Sol. 1-

1) d is real, finite & $d=|x-y|\geq 0$

2) $d(x,y)=0 \leftrightarrow |x-y|=0 \leftrightarrow x-y=0 \leftrightarrow x=y$ $\quad \forall\, x, y\in IR$

3) $d(x,y)=|x-y|=|-(y-x)|=|y-x|=d(y,x)$ $\quad \forall\, x, y\in IR$

4) $d(x,y)=|x-y|=|x-z+z-y|\leq|x-z|+|z-y|=d(x,z)+d(z,y)$ $\quad \forall\, x, y, z\in IR$

Then (IR, d) is a metric space.

* A ***norm*** on a vector space is a way of measuring distance between vectors.

Definition 1.11.: A ***norm*** on a linear vector space V over F is a function $\|\,.\,\|: V \to R$ with the properties that :

(1) $\|x\|\geq 0$ & $\|x\|=0 \leftrightarrow x=0$ (positive definite);

(2) $\|x+y\|\leq\|x\|+\|y\|$ for all $x, y\in V$ (triangle inequality);

(3) $\|\alpha x\|=|\alpha|\,\|x\|$ for all $x\in V$ and $\alpha\in F$.

In Definition 1.11(3) we are assuming that F is R or C and $|\,.\,|$ denotes the usual absolute value. If $\|\,.\,\|$ is a function with properties (2) and (3) only it is called a ***semi-norm***.

Definition 1.12. A ***normed linear space*** is a linear space V with a norm $\|\,.\,\|$ (sometimes we write $\|\,.\,\|_V$).

Theorem 1.13. If V is a normed space then:

1) $\|0\|=0$
2) $\|x\|=\|-x\|$ for every $x\in V$.
3) $\|x-y\|=\|y-x\|$ for every $x\in V$.
4) $|\,\|x\|-\|y\|\,|\leq\|x-y\|$ for every $x\in V$.

Proof:

Properties (1), (2) and (3) conclude directly from the definition, to prove property (4):

$x=(x-y)+y$

$\|x\|=\|(x-y)+y\|\leq\|x-y\|+\|y\| \to \|x\|-\|y\|\leq\|x-y\|$ ....(1)

Similarly:

$\|y\|-\|x\|\leq\|x-y\|$

$-(\|x\|-\|y\|)\leq\|x-y\| \to (\|x\|-\|y\|)\geq-\|x-y\|$ ....(2)

From (1) & (2), we get:

$-\|x-y\|\leq\|x\|-\|y\|\leq\|x-y\| \to |\,\|x\|-\|y\|\,|\leq\|x-y\|$

Examples 1.14.:- [H.W.6,7]

...ector space V is normed v.s. with the norm $\|x\|=|x|$ for all $x\in V$.

... $|x|\geq 0 \to \|x\|\geq 0$.

... $=0 \leftrightarrow |x|=0 \leftrightarrow x=0$

3) Let $x\in V$, $\alpha\in F$, then

$\|\alpha x\|=|\alpha x|=|\alpha|\,|x|=|\alpha|\,\|x\|$

4) Let $x,y\in V$, then:

$\|x+y\|=|x+y|\leq|x|+|y|=\|x\|+\|y\|$

[2] Let $V=R^n$ with the usual Euclidean norm

From (1) & (2), we get:

$-\| x\text{-}y \| \leq \| x \| - \| y \| \leq \| x\text{-}y \| \rightarrow |\, \| x \| - \| y \| \,| \leq \| x\text{-}y \|$

Examples 1.14.:- [H.W.6,7]

[1] The vector space V is normed v.s. with the norm $\| x \| = | x |$ for all $x \in$ V.

Proof:

1) Since $| x | \geq 0 \rightarrow \| x \| \geq 0$.
2) $\| x \| = 0 \leftrightarrow | x | = 0 \leftrightarrow x = 0$
3) Let $x \in$ V, $\alpha \in$ F , then

   $\| \alpha x \| = | \alpha x | = |\alpha|\, | x | = | \alpha |\, \| x \|$
4) Let $x,y \in$ V, then:

   $\| x + y \| = | x+y | \leq | x | + | y | = \| x \| + \| y \|$

[2] Let V = R^n with the usual Euclidean norm

$\| x \| = \| x \|_2 = (\sum_{j=1}^{n} | x_j |^2)^{1/2}$

proof:

1) Since $x_j^2 \geq 0$ for all $j = 1,2,\ldots,n \rightarrow \| x \| \geq 0$
2) $\| x \| = 0 \leftrightarrow (\sum_{j=1}^{n} | x_j |^2)^{1/2} = 0 \leftrightarrow \sum_{j=1}^{n} | x_j |^2 = 0$

   $\leftrightarrow x_j^2 = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x_j = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x = 0$
3) Let $x \in IR^n$, $\alpha \in$ IR:

   $\alpha x = \alpha\, (x_1, \ldots , x_n) = (\alpha x_1, \ldots , \alpha x_n)$

   $\| \alpha x \| = (\sum_{j=1}^{n} | \alpha x_j |^2)^{1/2} = | \alpha | (\sum_{j=1}^{n} | x_j |^2)^{1/2} = | \alpha |\, \| x \|$.
4) Let $x,y \in IR^n$:

   $x + y = (x_1, \ldots , x_n) + (y_1, \ldots , y_n) = (x_1 + y_1, \ldots , x_n + y_n)$

   $\| x+y \| = (\sum_{j=1}^{n} | x_j + y_j |^2)^{1/2}$

By using *MinKowski's inquality* where p=2, we have:

$\| x + y \| = (\sum_{i=1} | x_i + y_i |^2)^{1/2} \leq (\sum_{i=1} | x_i |^2)^{1/2} + (\sum_{i=1} | y_i |^2)^{1/2} = \| x \| + \| y \|$

[3] There are many other norms on R^n, called the *p*-norms. For $1 \leq p < \infty$ defined by:

$\sum^{n} | x_j |^p)^{1/p}$

s a norm on V (to check the triangle inequality use *MinKowski's Inequality*)

$\ldots y_j |^p)^{1/p} \leq (\sum_{j=1}^{n} | x_j |^p)^{1/p} + (\sum_{j=1}^{n} | y_j |^p)^{1/p}$

[4] There is another norm corresponding to $p = \infty$, defined by:

$\| x \|_\infty = \max_{1 \leq j \leq n} \{ | x_j | \}$

where $\| . \|$: $IR^n \rightarrow$ IR and $x = (x_1, \ldots, x_n)$.

[3] There are many other norms on R^n, called the *p*-norms. For $1 \le p < \infty$ defined by:

$$\| x \|_p = \left(\sum_{j=1}^{n} | x_j |^p\right)^{1/p}$$

Then $\| \,.\, \|_p$ is a norm on V (to check the triangle inequality use *MinKowski's Inequality*)

$$\left(\sum_{j=1}^{n} | x_j + y_j |^p\right)^{1/p} \le \left(\sum_{j=1}^{n} | x_j |^p\right)^{1/p} + \left(\sum_{j=1}^{n} | y_j |^p\right)^{1/p}$$

[4] There is another norm corresponding to $p = \infty$, defined by:

$$\| x \|_\infty = \max_{1 \le j \le n} \{| x_j |\}$$

where $\| \,.\, \|: IR^n \to IR$ and $x = (x_1, \ldots, x_n)$.

proof:

1) Since $| x_i | \ge 0$ for all $i=1, \ldots, n \to \| x \| \ge 0$.
2) $\| x \| = 0 \leftrightarrow \max \{| x_1 |, \ldots, | x_n |\} = 0 \leftrightarrow | x_i | = 0$ for all $i=1, \ldots, n$

   $\leftrightarrow x_i = 0$ for all $i=1, \ldots, n \;\; \leftrightarrow x=0$
3) Let $x \in IR^n$ and $\alpha \in IR$, then

   $\alpha x = \alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$

   $\| \alpha x \| = \max \{ | \alpha x_1 |, \ldots, | \alpha x_n |\}$

   $= \max \{ | \alpha | | x_1 |, \ldots, | \alpha | | x_n |\}$

   $= | \alpha | \max \{ | x_1 |, \ldots, | x_n |\}$

   $= | \alpha | \| x \|$
4) Let $x, y \in IR^n$

   $x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (, \ldots,)$

   $\| x + y \| = \max \{ | x_1+y_1 |, \ldots, | x_n+y_n | \}$

   $\le \max \{ | x_1 | + | y_1 |, \ldots, | x_n | + | y_n | \}$

   $\le \max \{ | x_1 |, \ldots, | x_n | \} + \max \{ | y_1 |, \ldots, | y_n | \}$

   $= \| x \| + \| y \|$

[5] Let X = C[a; b], and put $\| f \| = \sup_{t \in [a,b]} | f(t) |$. This is called the uniform or supremum norm.

proof :

1) Since $| f(t) | \ge 0$ for all $t \in [a, b] \to \| f \| \ge 0$.
2) $\| f \| = 0 \leftrightarrow \sup_{t \in [a,b]} | f(t) | = 0 \leftrightarrow | f(t) | = 0$ for all $t \in [a, b]$

   $(t) = 0$ for all $t \in [a, b] \;\; \leftrightarrow f = 0$.

   $\in X, \alpha \in IR$, then:

   $= \sup\{ | \alpha f(t) |: t \in [a, b]\}$

   $= \sup\{ | \alpha | | f(t) |: t \in [a, b]\}$

   $= | \alpha | \sup\{ | f(t) |: t \in [a, b]\}$

   $= | \alpha | \| f \|$.
4) $\| f + g \| = \sup\{ | (f + g)(t) |: t \in [a, b]\} = \sup\{ | (f(t) + g(t) |: t \in [a, b]\}$

   $\le \sup\{ | f(t) | + | g(t) |: t \in [a, b]\}$