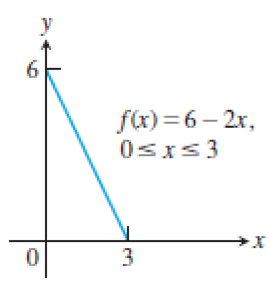
c)



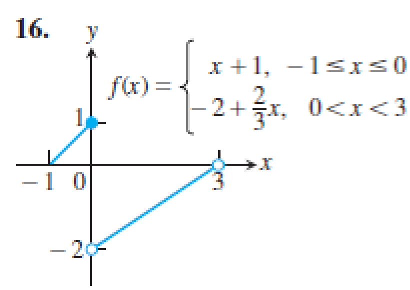
d)



e)



f)



4. a) Graph the function $f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1$. What symmetry does the graph have?

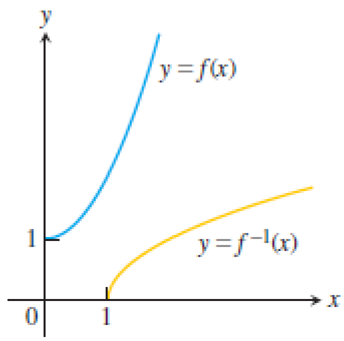
b) Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.

5. **a)** Graph the function $f(x) = 1/x$. What symmetry does the graph have?

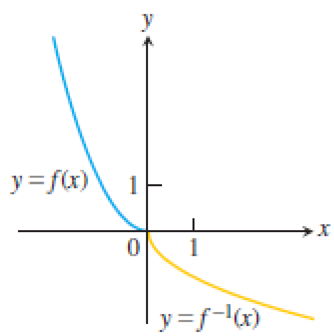
b) Show that f is its own inverse.

6. Each of following gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

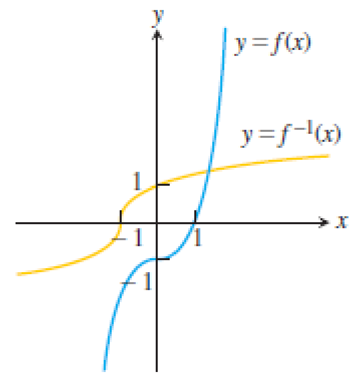
a) $f(x) = x^2 + 1, x \geq 0$



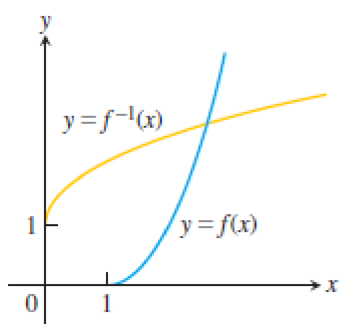
b) $f(x) = x^2, x \leq 0$



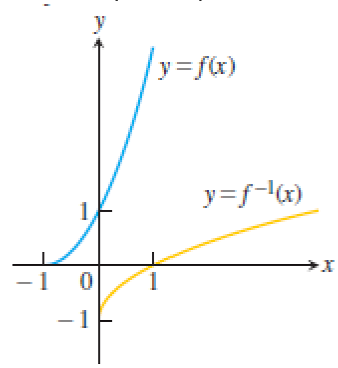
c) $f(x) = x^3 - 1$



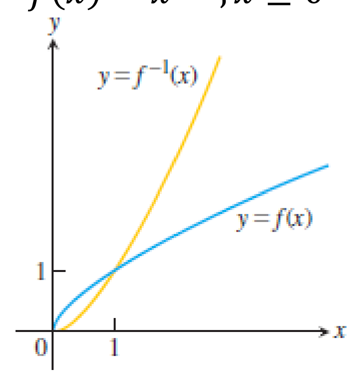
d) $f(x) = x^2 - 2x + 1, x \geq 1$



e) $f(x) = (x+1)^2, x \geq -1$



f) $f(x) = x^{2/3}, x \geq 0$



8. Each of following gives a formula for a function $y = f\ (x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f\ (x)\right) = x$.

a) $f(x) = x^5$ **b)** $f(x) = x^4, x \geq 0$ **c)** $f(x) = x^3 + 1$

d) $f(x) = (1/2)x - 7/2$ **e)** $f(x) = 1/x^2, x > 0$ **f)** $f(x) = 1/x^3, x \neq 0$

g) $f(x) = \frac{x+3}{x-2}$ **h)** $f(x) = \frac{\sqrt{x}}{\sqrt{x}-3}$ **i)**

9. Express the following logarithms in terms of ln 2 and ln 3.

a) $ln\ 0.75$ **b)** $ln(4/9)$ **c)** $ln(1/\ 2)$ **d)** $ln\sqrt[3]{9}$ **e)** $ln3\sqrt{2}$ **f)** $ln\sqrt{13.5}$

10. Use the properties of logarithms to write the expressions in following as a single term.

a) $\ln\sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ **b)** $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$ **c)** $\frac{1}{2}\ln(4t^4) - \ln b$

d) $\ln\sec\theta + \ln\cos\theta$ **e)** $\ln(8x+4) - 2\ln c$ **f)** $3\ln\sqrt[3]{t^2-1} - \ln(t+1)$

11. Find simpler expressions for the quantities in following:

a) $e^{\ln 7.2}$ **b)** $e^{-\ln x^2}$ **c)** $e^{\ln x - \ln y}$ **d)** $e^{\ln(x^2+y^2)}$

e) $e^{-\ln 0.3}$ **f)** $e^{\ln \pi x - \ln 2}$ **g)** $2\ln\sqrt{e}$ **h)** $\ln(\ln e^e)$

i) $\ln(e^{-x^2-y^2})$ **j)** $\ln(e^{\sec\theta})$ **k)** $\ln(e^{(e^x)})$ **l)** $\ln(e^{2\ln x})$

12. In following, solve for y in terms of t or x, as appropriate.

a) $\ln y = 2t + 4$ **b)** $\ln y = -t + 5$

c) $\ln(y - b) = 5t$ **d)** $\ln(c - 2y) = t$

e) $\ln(y-1) - \ln 2 = x + \ln x$ **f)** $\ln(y^2 - 1) - \ln(y+1) = \ln(\sin x)$

13. Express the ratios in following as ratios of natural logarithms and simplify.

a) $\frac{\log_2 x}{\log_3 x}$ **b)** $\frac{\log_2 x}{\log_8 x}$ **c)** $\frac{\log_x a}{\log_{x^2} a}$ **d)** $\frac{\log_9 x}{\log_3 x}$ **e)** $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **f)** $\frac{\log_a b}{\log_b a}$

14. In following, find the exact value of each expression.

a) $sin^{-1}\left(\frac{-1}{2}\right)$ **b)** $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c)** $sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ **d)** $cos^{-1}\left(\frac{1}{2}\right)$ **e)** $cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

f) $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **g)** $\arccos(-1)$ **h)** $\arccos(0)$ **i)** $\arcsin(-1)$ **j)** $\arcsin\left(\frac{-1}{\sqrt{2}}\right)$