CHAPTER (3)
NUCLEAR MODELS

At this point it is tempting to try to extend the ideas of the previous chapter to heavier nuclei.

The Liquid Drop Model and the Semi-Empirical Mass Formula

The analogy with a liquid, on the other hand, turns out to be extremely useful in understanding certain aspects of nuclear behavior. Let us see how the picture of a nucleus as a drop of liquid accounts for the observed variation of binding energy per nucleon with mass number. We start by assuming that the energy associated with each nucleon-nucleon bond has some value **U**. This energy is actually negative since attractive forces are involved, but is usually written as positive because binding energy is considered a positive quantity for convenience.

Because each bond energy **U** is shared by two nucleons, each has a binding energy of 1/2**U**. When an assembly of spheres of the same size is packed together into the smallest volume, as we suppose is the case of nucleons within a nucleus, each interior sphere has 12 other spheres in contact with it as shown in Figure (1). Hence each interior nucleon in a nucleus has a binding energy of (12)(1/2 **U**) or 6 **U**. If all A nucleons in a nucleus were in its interior, the total binding energy of the nucleus would be :

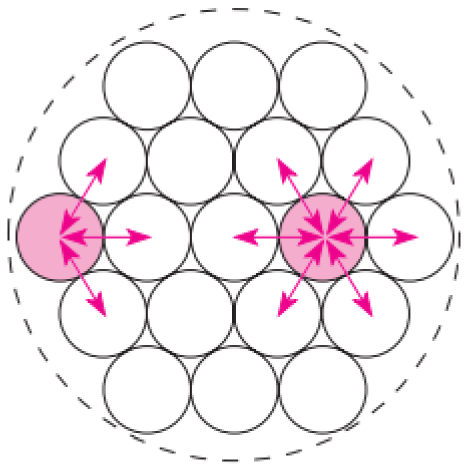

Figure 1 A nucleon at the surface of a nucleus interacts with fewer other nucleons than one in the interior of the nucleus and hence its binding energy is less. The larger the nucleus, the smaller the proportion of nucleons at the surface.

Volume energy $E_v = a_1 A$ ----- (1)

The energy E_v is called the volume energy of a nucleus and is directly proportional to A.

Actually, of course, some nucleons are on the surface of every nucleus and therefore have fewer than 12 neighbors as shown in Figure (1). The number of such nucleons depends on the surface area of the nucleus in question. A nucleus of radius R has an area of $4\pi R^2 = 4\pi(r_o)^2 A^{2/3}$. Hence the number of nucleons with fewer than the maximum number of bonds is proportional to $A^{2/3}$, reducing the total binding energy by:

Surface energy $E_s = a_2 A^{2/3}$ -------(2)

The negative energy E_s is called the surface energy of a nucleus. It is most significant for the lighter nuclei since a greater fraction of their nucleons are on the surface.

Because natural systems always tend to evolve toward configurations of minimum potential energy, nuclei tend toward configurations of maximum binding energy. Hence a nucleus should exhibit the same surface-tension effects as a liquid drop, and in the absence of other effects it should be spherical, since a sphere has the least surface area for a given volume.

The electric repulsion between each pair of protons in a nucleus also contributes toward decreasing its binding energy. The coulomb energy E_c of a nucleus is the work that must be done to bring together Z protons from infinity into a spherical aggregate the size of the nucleus. The potential energy of a pair of protons r apart is equal to:

$$V = -\frac{e^2}{4\pi\epsilon_0 r} \quad \text{-------- (3)}$$

Since there are Z (Z-1)/2 pairs of protons, then:

$$E_c = \frac{Z(Z-1)}{2} V = -\frac{Z(Z-1)e^2}{8\pi\epsilon_0}\left(\frac{1}{r}\right)_{av} \quad \text{-------- (4)}$$

where $(1/r)_{av}$ is the value of (1/r) averaged over all proton pairs. If the protons are uniformly distributed throughout a nucleus of radius R, $(1/r)_{av}$ is proportional to $1/R$ and hence to $1/A^{1/3}$, so that:

Coulomb energy $$E_c = -a_3 \frac{Z(Z-1)}{A^{1/3}} \quad \text{-------- (5)}$$

The coulomb energy is negative because it arises from an effect that opposes nuclear stability. This is as far as the liquid-drop model itself can go. Let us now see how the result compares with reality.

As it happens, the greater the number of nucleons in a nucleus, the smaller is the energy level spacing **€**, with **€** proportional to $1/A$. This means that the asymmetry energy E_a due to the difference between N and Z can be expressed as:

Asymmetry energy $$E_a = -\Delta E = -a_4 \frac{(A-2Z)^2}{A} \quad \text{-------- (6)}$$

The asymmetry energy is negative because it reduces the binding energy of the nucleus. These effects occurs when the neutrons in a nucleus outnumber the protons, which means that higher energy levels have to be occupied than would be the case if N and Z were equal.

The last correction term arises from the tendency of proton pairs and neutron pairs to occur. ***Even-even*** nuclei are the most stable and hence have higher binding energies than would otherwise be expected. Thus such nuclei as $^{4}_{2}He$, $^{12}_{6}C$, and $^{16}_{8}O$ appear as peaks on the empirical curve of binding energy per nucleon. At the other extreme, ***odd-odd*** nuclei have both unpaired protons and neutrons and have relatively low binding energies. The pairing energy E_p is positive for ***even-even*** nuclei, **0** for ***odd-even*** and ***even-odd*** nuclei, and negative for ***odd-odd*** nuclei, and seems to vary with A as $A^{-3/4}$. Hence:

Pairing energy

$$E_p = (\pm, 0)\frac{a_5}{A^{3/4}} \quad \text{-------- (7)}$$

$$E_p = \delta = \begin{cases} +\Delta, & \text{for even } N \text{ and } Z \\ 0, & \text{for odd } A \text{ (even/odd } N/Z \text{ or vice-versa)} \\ -\Delta, & \text{for odd } N \text{ and } Z \end{cases}$$

$$\text{Thus, } \Delta = \frac{a_5}{A^{3/4}}$$

The final expression for the binding energy of a nucleus of atomic number Z and mass number A, is:

Semi-empirical binding-energy formula

$$E_b = a_1A - a_2A^{2/3} - a_3\frac{Z(Z-1)}{A^{1/3}} - a_4\frac{(A-2Z)^2}{A}(\pm, 0)\frac{a_5}{A^{3/4}} \quad \text{------- (8)}$$

A set of coefficients that gives a good fit with the data is as follows:
a_1= 14.1MeV , a_2= 13.0MeV , a_3= 0.595MeV , a_4= 19.0MeV , and a_5= 33.5MeV .

Thus, the total binding energy E_b of a nucleus ought to be the sum of its volume, surface, coulomb, asymmetry, and pairing energies:

$$E_b = E_v + E_s + E_c + E_a + E_P \quad \text{-------- (9)}$$

Where, the first term is the *volume term* that accounts for the fact that the total binding energy is proportional to the number of nucleons (i.e. the sum of all nuclear interactions), while the second *surface effect term* reduces the binding energy because nucleons on the edge (or outer surface) of the nucleus are "missing" some nuclear interactions with non-existing neighbours. The third term is the loss in binding that is due to counteracting the *Coulomb* repulsion between protons. It equals the quantity calculated in equation (4), with the self-energy of the proton subtracted.

The fourth term is the *symmetry term* and accounts for the experimental fact that nature appears to favour similar numbers of protons and neutrons as much as possible. This term is consistent with the Pauli exclusion principle. Finally, the last term reflects the fact that nuclei are more stable when they possess even numbers of protons and neutrons. This is also a function of the Pauli exclusion principle since neutrons and protons are fermions.

Example 1:

The atomic mass of the zinc isotope $^{64}_{30}\text{Zn}$ is 63.929 u. Compare its binding energy with the prediction of Eq. (8).

Solution

The binding energy of $^{64}_{30}\text{Zn}$ is,

$$E_b = [(30)(1.007825\ \text{u}) + (34)(1.008665\ \text{u}) - 63.929\ \text{u}](931.49\ \text{MeV/u}) = 559.1\ \text{MeV}$$

The semiempirical binding energy formula, using the coefficients in the text, gives

$$E_b = (14.1\ \text{MeV})(64) - (13.0\ \text{MeV})(64)^{2/3} - \frac{(0.595\ \text{MeV})(30)(29)}{(64)^{1/3}} - \frac{(19.0\ \text{MeV})(16)}{64} + \frac{33.5\ \text{MeV}}{(64)^{3/4}} = 561.7\ \text{MeV}$$

The plus sign is used for the last term because $^{64}_{30}\text{Zn}$ is an even-even nucleus. The difference between the observed and calculated binding energies is less than 0.5 percent.

The existence of the *Coulomb term* and the *asymmetry term* means that for each ***A*** there is a nucleus of maximum binding energy found by setting $\partial \mathbf{B}/\partial \mathbf{Z} = \mathbf{0}$. As, the maximally bound nucleus has (***Z*** = ***N*** = ***A***/2) for low ***A*** where the asymmetry term dominates but the Coulomb term favors (***N*** > ***Z***) for large ***A***.

Example 2:

Isobars are nuclides that have the same mass number A. Derive a formula for the atomic number of the most stable isobar of a given A and use it to find the most stable isobar of $A = 25$.

Solution

To find the value of Z for which the binding energy E_b is a maximum, which corresponds to maximum stability, we must solve $dE_b/dZ = 0$ for Z. From Eq. (8) we have

$$\frac{dE_b}{dZ} = -\frac{a_3}{A^{1/3}}(2Z - 1) + \frac{4a_4}{A}(A - 2Z) = 0$$

$$Z = \frac{a_3 A^{-1/3} + 4a_4}{2a_3 A^{-1/3} + 8a_4 A^{-1}} = \frac{0.595A^{-1/3} + 76}{1.19A^{-1/3} + 152A^{-1}}$$

For $A = 25$ this formula gives $Z = 11.7$, from which we conclude that $Z = 12$ should be the atomic number of the most stable isobar of $A = 25$. This nuclide is $^{25}_{12}\text{Mg}$, which is in fact the only stable $A = 25$ isobar. The other isobars, $^{25}_{11}\text{Na}$ and $^{25}_{13}\text{Al}$, are both radioactive.

SHELL MODEL

Magic numbers in the nucleus

The ***shell model*** of the nucleus is an attempt to account for the existence of magic numbers and certain other nuclear properties in terms of nucleon behavior in a common force field.

The electrons in an atom may be thought of as occupying positions in "shells" designated by the various principal quantum numbers. The degree of occupancy of the outermost shell is what determines certain important aspects of an atom's behavior. For instance, atoms with 2, 10, 18, 36, 54, and 86 electrons have all their electron shells completely filled. Such electron structures have high binding energies and are exceptionally stable, which accounts for the chemical inertness of the rare gases.

The same kind of effect is observed with respect to nuclei. Nuclei that have 2, 8, 20, 28, 50, 82, and 126 neutrons or protons are more abundant than other nuclei of similar mass numbers, suggesting that their structures are more stable. Since complex nuclei arose from reactions among lighter ones, the evolution of heavier and heavier nuclei became retarded when each relatively inert nucleus was formed, which accounts for their abundance.

Other evidence also points up the significance in nuclear structure of the numbers 2, 8, 20, 28, 50, 82, and 126, which have become known as magic numbers. An example is the observed pattern of nuclear electric quadrupole moments, which are measures of how much nuclear charge distributions depart from sphericity. A spherical nucleus has no quadrupole moment, while one shaped like a football has a positive moment and one shaped like a pumpkin has a negative moment. Nuclei of magic ***N*** and ***Z*** are found to have zero quadrupole moments and hence are spherical, while other nuclei are distorted in shape.

Because the precise form of the potential-energy function for a nucleus is not known, unlike the case of an atom, a suitable function **U(r)** has to be assumed. Schrödinger's equation for a particle in a potential well of this kind is then solved, and it is found that stationary states of the system occur that are characterized by quantum numbers **n**, ***l***, and $\mathbf{m}_l$ whose significance is the same as in the analogous case of stationary states of atomic electrons. Neutrons and protons occupy separate sets of states in a nucleus because the latter interact electrically as well as through the specifically nuclear charge.

How Magic Numbers Arise **?**

The problem was finally solved independently by incorporate a spin-orbit interaction whose magnitude is such that the consequent splitting of energy levels into sublevels is many times larger than the analogous splitting of atomic energy levels. The exact form of the potential-energy function then turns out not to be critical, provided that it more or less resembles a square well.

The shell theory assumes that **LS** coupling holds only for the very lightest nuclei, in which the $\boldsymbol{l}$ values are necessarily small in their normal configurations. In this scheme, the intrinsic spin angular momenta $\boldsymbol{S_i}$ of the particles concerned (the neutrons form one group and the protons another) are coupled together into a total spin momentum $\boldsymbol{S}$. The orbital angular momenta $\boldsymbol{L_i}$ are separately coupled together into a total orbital momentum $\boldsymbol{L}$. Then $\boldsymbol{S}$ and $\boldsymbol{L}$ are coupled to form a total angular momentum **J** of magnitude J $\sqrt{J(J+1)}\hbar$.

After a transition region in which an intermediate coupling scheme holds, the heavier nuclei exhibit ***jj*** coupling. In this case the $\boldsymbol{S_i}$ and $\boldsymbol{L_i}$ of each particle are first coupled to form a $\mathbf{J_i}$ for that particle of magnitude $\sqrt{J(J+1)}\hbar$. The various $\mathbf{J_i}$ then couple together to form the total angular momentum **J**. The **jj** coupling scheme holds for the great majority of nuclei.

When an appropriate strength is assumed for the spin-orbit interaction, the energy levels of either class of nucleon fall into the sequence shown in Figure (2). The levels are designated by a prefix equal to the total quantum number n, a letter that indicates l for each particle in that level according to the usual pattern (s, p, d, f, g, . . .) corresponding, respectively, to $\boldsymbol{l}$ = (0, 1, 2, 3, 4, . . .), and a subscript equal to $\boldsymbol{j}$. The spin-orbit interaction splits each state of given $\boldsymbol{j}$ into $2\boldsymbol{j} + 1$ sub states, since there are $2\boldsymbol{j} + 1$ allowed orientations of $\mathbf{J_i}$. Large energy gaps appear in the spacing of the levels at intervals that are consistent with the notion of separate shells. The number of available nuclear states in each nuclear shell is, in ascending order of energy, 2, 6, 12, 8, 22, 32, and 44. Hence shells are filled when there are 2, 8, 20, 28, 50, 82, and 126 neutrons or protons in a nucleus.

The shell model accounts for several nuclear phenomena in addition to magic numbers. To begin with, the very existence of energy sublevels that can each be occupied by two particles of opposite spin explains the tendency of nuclear abundances to favor even $\boldsymbol{Z}$ and even $\boldsymbol{N}$. The ***shell model*** can also predict nuclear angular momenta. In *even-even* nuclei, all the protons and neutrons should pair off to cancel out one another's spin and orbital angular momenta. Thus *even-even* nuclei ought to have zero nuclear angular momenta, as observed. In *even-odd* and *odd-even* nuclei, the half-integral spin of the single **"extra"** nucleon should be combined with the integral angular momentum of the rest of the nucleus for a half-integral total angular momentum. *Odd-odd* nuclei each have an extra neutron and an extra proton whose half-integral spins should yield integral total angular momenta. Both these predictions are experimentally confirmed.

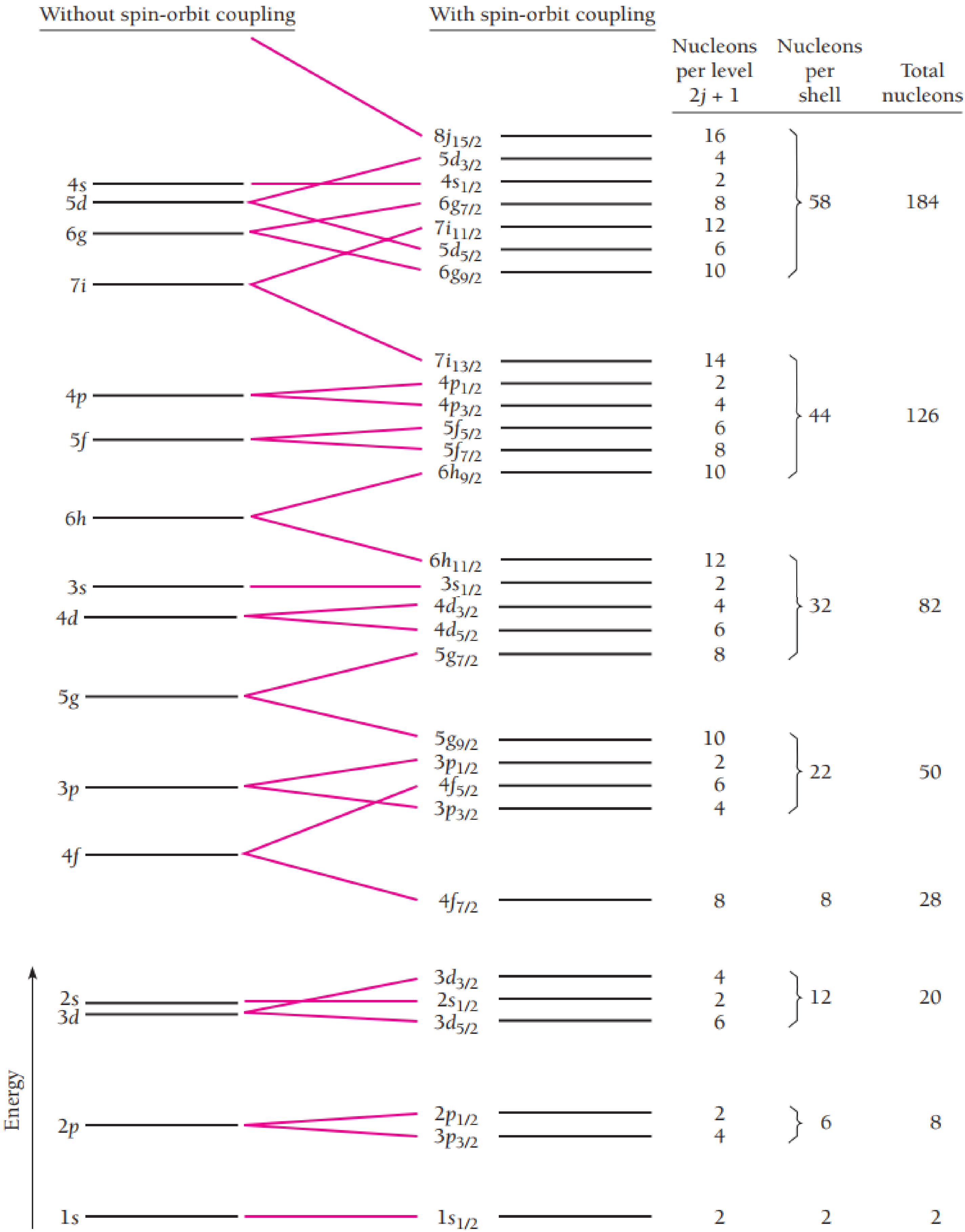

Fig. (2): Nucleon orbital's in a model with a spin-orbit interaction. The two left -most columns show the magic numbers and energies for a pure harmonic potential .The splitting of different values of the orbital angular momentum ***l*** can be arranged by modifying the central potential. Finally, the spin-orbit coupling splits the levels so that they depend on the relative orientation of the spin and orbital angular momentum. The number of nucleons per level (2***j*** + 1) and the resulting magic numbers are shown on the right.

For example, ${}^{17}_{9}F_8$ and ${}^{17}_{8}O_9$ have one unpaired nucleon outside a doubly magic ${}^{16}_{8}O_8$ core. Figure (2) tells us that the unpaired nucleon is in a $\boldsymbol{l}$ =2, $\boldsymbol{j}$ =5/2. The spin parity of the nucleus is predicted to by $5/2^+$ since the parity of the orbital is $(-1)^{\boldsymbol{l}}$. This agrees with observation. The first excited states of ${}^{17}_{9}F_8$ and ${}^{17}_{8}O_9$, corresponding to raising the unpaired nucleon to the next higher orbital, are predicted to be $1/2^+$, once again in agreement with observation.

On the other hand, ${}^{15}_{8}N_7$ and ${}^{15}_{8}O_7$ have one **"hole"** in their ${}^{16}O$ core. The ground state quantum numbers should then be the quantum numbers of the hole which are $\boldsymbol{l}$ = 1 and $\boldsymbol{j}$ =1/2 according to Figure (2). The quantum numbers of the ground state are then predicted to be $1/2^-$, in agreement with observation.

Note:

$$s,\ p,\ d,\ f,\ g,\ h,\ i,\ k, \ldots$$
$$l = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, \ldots$$
$$\text{occupancy} = 2(2l+1) = 2, 6, 10, 14, 18, 22, 26, 30, \ldots$$

Example:

${}^{15}_{8}O_7$ has spin $\frac{1}{2}$, because the last unpaired neutron is in a level with $j = 1/2$,

${}^{17}_{8}O_9$ has spin $\frac{5}{2}$ because the last unpaired neutron is in a level with $j = 5/2$.

For each of the liquid drop model and shell model have a specific applications, all of them succeed in the interpretation of some phenomena and fails to explain other phenomena. So it became logical to consider each of these models is complementary to another in a single model called the ***collective model*** as a model that combines the two models.

Collective Model

This model views the nucleus as having a hard core of nucleons in filled shells, as in the shell model, with outer valence nucleons that behave like the surface molecules of a liquid drop. In addition to the successes of each of the two models, this model has succeeded in formulating an equation to calculate the rotational energy levels to the even-even nuclei, i.e. the energy levels of deformed nuclei are very complicated, because there is often coupling between the various modes of excitation, but nevertheless many predictions of the collective model are confirmed experimentally.

$$E_{rot.} = \frac{\hbar^2}{2I} J(J+1) \quad \text{-------- (10)}$$

where,

I is the moment of inertia to the nucleus. **J** is the total angular momentum to the nucleus.

Reconciling the Models

If the nucleons in a nucleus are so close together and interact so strongly that the nucleus can be considered as analogous to a ***liquid drop***, how can these same nucleons be regarded as moving independently of each other in a common force field as required by the ***shell model***? It would seem that the points of view are mutually exclusive, since a nucleon moving about in a liquid-drop nucleus must surely undergo frequent collisions with other nucleons.

Both the ***liquid-drop*** and ***shell models*** of the nucleus are, in their very different ways, able to account for much that is known of nuclear behavior. The ***collective model*** of combines features of both models in a consistent scheme that has proved quite successful. The ***collective model*** takes into account such factors as the non-spherical shape of all but even-even nuclei and the centrifugal distortion experienced by a rotating nucleus. The detailed theory is able to account for the spacing of excited nuclear levels inferred from the gamma-ray spectra of nuclei and in other ways.

SUMMARY

1- The basic assumption of the liquid-drop model is that each nucleon in a nucleus interacts only with its nearest neighbors, like a molecule in a liquid.

2- The first term is a *volume* term which reflects the nearest-neighbor interactions, and which by itself would lead to a constant binding energy per nucleon $B/A \sim 16$ MeV.

3- The term a_s, which lowers the binding energy, is a *surface* term. Internal nucleons feel isotropic interactions whereas nucleons near the surface of the nucleus feel forces coming only from the inside. Therefore this is a *surface tension* term, proportional to the area $4\pi R^2 \sim A^{2/3}$.

4- The term a_c is the *Coulomb repulsion* term of protons, proportional to Q^2/R, i.e. $\sim Z^2/A^{1/3}$. This term is calculable. It is smaller than the nuclear terms for small values of Z. It favors a neutron excess over protons.

5- Conversely, the *asymmetry* term a_a favors symmetry between protons and neutrons (isospin). In the absence of electric forces, $Z = N$ is energetically favorable.

6- There are an abnormally high number of stable nuclides whose proton and/or neutron numbers equals the magic numbers 2,8,20,28,50,82,126.

7- Further evidence for such magic numbers is provided by the very high binding energy of nuclei with both Z and N being magic.

8- The abnormally high or low alpha and beta particle energies emitted by radioactive nuclei according to whether the daughter or parent nucleus has a magic number of neutrons. Similarly.

9- Nuclides with a magic number of neutrons are observed to have a relatively low probability of absorbing an extra neutron, i.e. they have lowest of absorption cross sections for neutrons (neutron-capture cross sections).

EXERCISES

1- Which nucleus would you expect to be more stable, ${}^{7}_{3}Li$ or ${}^{8}_{3}Li$; ${}^{13}_{6}C$ or ${}^{15}_{6}C$?

2- Find the binding energy for ${}^{6}{}_{3}Li$, ${}^{8}{}_{4}Be$, ${}^{17}{}_{8}O$, ${}^{208}{}_{82}Pb$ where:
${}^{6}Li$=6.015124u, ${}^{8}Be$=8.02502u, ${}^{17}O$=17.00453u and ${}^{208}Pb$=208.04754u .

3- Find the binding energy per nucleon in ${}^{20}_{10}Ne$ and in ${}^{56}_{26}Fe$.

4- Find the binding energy per nucleon in ${}^{79}_{35}Br$ and in ${}^{197}_{79}Au$.

5- The binding energy of ${}^{24}_{12}Mg$ is 198.25 MeV. Find its atomic mass.

6- Which isobar of A = 75 does the liquid-drop model suggest is the most stable?

7- Use the liquid-drop model to establish which of the mirror isobars ${}^{127}{}_{52}Te$ and ${}^{127}{}_{53}I$ decays into the other. What kind of decay occurs?

8- Calculate the volume energy (that represents nuclear attractive forces) and coulomb energy (repulsive forces) for ${}^{4}He$, ${}^{12}C$, ${}^{60}Ni$, ${}^{137}Ba$, ${}^{151}Eu$, ${}^{182}W$, ${}^{197}Au$, ${}^{206}Pb$, ${}^{238}U$, ${}^{252}Cf$, ${}^{257}Fm$ and ${}^{264}Ha$. Based on this, explain why the periodic table cannot be extended indefinitely.