5. Polynomials:

A polynomial is a mathematical expression constructed with constants and one variable x involving only non-negative integer powers of x, using the four operations:

That is a polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Where the a's are the coefficients of the polynomial (real numbers or complex numbers).

Definition:

The **leading coefficient** of the polynomial is $a_n \neq 0$ and a_0 *is constant*.

Definition:

The **degree of a polynomial** is the highest power of x in its expression.

Constant (non-zero) polynomials, linear polynomials, quadratics, cubics and quartics are polynomials of degree 0, 1, 2 , 3 and 4 respectively.

Polynomial	Example	Degree
Constant	1	0
Linear	$2x+1$	1
Quadratic	$3x^2+2x+1$	2
Cubic	$4x^3+3x^2+2x+1$	3
Quartic	$5x^4+4x^3+3x^2+2x+1$	4

Example:

Decide whether the following function polynomial, if it is write in standard form, state its degree and write the leading coefficient ?:

1- $f(x) = \frac{1}{2}x^2 - 3x^4 - 7$

The function is a polynomial function.

Its standard form is $f(x) = -3x^4 + \frac{1}{2}x^2 - 7.$

It has degree 4, so it is a quartic function.

The leading coefficient is -3.

2- $f(x) = 3^x + x^3$

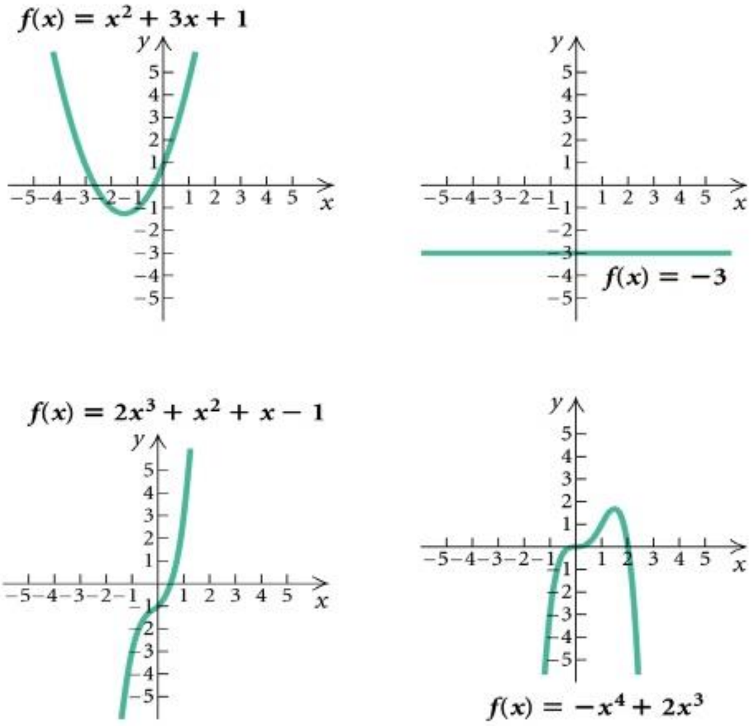
3- $f(x) = 4x^3 + \sqrt{x} - 1$

4- $f(x) = 5x^4 - 2x^2 + 3/x$

5- $f(x) = 0.5x + \pi x^2 - \sqrt{2}$

H.W.

Graph:



(5.1) Properties of Polynomials

Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = \sum_{i=0}^{n} a_i x^i$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0 = \sum_{i=0}^{m} b_i x^i$$

be two polynomials. Then

(i) $p(x)=q(x) \Leftrightarrow n=m \text{ and } a_n = b_m, a_{n-1} = b_{m-1}, \ldots a_0 = b_0$.

(ii) If $m \le n$ then $p(x) + q(x)$ is a polynomial of degree $\le n$.

(iii) $p(x).q(x)$ is a polynomial of degree n + m

(iv) If $p(x) \neq 0$ and $p(x) \cdot q(x) = 0$ then $q\,(x) = 0$
(v) If $q\,(x) \;\neq 0$ and $f(x) \cdot q(x) = p(x){\cdot}q(x)$ then $f(x) = p(x)$.

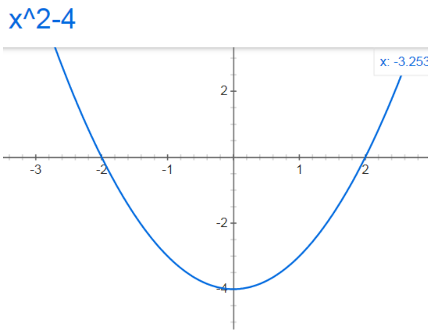
Example:
Let $f(x) = 1 + 2x^2$ and $g(x) = 3 + 4x - 2x^2$ find f+g , f.g and their degree

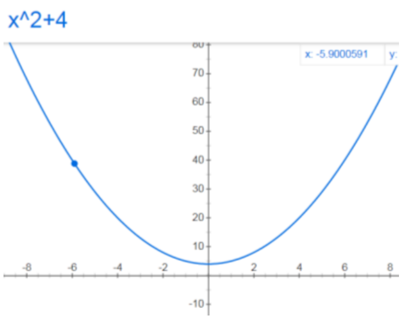
H.W.

Definition
A number r is said to be a zero or root of a function f(x) if f(r) = 0, and this occurs when (x−r) is a factor of f(x)

The zeros of (x) are the solutions of the equation $(x) = 0$. So if the coefficients of a polynomial (x) are real numbers, then the real zeros of (x) are just the x intercepts of the graph of (x). For example, the real zeros of the polynomial $P(x) = x^2 - 4$ are 2 and -2, the x- intercepts of the graph of (x). However, a polynomial may have zeros that are not x - intercepts.

$Q(x) = x^2 + 4$, for example, has zeros $2i$ and $-2i$, but its graph has no x – intercepts.





(5.2) Polynomial Division (Long Division)
The process is very like the long division of numbers. Example below will illustrate the process.

Example
Divide $P(x) = 3x^3 - 5 + 2x^4 - x$ by $(2 + x)$.

Solution: First, rewrite the dividend (x) in descending powers of x, inserting 0 as the coefficient for any missing terms of degree less than 4:
$P(x) = 2x^4 + 3x^3 + 0x^2 - x - 5.$

Similarly, rewrite the divisor $(2 + x)$ in the form $(x + 2)$. Then divide the first term x of the divisor into the first term $2x^4$ of the dividend. The result will be

$2x^3$, Multiply it by the divisor, obtaining $2x^4 + 4x^3$. Line up like terms, subtract as in arithmetic, and bring down $0x^2$. Repeat the process until the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{lrl}
 & 2x^3 - x^2 + 2x - 5 & \text{Quotient} \\
\text{Divisor} \quad x + 2 \,\big) & 2x^4 + 3x^3 + 0x^2 - x - 5 & \text{Dividend} \\
 & \underline{2x^4 + 4x^3} \qquad\qquad\qquad\quad & \text{Subtract} \\
 & -x^3 + 0x^2 \qquad\qquad & \\
 & \underline{-x^3 - 2x^2} \qquad\qquad & \text{Subtract} \\
 & 2x^2 - x \qquad & \\
 & \underline{2x^2 + 4x} \qquad & \text{Subtract} \\
 & -5x - 5 & \\
 & \underline{-5x - 10} & \text{Subtract} \\
 & 5 & \text{Remainder}
\end{array}
$$

Then

$$\frac{2x^4 + 3x^3 - x - 5}{x + 2} = 2x^3 - x^2 + 2x - 5 + \frac{5}{x + 2}$$

H.W: check your answer using multiplication.

(5.3) Division Algorithm Theorem for Polynomials:

(Dividend = Quotient × Divisor + Remainder)

Suppose f(x) and g(x) are the two polynomials, where g(x)≠0, there are unique polynomials q(x) and r(x) such that : f(x) = q(x) g(x) + r(x) where r(x) is the remainder polynomial and degree r(x) < degree g(x).

That is this leads to : For each polynomial $P(x)$ of degree greater than 0 and each number r, there exists a unique polynomial $Q(x)$ of degree less than $P(x)$ and a unique number R such that

$$P(x) = (x - r)Q(x) + R.$$

The polynomial $P(x)$ is called the **dividend**, $Q(x)$ is the **quotient,** $(x - r)$ is the **divisor,** and R is the **remainder.** Note that R may be 0.

Verification of Division Algorithm

Take the above example and verify it.:

$$P(x) = 2x^4 + 3x^3 + 0x^2 - x - 5 = (x + 2)(2x^4 + 4x^3) + 5$$