i) **C.I for Variance σ^2**

- **If the mean is known:**

$$\left(\frac{(n)\,S^2}{X^2{}_{1-\frac{\alpha}{2}}} , \frac{(n)\,S^2}{X^2{}_{\frac{\alpha}{2}}} \right)$$

The C.I of $(1-\alpha)\%$ for σ^2 is given by the following probability:

$$Pr\left(\frac{(n)S^2}{X^2{}_{1-\frac{\alpha}{2}}(n)} < \sigma^2 < \frac{(n)S^2}{X^2{}_{\frac{\alpha}{2}}(n)} \right) = 1-\alpha$$

Where $X^2{}_{\frac{\alpha}{2}}$, $X^2{}_{1-\frac{\alpha}{2}}$ are the X^2 values obtained from X^2 distribution table with (n) degrees of freedom and level of significant $1-\frac{\alpha}{2}$, $\frac{\alpha}{2}$ respectively.

<u>Ex.</u> let S^2 =9 denoted the variance of ar.s of size 25 from N(10 , σ^2) , find 95% c.I. for σ^2 ?

<u>Sol.</u>

$$1-\alpha = 0.95 \;\Rightarrow\; \alpha = 0.05 \;\Rightarrow 1-\frac{\alpha}{2} = 1-0.025 = 0.975$$

From X^2 (Chi square table), we find that:

$$X^2{}_{\frac{\alpha}{2}}(n) = X^2{}_{0.025}(25) = 13.1197$$

$$X^2{}_{1-\frac{\alpha}{2}}(n) = X^2{}_{0.975}(25) = 40.6465$$

$$Pr\left(\frac{(n)S^2}{X^2{}_{1-\frac{\alpha}{2}}(n)} < \sigma^2 < \frac{(n)S^2}{X^2{}_{\frac{\alpha}{2}}(n)} \right) = 1-\alpha$$

$$Pr\left(\frac{(25)9}{X^2{}_{0.975}(25)} < \sigma^2 < \frac{(25)9}{X^2{}_{0.0255}(25)} \right) = 0.95$$

$$Pr\left(\frac{216}{40.6465} < \sigma^2 < \frac{216}{13.1197} \right) = 0.95$$

$$p_r[5.5355 \;<\; \sigma^2 \;<17.1498] = 0.95$$

$$C.L = 5.5355, CU = 17.1498$$

- **If the mean is unknown:**

The C.I of $(1-\alpha)\%$ for σ^2 is given by the following probability:

$$Pr\left(\frac{(n-1)S^2}{X^2{}_{1-\frac{\alpha}{2}}(n-1)} < \sigma^2 < \frac{(n-1)S^2}{X^2{}_{\frac{\alpha}{2}}(n-1)}\right) = 1-\alpha$$

Where $X^2{}_{\frac{\alpha}{2}}$, $X^2{}_{1-\frac{\alpha}{2}}$ are the X^2 values obtained from X^2 distribution table with (n-1) degrees of freedom and level of significant $1-\frac{\alpha}{2}$, $\frac{\alpha}{2}$ respectively.

Ex. Let $x_1, x_2, \cdots, x_{10}$ be a r.s from normal population of N (μ, σ^2) where both μ and σ^2 are unknown, suppose that $\sum_{i=1}^{10} x_i = 159$, and $\sum_{i=1}^{10} x_i^2 = 2531$, compute C.I. for σ^2 ? Where $X^2{}_{0.025}(9) = 19.02$, and $X^2{}_{0.975}(9) = 19.02$

Sol.

$$\sum (x_i - \bar{x})^2$$

$$= \sum (x^2{}_i - 2x_i\,\bar{x} + \bar{x}^2)$$

$$= \sum x^2{}_i - 2\sum x_i\bar{x} + \sum \bar{x}^2$$

$$= \sum x^2{}_i - 2\frac{n}{n}\sum x_i\bar{x} + n\bar{x}^2$$

$$= \sum x^2{}_i - 2n\frac{\sum x_i}{n}\,\bar{x} + n\bar{x}^2$$

$$\sum x^2{}_i - 2n\,\bar{x}^2 + n\bar{x}^2$$

$$S^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$$

$$(n-1)S^2 = \sum (x_i - \bar{x})^2$$

$$(n-1)S^2 = \sum x_i^2 - n\bar{x}^2$$

$$(n-1)S^2 = 2531 - 10\left(\frac{159}{10}\right)^2 = 2.90$$

$$1-\alpha = 0.95 \;\Rightarrow\; \alpha = 0.05 \;\Rightarrow\; 1-\frac{\alpha}{2} = 1-0.025 = 0.975$$

$$Pr\left(\frac{(n-1)S^2}{X^2{}_{1-\frac{\alpha}{2}}(n-1)} < \sigma^2 < \frac{(n-1)S^2}{X^2{}_{\frac{\alpha}{2}}(n-1)}\right) = 1-\alpha$$

$$Pr\left(\frac{2.90}{19.02} < \sigma^2 < \frac{2.90}{2.70}\right) = 0.95$$

$$Pr(0.152 < \sigma^2 < 1.074) = 0.95$$

ii) C.I. for the Ration of Two Variances

Let S_1^2 *and* S_2^2 be the variance of two independent random samples of size n_2 *and* n_2 respectively.

Let $v_1 = n_1 - 1$ *and* $v_2 = n_2 - 1$ be the degrees of freedom then $(1 - \alpha)\%$ C.I. for the ratio $\frac{\delta_1^2}{\delta_2^2}$ is given by:

$$Z_1 = \frac{nS_1^2}{\delta_1^2} \sim X^2(n_1 - 1)$$

$$Z_2 = \frac{nS_2^2}{\delta_2^2} \sim X^2(n_2 - 1)$$

$$F = \frac{Z_1/(n_1 - 1)}{Z_2/(n_2 - 1)}$$

$$Pr\left(\frac{S_1^2}{S_2^2}\frac{1}{f_{\frac{\alpha}{2}}(v_1, v_2)} < \frac{\delta_1^2}{\delta_2^2} < \frac{S_1^2}{S_2^2}\, f_{\frac{\alpha}{2}}(v_2, v_1)\right) = 1 - \alpha$$

The values of $f_{\frac{\alpha}{2}}(v_1, v_2)$ and $f_{\frac{\alpha}{2}}(v_2, v_1)$ obtained from the F distribution table.

<u>Ex.</u> Find 98% C.I. for $\frac{\delta_1^2}{\delta_2^2}$ if it is known that $n_1 = 25$, $n_2 = 16$, $S_1 = 8$, $S_2 = 7$?

<u>Slo.</u> We have

$$1 - \alpha = 0.98 \Rightarrow \alpha = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01$$

$$v_1 = n_1 - 1 \text{ and } v_2 = n_2 - 1$$

$$v_1 = 24 \; and \; v_2 = 15$$

$$f_{\frac{\alpha}{2}}(v_1, v_2) \;\Rightarrow\; f_{0.01}(24{,}15) = 3.29$$

$$f_{\frac{\alpha}{2}}(v_2, v_1) \;\Rightarrow\; f_{0.01}(15{,}24) = 2.89$$

$$Pr\left(\frac{S_1^2}{S_2^2}\,\frac{1}{f_{\frac{\alpha}{2}}(v_1, v_2)} < \frac{\delta_1^2}{\delta_2^2} < \frac{S_1^2}{S_2^2}\, f_{\frac{\alpha}{2}}(v_2, v_1)\right) = 1 - \alpha$$

$$Pr\left(\frac{64}{49}\,\frac{1}{3.29} < \frac{\delta_1^2}{\delta_2^2} < \frac{64}{49}\,(2.89)\right) = 0.98$$

$$Pr\left(0.397 < \frac{\delta_1^2}{\delta_2^2} < 3.775\right) = 0.98$$

$$C.I. = (0.397{,}3.775)$$

$$C.L = 0.397 \; and \; C.U = 3.775$$