$\Rightarrow |\, \| x_n - y_n \| - \| x_0 - y_0 \| \,| \to 0 \;\Rightarrow\; \| x_n - y_n \| \to \| x_0 - y_0 \|$

4- $\|\alpha x_n - \alpha x_0\| = \|\alpha(x_n - x_0)\| = |\alpha|\, \| x_n - x_0\|$

since $\| x_n - x_0\| \to 0$ where $n \to \infty \Rightarrow \|\alpha x_n - \alpha x_0\|$ where $n \to \infty \Rightarrow \alpha x_n \to \alpha x_0$

Theorem 1.22.: If the sequence (x_n) is convergent in the normed space X then its convergent point is unique.

proof:

suppose that $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$, and let $\| x - y \| = \varepsilon \to \varepsilon > 0$

since $x_n \to x \Rightarrow \exists k_1 \in Z^+$ s.t. $\| x_n - x\| < \varepsilon/2$, $\forall n > k_1$

and $x_n \to y \Rightarrow \exists k_2 \in Z^+$ s.t. $\| x_n - y\| < \varepsilon/2$, $\forall n > k_2$

put $k = \max\{ k_1, k_2\}$. Then $\| x_n - x\| < \varepsilon/2$, $\| x_n - y\| < \varepsilon/2$ $\forall n > k$.

$\varepsilon = \| x - y \| = \| (x - x_n) + (x_n - y) \| \leq \| (x_n - x) \| + \| (x_n - y) \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$!

and this contradiction then $x = y$.

Definition 1.23. A sequence (x_n) in a normed space X is a ***Cauchy convergent sequence*** if:

$\forall \varepsilon > 0\ \exists\ k(\varepsilon) \in Z^+$ such that $\| x_n - x_m \| < \varepsilon \quad \forall n,m > k(\varepsilon)$

Theorem 1.24.: Every convergent sequence is a Cauchy convergent sequence.

proof:

Suppose that (x_n) is a convergent sequence in the normed space X, then $\exists\, x \in X$ s.t. $x_n \to x$

Let $\varepsilon > 0$, since $x_n \to x \Rightarrow \exists\ k \in Z^+$ s.t. $\| x_n - x \| < \varepsilon/2 \quad \forall n > k$

If $n,m \geq k$,then $\| x_n - x_m \| = \| (x_n - x) + (x - x_m) \| \leq \| x_n - x\| + \| x - x_m \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Then (x_n) is a Cauchy sequence.

Remark:

The converse to above theorem may not be true. For example:

Let X= IR-{0}, $(x_n) = (1/n)$

(x_n) Cauchy convergent sequence in IR

Since IR complete $\Rightarrow (x_n) = (1/n) \to 0$ convergent in IR

But (x_n) not convergent in IR-{0} , since $0 \notin$ IR-{0}.

Definition 1.25.: Let X be a normed space, $x_0 \in X$, a function f is said to be ***continuous*** at x_0 if:

$\forall\, \varepsilon > 0, \exists\, \delta(x_0, \varepsilon) > 0$ s.t. $\| f(x) - f(x_0) \| < \varepsilon$ whenever $\| x - x_0 \| < \delta$.

Theorem 1.26. : Let X, Y be two Normed space, a function $f : X \to Y$ continuous at $x_0 \in X$ iff for each sequence (x_n) in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Definition 1.27.: Let X be a normed space, a function $f : X \to$ IR is called ***bounded*** if:

$\exists\, M > 0$ s.t. $\| f(x) \| \leq M, \forall x \in X$.

Definition 1.28.: Let (x_n) be a sequence in a normed space X, say (x_n) is ***bounded sequence*** in ... > 0 s.t. $\| x_n \| \leq M, \forall n \in Z^+$.

...**.29.**: If (x_n) is Cauchy convergent sequence in a normed space X then it is bounded.

... a Cauchy sequence in X

Given $\varepsilon = 1, \exists\ k \in Z^+$ s.t. $\| x_n - x_m \| < 1, \forall n, m > k$.

Let $m = k+1 \Rightarrow \| x_n - x_{k+1} \| < 1$

Since $|\, \| x_n\| - \| x_{k+1} \| \,| \leq \| x_n - x_{k+1} \| < 1$

$\Rightarrow |\, \| x_n\| - \| x_{k+1} \| \,| < 1 \Rightarrow \| x_n\| < 1 + \| x_{k+1} \|, \forall n > k$

Put $M = \max\{ \| x_1\|, \| x_2\|, \dots, \| x_k\|, \| x_{k+1}\|\} \Rightarrow \| x_n \| \leq M , \forall n \in Z^+$.

Theorem 1.26. : Let X, Y be two Normed space, a function $f : X \to Y$ continuous at $x_0 \in X$ iff for each sequence (x_n) in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Definition 1.27.: Let X be a normed space, a function $f : X \to IR$ is called ***bounded*** if:

$\exists M > 0$ s.t. $\| f(x) \| \leq M, \forall x \in X$.

Definition 1.28.: Let (x_n) be a sequence in a normed space X, say (x_n) is ***bounded sequence*** in X if : $\exists M > 0$ s.t. $\| x_n \| \leq M, \forall n \in Z^+$.

Theorem 1.29.: If (x_n) is Cauchy convergent sequence in a normed space X then it is bounded.

proof:

Let (x_n) be a Cauchy sequence in X

Given $\varepsilon=1$, $\exists k \in Z^+$ s.t. $\| x_n - x_m \| < 1, \forall n, m > k$.

Let $m = k+1 \Rightarrow \| x_n - x_{k+1} \| < 1$

Since $| \| x_n \| - \| x_{k+1} \| | \leq \| x_n - x_{k+1} \| < 1$

$\Rightarrow | \| x_n \| - \| x_{k+1} \| | < 1 \Rightarrow \| x_n \| < 1 + \| x_{k+1} \|, \forall n > k$

Put $M = \max \{ \| x_1 \|, \| x_2 \|, \ldots, \| x_k \|, \| x_{k+1} \| \} \Rightarrow \| x_n \| \leq M, \forall n \in Z^+$.

Theorem1.30. : Every convergent sequence in the normed space X is bounded.

proof:

Let (x_n) be a convergent sequence in X $\Rightarrow (x_n)$ a Cauchy convergent sequence in X (by 1.24)

$\Rightarrow (x_n)$ bounded (by 1.29).

Definition 1.31. : Let X is a normed space, $x_0 \in X$, $r > 0$, define:

1) $B_r(x_0)=\{ x \in X: \| x - x_0 \| < r \}$ is called ***open ball*** of center x_0 and radius r.
2) $D_r(x_0)=\{ x \in X: \| x - x_0 \| \leq r \}$ is called ***closed ball*** of center x_0 and radius r.
3) $B_1(0)=\{ x \in X: \| x \| < 1 \}$ is called ***open unite*** of center 0 and radius 1.
4) $D_1(0)=\{ x \in X: \| x \| \leq 1 \}$ is called ***closed unite*** of center 0 and radius 1.

Definition 1.32.: Let $\| . \|_1, \| . \|_2$ be two norms on vector space X, $\| . \|_1$ is said to be ***equivalent*** to $\| . \|_2$ ($\| . \|_1 \sim \| . \|_2$) if there exist a and b positive real numbers such that:

$$a \| . \|_2 \leq \| . \|_1 \leq b \| . \|_2$$

Example: Let $X=IR^n$,

$\| x \| = \sum_{i=1}^{n} | x_i |, \forall x \in IR^n$

$\| x \|_e = \sum_{i=1}^{n} | x_i^2 |^{\frac{1}{2}}, \forall x \in IR^n$

Then $\| x \| \sim \| x \|_e$

[illegible] $\sum x_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$, $\forall x_i, y_i \in IR^n$ (by using Cauchy – Schwars inquality)

[illegible] $i = 1, 2, \ldots, n$.

$\Rightarrow \sum_{i=1}^{n} | y_i | \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} 1)^{\frac{1}{2}}$

$\| x \| \leq \| x \|_e . \sqrt{n}$

$\frac{1}{\sqrt{n}} \| x \| \leq \| x \|_e$ (i.e. $a= \frac{1}{\sqrt{n}}$) ….. (1)

$\| x \|_e = \sum_{i=1}^{n} | x_i^2 |^{\frac{1}{2}}$, $\forall x \in IR^n$

Then $\| x \| \sim \| x \|_e$

proof:

$\sum_{i=1}^{n} | x_i y_i | \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$, $\forall x_i, y_i \in IR^n$ (by using Cauchy – Schwars inquality)

Put $y_i=1$, $\forall\, i = 1, 2, \ldots, n$.

$\Rightarrow \sum_{i=1}^{n} | y_i | \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} 1)^{\frac{1}{2}}$

$\| x\| \leq \| x \|_e \cdot \sqrt{n}$

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e$ (i.e. $a= \frac{1}{\sqrt{n}}$) ….. (1)

But $\| x \|_e \leq \| x\|$ (i.e. b=1)

From (1) & (2), we have:

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e \leq \| x\|$

Then $\| x \| \sim \| x \|_e$

Theorem 1.33.: On a finite dimensional normed space, all norms are equivalent.

Examples:

1- $X=IR^2$, $\| \cdot \|_e$, $\| \cdot \|_2$, $\| \cdot \|_3$ are equivalent.
2- $X= IR^n$, $\| \cdot \|_e$, $\| \cdot \|_2$, $\| \cdot \|_3$ are equivalent.

Chapter Two: Banach spaces

Definition 2.1. A normed linear space X is said to be ***complete*** if all Cauchy convergent sequences in X are convergent in X. The complete normed space is called ***Banach space***.

Examples 2.2.

[illegible]ace F^n with the norm $\| x \| = (\sum_{i=1}^{n} | x_i |^2)^{\frac{1}{2}}$, $\forall x=(x_1,x_2,\ldots,x_n) \in F^n$ is a Banach space.

[illegible] is a normed space ,

[illegible] Cauchy sequence in $F^n \Rightarrow x_m \in F^n \Rightarrow x_m=(x_{1m},x_{2m},\ldots,x_{nm})$

[illegible] $\Rightarrow \exists\, k \in Z^+$ s.t. $\| x_m - x_l \| < \varepsilon$ $\forall\, m.l > k$

$\Rightarrow \| x_m - x_l \|^2 < \varepsilon^2$ $\forall\, m.l > k$ ….. (1)

$x_m - x_l = (x_{1m}-x_{1l}, x_{2m}-x_{2l}, \ldots, x_{nm}-x_{nl})$

$\| x_m - x_l \|^2 = \sum_{i=1}^{n} | x_{im} -$ [illegible] …… (2)

Chapter Two: Banach spaces

Definition 2.1. A normed linear space X is said to be ***complete*** if all Cauchy convergent sequences in X are convergent in X. The complete normed space is called ***Banach space***.

Examples 2.2.

[1] The space F^n with the norm $\| x \| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$, $\forall x=(x_1,x_2,\ldots,x_n) \in F^n$ is a Banach space.

Proof: F^n is a normed space ,

let $\{x_n\}$ is Cauchy sequence in $F^n \Rightarrow x_m \in F^n \Rightarrow x_m=(x_{1m},x_{2m},\ldots,x_{nm})$

let $\varepsilon > 0 \Rightarrow \exists\, k \in Z^+$ s.t. $\|x_m - x_l\| < \varepsilon \quad \forall\, m.l > k$

$\Rightarrow \|x_m - x_l\|^2 < \varepsilon^2 \quad \forall\, m.l > k$ ..... (1)

$x_m - x_l = (x_{1m}-x_{1l}, x_{2m}-x_{2l}, \ldots, x_{nm}-x_{nl})$

$\|x_m - x_l\|^2 = \sum_{i=1}^{n} |x_{im} - x_{il}|^2$ ...... (2)

from (1) & (2) , we get:

$\sum_{i=1}^{n} |x_{im} - x_{il}|^2 < \varepsilon^2 \quad \forall\, m.l \geq k$

then

$|x_{im} - x_{il}|^2 < \varepsilon^2 \quad \forall\, m.l \geq k \Rightarrow |x_{im} - x_{il}| < \varepsilon \quad \forall\, m.l \geq k$

$\Rightarrow \forall i$, $\{x_{im}\}$ is a Cauchy sequence in F

Since F is complete (because F is IR or C)

$\Rightarrow \forall i, \exists\, x_i \in F$ s.t. $x_{im} \to x_i$

Put $x=(x_1,x_2,\ldots,x_n) \Rightarrow x \in F$, T.P. $x_m \to x$.

Let $\varepsilon > 0$, $\forall\, m > k$, we get:

$\|x_m - x\|^2 = \sum_{i=1}^{n} |x_{im} - x_i|^2 < \varepsilon^2 \Rightarrow \|x_m - x\| < \varepsilon \quad \forall m > k \Rightarrow \{x_m\}$ convergent $\Rightarrow F^n$ is complete

Since F^n is normed space $\Rightarrow F^n$ is a Banach space

[2] H.W. The space l^p $(1 \leq p < \infty)$ with the norm $\| x \| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, $x=(x_1,x_2,\ldots) \in l^p$, is a Banach space.

[3] The space l^∞ with the norm $\| x \| = \sup_i |x_i|$ is a Banach space.

Proof:

l^∞ is a normed space

Let $\{x_m\}$ is a Cauchy sequence in $l^\infty \Rightarrow x_m \in l^\infty \Rightarrow x_m=(x_{1m},x_{2m},\ldots,x_{nm},\ldots)$

[illegible] $\exists\, k \in Z^+$ s.t.

[illegible] ε , $\forall m,l > k$ ....(1)

[illegible] $- x_{1l}, \ldots, x_{nm}-x_{nl}, \ldots)$

[illegible] $= \sup_i |x_{im} - x_{il}|$ ....(2)

From (1) and (2), we have:

$\sup_i |x_{im} - x_{il}| < \varepsilon \quad , \forall m , l > k$

then for all i , $|x_{im} - x_{il}| < \varepsilon \quad , \forall m , l > k$ ...... (3)

$\Rightarrow \forall i$, then $\{x_{im}\}$ is Cauchy sequence in F