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[3] $l_\infty = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1} |\alpha_n|^p \le m\}$ is a vector space over R or C.

[4] C[a, b]={f: [a, b]→R : f is continuous and C[a, b]} is a vector space over R or C.

[5] L^p[a, b]= {f:[a, b] →R , f is Lebesgue integrable on [a, b] s.t. $\int_a^b |f(x)|\,dx < \infty$ } is a vector space over R or C.

[6] Let V be the set M(m, n)(C) of complex{valued m x n matrices, with usual addition of matrices and scalar multiplication.

Sol.

[1] Let $x = (\alpha_n)_{n=1}^{\infty}$, $y = (\beta_n)_{n=1}^{\infty} \in$S, λ is a scalar, then

1. $x + y = (\alpha_n)_{n=1}^{\infty} + (\beta_n)_{n=1}^{\infty} = (\alpha_n + \beta_n)_{n=1}^{\infty} \in S$
2. $\lambda(\alpha_n)_{n=1}^{\infty} = (\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n, ...) = (\lambda\alpha_n)_{n=1}^{\infty} \in S$

Definition 1.3

Let V be a vector space. A non-empty set U ⊂ V is a *linear subspace* of V if U is itself a vector space (with the same vector addition and scalar multiplication as in V). This is equivalent to the condition that:

$\alpha x + \beta y \in$ U, for all α, $\beta \in$F and x, $y \in$U

(which is called the *subspace test).*

Example 1.4.

[1]The set of vectors in R^n of the form (x_1, x_2, x_3, 0,…, 0) forms a three-dimensional linear subspace.

[2] The set of polynomials of degree ≤ r forms a linear subspace of the set of polynomials of degree ≤ n for any $r \le n$.

Definition 1.5. Linear independence and dependence of a given set M of vectors $x_1,…,x_r$ (r ≥1) in a vector space V are defined by means of the equation

$\alpha_1 x_1 + \alpha_2 x_2 + … + \alpha_r x_r = 0$ …. (*)

where α_1, α_2, …, α_r are scalars. Clearly, equation (*) holds for $\alpha_1 = \alpha_2 = … = \alpha_r = 0$. If this is the only r-tuple of scalars for which (*) holds, the set M is said to be *linearly independent*. M is said to be *linearly dependent* if M is not linearly independent, that is , if (*) also holds for some r-tuple of scalars, not all zero.

Definition 1.6.: Let V be a vector space over a field F, $x \in$V is called linear combination of $x_1, x_2, …, x_n \in$V if $x = \lambda_1 x_1 + \lambda_2 x_2 + … + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \lambda_i \in F,\ 1 \le i \le m$.

1.7.: Let V be a vector space over a field F, and let S={ $x_1, x_2, …, x_n$}⊆V, S is said to

d V if $x = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \forall x_i \in S, \lambda_i \in$ F, $1 \le i \le m$.

1.8.: Let V be a vector space over a field F, and A be a non-empty subset of V (ϕ≠A⊆ V), A is said to be basis of V if :

1- A linearly independent set.
2- A generated V

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Definition 1.6.: Let V be a vector space over a field F, $x \in V$ is called linear combination of $x_1, x_2, \ldots, x_n \in V$ if $x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \lambda_i \in F,\ \ 1 \le i \le m.$

Definition 1.7.: Let V be a vector space over a field F, and let $S = \{x_1, x_2, \ldots, x_n\} \subseteq V$, S is said to be *generated* V if $x = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \forall x_i \in S, \lambda_i \in F,\ \ 1 \le i \le m.$

Definition 1.8.: Let V be a vector space over a field F, and A be a non-empty subset of V ($\phi \ne A \subseteq V$), A is said to be basis of V if :

1- A linearly independent set.
2- A generated V.

Definition 1.9. A vector space V is said to be *finite dimensional* if there is a positive integer *n* such that X contains a linearly independent set of *n* vectors whereas any set of *n*+1 or more vectors of X is linearly dependent. *n* is called the dimension of X, written *n*=dim X. By definition, X={0} is finite dimensional and dim X=0. If X is not finite dimensional, it is said to be infinite dimensional.

Examples : dim R=1, dim R^2=2, dim R^n=*n* .

Remarks

1- Let V(F) be a finite dimensional V.S. over a field F, and let w subspace of V(F), then dim W $\le$ dim V, If dim W=dim V then W=V.
2- Let ($\phi \ne S \subseteq V$) then if $0 \in S$ then S is linear dependent subspace.
3- The singleton {x} is linear dependent iff $x \ne 0$.
4- Any subset of linear dependent set is linear dependent.
5- Any set containing a linearly dependent subset is linearly dependent too.

Definition 1.10 : A ***metric space*** is a pair (X, *d*), where X is a set and *d* is a metric on X (or distance function on X), that is, a function defined on X x X such that for all $x, y, z \in X$, we

-valued, finite and nonnegative function.
0 if and only if $x=y$
$= d(y, x)$ (Symmetry).
(4) $d(x, y) \le d(x, z) + d(z, y)$ (Triangle inequality).

Examples (H.W. 2-6)

1) **Real line IR**: this is the set of all real numbers, taken with the usual metric defined by:

$$d(x, y) = |x - y| \quad \forall x, y \in IR$$

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Definition 1.10 : A ***metric space*** is a pair (X, *d*), where X is a set and *d* is a metric on X (or distance function on X), that is, a function defined on X x X such that for all $x, y, z \in$ X, we have:

(1) *d* is real-valued, finite and nonnegative function.

(2) $d(x, y)=0$ if and only if $x=y$

(3) $d(x, y) = d(y, x)$ (Symmetry).

(4) $d(x, y) \leq d(x, z)+d(z, y)$ (Triangle inequality).

Examples (H.W. 2-6)

1) **Real line IR**: this is the set of all real numbers, taken with the usual metric defined by:

$$d(x, y) = |x-y| \quad \forall x,y \in IR$$

then (R, d) is metric space.

2) **Euclidean plane IR²**: The metric space IR^2, with Euclidean metric:

if $x=(x_1, x_2)$, $y=(y_1,y_2)$, then:

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

then (R^2, d) is metric space.

3) **Euclidean Space IRⁿ**: If $x=(x_1, x_2, \ldots, x_n)$, $y=(y_1,y_2, \ldots, y_n)$, then:

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^n}$$

then (R^n, d) is metric space.

4) **Function space C[a, b]:** As a set X we take the set of all real-valued functions x, y, ... which are functions of an independent real variable t and are defined and continuous on a given closed interval J = [a, b]. Choosing the metric defined by

$$d(x, y)=\max_{t\in J}|x(t)-y(t)|$$

then (C[a, b], d) is metric space.

5) **Discrete metric space**: We take any set X and on it the so-called discrete metric for X, defined by: $d(x, x) = 0$, $d(x,y)=1$ $(x \neq y)$.

This space (X, *d*) is called a discrete metric space.

6) **Space B(A) of bounded functions**: By definition, each element $x \in$ B(A) is a function defined and bounded on a given set A, and the metric is defined by:

$$d(x, y)=\sup_{t\in A}|x(t)-y(t)|$$

Sol. 1-

...l, finite & $d=|x-y| \geq 0$

$\leftrightarrow |x-y| =0 \leftrightarrow x-y=0 \leftrightarrow x=y$ $\forall x, y \in IR$

$x-y| = |-(y-x)| = |y-x| = d(y, x)$ $\forall x, y \in IR$

$x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x, z)+d(z, y)$ $\forall x, y, z \in IR$

Then (IR, d) is a metric space.

* A ***norm*** on a vector space is a way of measuring distance between vectors.

Definition 1.11.: A ***norm*** on a linear vector space V over F is a function $\|.\| : V \to R$ with the properties that :