proof :

1) Since $|f(t)| \geq 0$ for all $t \in$ [a, b] $\rightarrow \|f\| \geq 0$.
2) $\|f\| = 0 \leftrightarrow \sup_{t\in[a,b]} |f(t)|=0 \leftrightarrow |f(t)|=0$ for all $t \in$ [a, b]

   $\leftrightarrow f(t)=0$ for all $t \in$ [a, b] $\leftrightarrow f=0$.
3) Let $f \in$ X, $\alpha \in$ IR, then:

   $\|\alpha f\| = \sup\{ |\alpha f(t)| : t \in [a, b]\}$

   $= \sup\{ |\alpha||f(t)| : t \in [a, b]\}$

   $=|\alpha| \sup\{ |f(t)| : t \in [a, b]\}$

   $= |\alpha| \|f\|$.
4) $\|f+g\| = \sup\{ |(f+g)(t)| : t \in [a, b]\} = \sup\{ |(f(t) + g(t)| : t \in [a, b]\}$

   $\leq \sup\{ |f(t)| + |g(t)| : t \in [a, b]\}$

   $\leq \sup\{ |f(t)| : t \in [a, b]\} + \sup\{ |g(t)| : t \in [a, b]\} = \|f\| + \|g\|$.

[6] Let X = C[a; b], and choose $1 \leq p < \infty$. Then (using the integral form of Minkowski's inequality) we have the *p*-norm

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}$$

[7] Let V be the set of Riemann-integrable functions $f : (0; 1) \rightarrow$ R which are square-integrable. Let $\|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2} < \infty$. Then V is a normed linear space.

Definition 1.15. A set C in a linear space is *convex* if for any two points $x, y \in$ C, $tx + (1 - t)y \in$ C for all t $\in$ [0; 1].

Definition 1.16. A norm $\|.\|$ is *strictly convex* if $\|x\| = 1$, $\|y\| = 1$, $\|x+y\| = 2$ together imply that $x = y$.

Definition 1.17. :If (X; $\|.\|_X$) and (Y; $\|.\|_Y$) are normed linear spaces, then the *product*
X x Y = { $(x, y) \setminus x \in X; y \in Y$ }
is a linear space which may be made into a normed space in many different ways, a few of which follow.

Example 1.18.

[1] $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$.

proof:

1) $\|(x,y)\| = 0 \leftrightarrow \max\{\|x\|_X, \|y\|_Y\}=0 \leftrightarrow \|x\|_X=0, \|y\|_Y=0 \leftrightarrow x=0, y=0 \leftrightarrow (x, y)=0$

2) let (x_1,y_1) , $(x_2,y_2) \in$ XxY , then

$(x_1,y_1) + (x_2,y_2)=(x_1+ x_2, y_1+ y_2)$

... $y_1+ y_2)\| = \max\{\|x_1+ x_2\|_X, \|y_1+ y_2\|_Y)\} \leq \max\{\|x_1\|_X+\|x_2\|_X, \|y_1\|_Y+\|y_2\|_Y)$

$\leq \max\{\|x_1\|_X, \|y_1\|_Y\}+\max\{\|x_2\|_X, \|y_2\|_Y\}=\|(x_1, y_1)\|+\|(x_2, y_2)\|$

... $\in$ XxY and $\alpha \in$F, then

... $(\alpha x,\alpha y)\| = \max\{ \|\alpha x\|_X, \|\alpha y\|_Y\}= \max \{|\alpha| \|x\|_X, |\alpha| \|y\|_Y\}$

$=|\alpha| \max \{\|x\|_X, \|y\|_Y\}=|\alpha| \|(x, y)\|$

[2] H.W. $\|(x, y)\| = (\|x\|_X + \|y\|_Y)^{1/p}$;

Theorem 1.19. Every normed linear space is metric space.

proof:

let (X, $\|.\|$) is a normed space. We define the function d: XxX $\rightarrow$ IR as:

proof:

1) $\|(x,y)\| = 0 \leftrightarrow \max\{\|x\|_X, \|y\|_Y\} = 0 \leftrightarrow \|x\|_X = 0, \|y\|_Y = 0 \leftrightarrow x = 0, y = 0 \leftrightarrow (x, y) = 0$

2) let (x_1,y_1), $(x_2,y_2) \in X \times Y$, then

$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$

$\|(x_1 + x_2, y_1 + y_2)\| = \max\{\|x_1 + x_2\|_X, \|y_1 + y_2\|_Y\} \leq \max\{\|x_1\|_X + \|x_2\|_X, \|y_1\|_Y + \|y_2\|_Y\}$

$\leq \max\{\|x_1\|_X, \|y_1\|_Y\} + \max\{\|x_2\|_X, \|y_2\|_Y\} = \|(x_1, y_1)\| + \|(x_2, y_2)\|$

3) let $(x, y) \in X \times Y$ and $\alpha \in F$, then

$\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \max\{\|\alpha x\|_X, \|\alpha y\|_Y\} = \max\{|\alpha|\,\|x\|_X, |\alpha|\,\|y\|_Y\}$

$= |\alpha| \max\{\|x\|_X, \|y\|_Y\} = |\alpha|\,\|(x, y)\|$

[2] H.W. $\|(x, y)\| = (\|x\|_X + \|y\|_Y)^{1/p}$;

<u>Theorem 1.19.</u> Every normed linear space is metric space.

proof:

let $(X, \|.\|)$ is a normed space. We define the function $d: X \times X \to IR$ as:

$d(x,y) = \|x-y\|$ for all $x,y \in X$, since this function satisfies all the conditions of metric :

1) let $x,y \in X \to x-y \in X$ (since X is vector space) $\to \|x-y\| \geq 0 \to d(x, y) \geq 0$.

2) $d(x, y) = 0 \leftrightarrow \|x-y\| = 0 \leftrightarrow x-y = 0 \leftrightarrow x = y$

3) $d(x, y) = \|x-y\| = \|y-x\| = d(y, x)$

4) let $x,y,z \in X$:

$\|x-y\| = \|(x-z)+(z-y)\| \leq \|x-z\| + \|z-y\| \to d(x, y) \leq d(x, z) + d(z, y)$

<u>Remark</u> : The converse may be not true, for example:

If X be a v.s., define $d: X \times X \to IR$ as:

$$d(x,y) = \begin{cases} 0 & x = y \\ 2 & x \neq y \end{cases}$$

And define $\|.\|: X \to IR$ as $\|x\| = d(x, 0)$

$(X, \|.\|)$ fails to be normed space.

Since if $x \neq 0 \to \|x\| = d(x, 0) = 2$

$\|2x\| = d(2x, 0) \to |2|\,\|x\| = 2 \to 2.2 = 2 \to 4 = 2$ C!

<u>Definition 1.20.</u>: Let $X = (X; \|.\|_X)$ be a normed linear space. A sequence of vectors (x_n) in X is said to ***convergent*** if:

$$\exists x \in X \text{ s.t.}, \forall \varepsilon > 0 \; \exists k(\varepsilon) \in Z^+ \text{ s.t. } \|x_n - x\| < \in \quad \forall n > k.$$

And we say x is the convergent point for the sequence (x_n) and write $x_n \to x$ when $n \to \infty$, this means $x_n \to x \leftrightarrow \|x_n - x\| \to 0$. If (x_n) not convergent is called ***divergent***.

<u>.21.</u>: Let X be a normed space, (x_n), (y_n) be a sequence in X such that $x_n \to x_0$,

$\to x_0 \pm y_0$

$\| \to \|x_0\|$

3- $\|x_n - y_n\| \to \|x_0 - y_0\|$

4- $\alpha x_n \to \alpha x_0 \quad \forall \alpha \in F$

Proof:

1- Since $x_n \to x_0$, $y_n \to y$, then:

$\exists\, x\in X$ s.t. , $\forall\varepsilon>0$ $\exists\, k(\varepsilon)\in Z^+$ s.t. $\|x_n - x\| < \epsilon$ $\forall\, n > k$.

And we say x is the convergent point for the sequence (x_n) and write $x_n\to x$ when $n\to\infty$, this means $x_n\to x \leftrightarrow \|x_n - x\| \to 0$. If (x_n) not convergent is called ***divergent***.

Theorem 1.21.: Let X be a normed space, (x_n), (y_n) be a sequence in X such that $x_n\to x_0$, $y_n\to y$, then:

1- $x_n\pm y_n\to x_0\pm y_0$

2- $\|x_n\| \to \|x_0\|$

3- $\|x_n - y_n\| \to \|x_0 - y_0\|$

4- $\alpha x_n \to \alpha x_0$ $\quad\forall\alpha\in F$

Proof:

1- Since $x_n\to x_0$, $y_n\to y$, then:

if $\varepsilon > 0$

$\exists\, k_1(\varepsilon)\in Z^+$ s.t. $\|x_n - x_0\| < \varepsilon/2$, $\forall\, n > k_1(\varepsilon)$

$\exists\, k_2(\varepsilon)\in Z^+$ s.t. $\|y_n - y_0\| < \varepsilon/2$, $\forall\, n > k_2(\varepsilon)$

Define $k_3(\varepsilon)=\max\{k_1(\varepsilon),k_2(\varepsilon)\}$

$$\|(x_n+y_n)-(x_0+y_0)\| = \|x_n+y_n-x_0-y_0\| = \|(x_n-x_0)+(y_n-y_0)\|$$
$$\le \|x_n-x_0\| + \|y_n-y_0\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,\ \forall\, n > k_3(\varepsilon)$$

$\to x_n+y_n\to x_0+y_0$

2- Since $x_n\to x_0$ T.P. $\|x_n\| \to \|x_0\|$ i.e. T.P. $|\,\|x_n\| - \|x_0\|\,| \to 0$

By Theorem (1.13.)-4 : $|\,\|x_n\| - \|x_0\|\,| \le \|x_n - x_0\|$ ..... (1)

Since $x_n\to x_0 \to \|x_n - x_0\| \to 0$ .....(2)

By (1) & (2) we get: $|\,\|x_n\| - \|x_0\|\,| \to 0$

Then $\|x_n\| \to \|x_0\|$

3- T.P. $\|x_n - y_n\| \to \|x_0 - y_0\|$, i.e. T.P. $|\,\|x_n - y_n\| - \|x_0 - y_0\|\,| \to 0$

Since $x_n\to x_0 \Rightarrow \|x_n - x_0\| \to 0$

& $y_n\to y_0 \Rightarrow \|y_n - y_0\| \to 0$

$$|\,\|x_n - y_n\| - \|x_0 - y_0\|\,| \le \|x_n - y_n - x_0 + y_0\|$$
$$\le \|x_n - x_0\| + \|y_n - y_0\|$$

$\Rightarrow |\,\|x_n - y_n\| - \|x_0 - y_0\|\,| \to 0 \Rightarrow \|x_n - y_n\| \to \|x_0 - y_0\|$

4- $\|\alpha x_n - \alpha x_0\| = \|\alpha(x_n - x_0)\| = |\alpha|\,\|x_n - x_0\|$

since $\|x_n - x_0\|\to 0$ where $n\to\infty \Rightarrow \|\alpha x_n - \alpha x_0\|$ where $n\to\infty \Rightarrow \alpha x_n \to \alpha x_0$

.22.: If the sequence (x_n) is convergent in the normed space X then its convergent

que.

suppose that $x_n\to x$ and $x_n\to y$ s.t. $x\neq y$, and let $\|x-y\| = \varepsilon \to \varepsilon>0$

since $x_n\to x \Rightarrow \exists k_1\in Z^+$ s.t. $\|x_n - x\| < \varepsilon/2$, $\forall\, n > k_1$

and $x_n\to y \Rightarrow \exists k_2\in Z^+$ s.t. $\|x_n - y\| < \varepsilon/2$, $\forall\, n > k_2$

put $k=\max\{k_1, k_2\}$. Then $\|x_n - x\| < \varepsilon/2$, $\|x_n - y\| < \varepsilon/2$ $\forall\, n > k$.

$\varepsilon = \|x-y\| = \|(x - x_n) + (x_n - y)\| \le \|(x_n - x)\| + \|(x_n - y)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$!