Functional Analysis

What is Functional Analysis?

Functional analysis represents one of the most important branches of mathematical sciences. Together with abstract algebra and mathematical logics it serves as a foundation of many other branches of mathematics.

Functional analysis is, in particular, widely used in probability theory and random functions theory and their numerous applications. Functional analysis serves also as a powerful tool in modern control and information sciences. The main subject of mathematical analysis represents scalar and finite-dimensional vector functions of scalar or finite-dimensional vector variables. Functional analysis is studying more general functions whose arguments and values may be the elements of any sets. While studying functions in mathematical analysis and linear algebra geometrical presentations are widely used; a function is considered as the mapping of one finite-dimensional space into another finite-dimensional space.

For instance, the scalar function of one scalar variable represents the mapping of the real axis R into the real axis R. The scalar function of two (three) scalar variables represents the mapping of the plane R^2 (the three-dimensional space R^3 respectively) into R. While studying more general functions whose arguments and values may be the elements of any sets wonderful analogies appear between many properties of functions and the visual geometric properties of more simple functions.

You meet such analogies in linear algebra where the spaces of any finite dimensions are considered (the n-dimensional spaces R^n at any finite n). In particular, the properties of linear functions in R^n are absolutely identical with the properties of linear functions in one-, two-- and three-dimensional spaces. These properties of functions caused the generalization of the notion of a space and wide application of intuitive geometrical presentations and geometrical terminology while studying any functions.

Functional analysis was born in the works of Italian mathematician Vito Volterra (Volterra 1913, Volterra and Peres 1935). He was the first who considered functions as the points of some space. The spaces whose points are functions are called function spaces.

Volterra defined also a real function whose argument represents the set of all the values of a continuous function in the interval [a, b]. Such a function he called a functional. This was the reason to call the branch of mathematics studying functionals a functional analysis.

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In this course we studied the following subjects:

1- Vector Spaces: Finite and Infinite Dimentional, Metric Spaces, Norms & Normed

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In this course we studied the following subjects:

1- Vector Spaces: Finite and Infinite Dimentional, Metric Spaces, Norms & Normed Spaces.
2- Banach Spaces: Some Important Inequalities(Cauchy, Holder and Minkowski's inequalities), Examples of Banach Spaces, Quotient Space of a Normed Linear Space, Continuous and Bounded Linear Transformations, Norm of Bounded Linear Transformations, Linear Operator on a Normed Space. Equivalent Norms, Continuous Linear Functional, Dual Spaces, The Hahan-Banach Theorem.
3- Hilbert Spaces: Definitions, Pre-Hilbert Spaces, Chauchy- Schwarz Inequality, orthogonal, Gram- Schmidt Theorem.

References:

1- Introductionary Functional Analysis and Application, By E. Kreyzig, 1978.
2- Introduction to Hilbert Space, by S. K. Berberian, 1976.

Chapter One: Vector Space

Definition 1.1.

...*or space over* F is a non-empty set V together with two functions, one from V x V to ...ther from F x V to V, denoted by $x + y$ and αx respectively, for all $x, y \in V$ and α ...at, for any $\alpha, \beta \in$ F and any $x, y, z \in V$,

...x, $x+(y+z)=(x+y)+z$;

(b) there exists a unique $0 \in V$ (independent of x) such that $x + 0 = x$;

(c) there exists a unique $-x \in V$ such that $x + (-x) = 0$;

(d) $1x = x$, $\alpha(\beta x) = (\alpha\beta)x$;

(e) $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$.

Definition 1.1.

A *vector space over* F is a non-empty set *V* together with two functions, one from *V* x *V* to *V*, and the other from F x *V* to *V*, denoted by $x + y$ and αx respectively, for all $x, y \in V$ and $\alpha \in$ F, such that, for any $\alpha, \beta \in$ F and any $x, y, z \in V$,

(a) $x+y=y+x$, $x+(y+z)=(x+y)+z$;

(b) there exists a unique $0 \in V$ (independent of x) such that $x + 0 = x$;

(c) there exists a unique $-x \in V$ such that $x + (-x) = 0$;

(d) $1x = x$, $\alpha(\beta x) = (\alpha\beta)x$;

(e) $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$.

If F = *R* (respectively, F = *C*) then V is a *real* (respectively, *complex*) vector space. Elements of F are called *scalars*, while elements of V are called *vectors*. The operation $x + y$ is called *vector addition*, while the operation ax is called *scalar multiplication*.

Some important inequalities

1- *Holder's inequality* : if $p, q \in$ IR such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{i=1} |x_i y_i| \le \left(\sum_{i=1} |x_i|^p\right)^{1/p} \left(\sum_{i=1} |y_i|^q\right)^{1/q}$$

2- If p=2 then q=2 and:

$$\sum_{i=1} |x_i y_i| \le \left(\sum_{i=1} |x_i|^2\right)^{1/2} \left(\sum_{i=1} |y_i|^2\right)^{1/2}$$

and is called *Cauchy - Schwar's inquality*.

3- *MinKowsk's inquality*: if $p \ge 1$, then:

$$\left(\sum_{i=1} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1} |x_i|^p\right)^{1/p} + \left(\sum_{i=1} |y_i|^p\right)^{1/p}$$

Example 1.2. [H.W.2-6]

[1] $S=\{x= \alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n\}$ is a vector space over *R* or *C* (sequence space).

[2] $l_p = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n|^p < \infty\}$, l_p is a vector space over *R* or *C* $(1 \le p \le \infty)$

[3] $l_\infty = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n|^p \le m\}$ is a vector space over *R* or *C*.

[4] $C[a, b]=\{f: [a, b]\to R : f$ is continuous and $C[a, b]\}$ is a vector space over *R* or *C*.

[5] $L^p[a, b]= \{f:[a, b] \to R$, f is Lebesgue integrable on [a, b] s.t. $\int_a^b |f(x)|\,dx < \infty$ } is a vector space over *R* or *C*.

…e the set M(m, n)(C) of complex{valued m x n matrices, with usual addition of … and scalar multiplication.

… $(\alpha_n)_{n=1}^{\infty}$, $y = (\beta_n)_{n=1}^{\infty} \in$ S, λ is a scalar, then

1. $x + y = (\alpha_n)_{n=1}^{\infty} + (\beta_n)_{n=1}^{\infty} = (\alpha_n + \beta_n)_{n=1}^{\infty} \in S$

2. $\lambda(\alpha_n)_{n=1}^{\infty} = (\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n, \ldots) = (\lambda\alpha_n)_{n=1}^{\infty} \in S$

Definition 1.3

Let V be a vector space. A non-empty set U ⊂ V is a *linear subspace* of V if U is itself a