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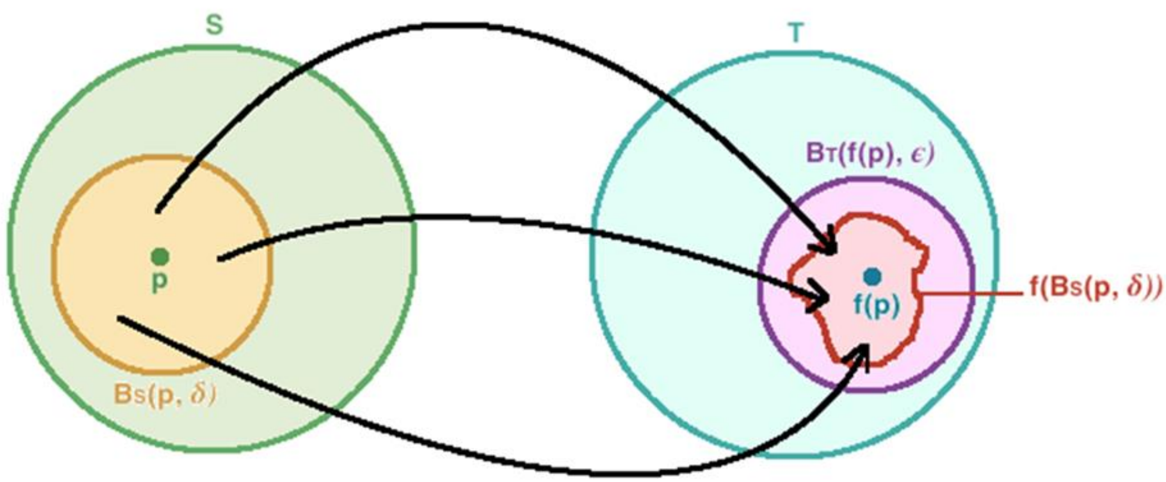

Chapter five

The Continuity in the Metric Spaces

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Chapter 5\The continuity

In this chapter, we recall the definition of continuity in of a function define on a metric space into another one and some its properties.

Definition

Let (X, d_1) and (Y, d_2) be two metric spaces. Suppose that $S \subseteq X, p \in S$ and $f: S \to Y$. We say that f continuous at $x = p$ iff $\forall \varepsilon > 0, \exists \delta > 0 \ni$ $d_2(f(x), f(p)) < \varepsilon$ whenever $d_1(x, p) < \delta$.

If f is continuous at x,$\forall x \in S$ then f is continuous on S.

Notation

By definition, if f continuous at p, then:-

1.$\forall x \in S, d_1(x, p) < \delta \Rightarrow x \in B(p, \delta)$

$$\Rightarrow f(x) \in f(B(p, \delta))$$

2.$\forall \varepsilon > 0, d_2(f(x), f(p)) < \varepsilon \Rightarrow f(x) \in B(f(p), \varepsilon)$

$$\Rightarrow f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$$

By 1 and 2 $\Rightarrow$

$$f \text{ is continuous at } x = p \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$$

Examples

Prove that $f: R \to R, f(x) = x^2$ $is\ cont. \forall x \in X, X = Y = R, d_1 = d_2 = |x - y|$.

Solution

Let $\varepsilon > 0$and $p \in R$ to prove that f continuous at $x = p$?

To find $\delta > 0 \ni |f(x) - f(p)| < \varepsilon \; if \; |x - p| < \delta$. Start with

$|f(x) - f(p)| = |x^2 - p^2|$

$$= |(x-p)(x+p)|$$

$$= |x-p||x+p|$$

$$\leq |x-p|(|x| + |p|) \qquad \dots (1)$$

since $\big||x| - |p|\big| \leq |x - p| < \delta$, then $\big||x| - |p|\big| < \delta$

$\Rightarrow -\delta < |x| - |p| < \delta \Rightarrow |x| - |p| < \delta$

$\Rightarrow |x| < \delta + |p| \dots (2)$

Substitution (2) in (1)

$|f(x) - f(p)| \leq |x - p|(|x| + |p|)$

$< |x - p|(|p| + |p| + \delta)$

$< \delta(2|p| + \delta)$

$\leq \delta(2|p| + 1)$ (since 0<δ<1)

Choose $\delta = min\left\{1, \frac{\varepsilon}{2|p|+1}\right\}$

$\Rightarrow |f(x) - f(p)| < \frac{\varepsilon}{2|p|+1} \cdot (2|p| + 1) = \varepsilon$

$\therefore f$ is continuous at $x = p \quad \forall p \in R \Rightarrow f$ continuous on R.

Exercise

1. $Let \; f: R \to R \ni f(x) = x^2$. Prove that $f \; conts. at \; x = 3 (by \; def.)$?

2. $Let \; f: R \to R \; \ni f(x) = \begin{cases} x sin\left(\frac{1}{x}\right) & if \; x \neq 0 \\ 0 & if \; x = 0 \end{cases}. Prove \; that \; f \; conts. at$

$x = 0(by\ def.)?$

3. $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & , if x \neq 0 \\ 0 & , if\ x = 0 \end{cases}$ Show that f cont. at x= 0.

$if |x - a| < \delta\ then\ |f(x) - f(0)| = \left|x\sin\left(\frac{1}{x}\right) - 0\right| \leq |x| < \delta = \varepsilon, take\ \delta = \varepsilon$

4. $show\ that\ f\ cont. at\ x = 1, where\ f(x) = x^2 - 5x + 7$

$if |x - 1| < \delta \quad then\ |f(x) - f(1)| = |x^2 - 5x + 7 - 3|$

$$= |x^2 - 5x + 4|$$

$$\leq |x - 1|^2 + 3|x - 1|$$

$$< |x - 1| + 3|x - 1|, \quad \begin{pmatrix} choose\ 0 < \delta < 1 \\ \rightarrow |x - 1| < \delta \\ \rightarrow |x - 1|2 \end{pmatrix}$$

$$= 4|x - 1|$$

$$< 4\delta \quad , take\ \delta = \frac{\varepsilon}{4}$$

5. If $f: R \rightarrow R\ such\ that\ |f(x)| \leq M|x|, \forall x \in R\ and\ M\ is\ fixed. Show\ that\ f\ is\ cont. at\ 0.$

$If\ |x - 0| < \delta\ then\ |f(0)| \leq M|0| \rightarrow f(0) = 0$

$|f(x) - f(0)| \leq M|x| \leq \delta < \varepsilon \quad , take\ \delta = \frac{\varepsilon}{M}$

<u>Theorem (1)</u>

Let $(X, d_1)\ and\ (Y, d_2)$ are two metric spaces $f: X \rightarrow Y\ be$ function, then:-

f is continuous at $p \in X \iff \lim_{n\to\infty} f(x_n) = f(p)$, for each seqence $\langle x_n \rangle in\ X$

where $x_n \to p$

Proof

$(\Rightarrow)$ *Suppose that* f *is cont. at* p *and* $\langle x_n \rangle$ *seq. in* X *such that* $x_n \to p$

$\lim\limits_{n\to\infty} f(x_n) = f(p)$?

$i.e.$, to prove that $\exists k \in N \ni d_2\big(f(x_n), f(p)\big) < \varepsilon \quad \forall n > k$?

since f is cont. at p, then $\forall \varepsilon > 0\ \exists k \in N \ni d_2\big(f(x_n), f(p)\big) < \varepsilon$.

[by hypothesis]

Now, $if\ d_1(x, p) < \delta, \forall x$. And, Since $x_n \to p$

then $\exists k \in N \ni d_1(x_n, p) < \delta \quad \forall n > k$

Since f *is* cont. $\Rightarrow$ at this k we get:

$d_2\big(f(x_n), f(p)\big) < \varepsilon \quad \forall n > k \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(p)$

$(\Leftarrow)$

Suppose that $f(x_n) \to f(p)$ *for each* $x_n \to p$. To prove that f is cont. at p?

If not, mean, we suppose that f is not cont. at p. Given $\varepsilon > 0$

$\Rightarrow \exists x \in X \ni d_2\big(f(x), f(p)\big) \geq \varepsilon$ and $d_1(x, p) < \delta$

choose $\delta_1 = 1 \Rightarrow \exists x_1 \in X \ni d_2\big(f(x_1), f(p)\big) \geq \varepsilon$ and $d_1(x_1, p) < \delta_1 = 1$

choose $\delta_2 = \frac{1}{2} \Rightarrow \exists x_2 \in X \ni d_2\big(f(x_2), f(p)\big) \geq \varepsilon$ and $d_1(x_2, p) < \delta_2 = \frac{1}{2}$

choose $\delta_3 = \frac{1}{3} \Rightarrow \exists x_3 \in X \ni d_2\big(f(x_3), f(p)\big) \geq \varepsilon$ and $d_1(x_3, p) < \delta_3 = \frac{1}{3}$

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choose $\delta_n = \frac{1}{n} \Rightarrow \exists x_n \in X \ni d_2\big(f(x_n), f(p)\big) \geq \varepsilon$ and $d_1(x_n, p) < \delta_n = \frac{1}{n}$

$since \ \frac{1}{n} \to 0 \ as \ n \to \infty \Rightarrow d_1(x_n, p) < \frac{1}{n} \to 0$

$\Rightarrow \langle x_n \rangle \to p$, but $d_2\big(f(x_n), f(p)\big) \geq \varepsilon \Rightarrow \langle f(x_n) \rangle$ doesnot convergent to $f(p)$

$\Rightarrow C!$

$\Rightarrow f \ cont. at \ p.$

Examples

1. Let $f: R \to R \ni f(x) = \begin{cases} 1 \ if \ x > 0 \\ 0 \ if \ x = 0 \\ -1 \ if \ x < 0 \end{cases}$

   Use Theorem (1)to prove that f is not cont. at p=0?

Solution: let $\langle x_n \rangle = \langle \frac{1}{n} \rangle \Rightarrow \frac{1}{n} \to 0 \ as \ n \to \infty \Rightarrow$

$\forall n, f(x_n) = f\left(\frac{1}{n}\right) = 1 \Rightarrow f(x_n) = f\left(\frac{1}{n}\right) = 1,1,1,\ldots$ converges to $1 \neq f(0) = 0$

$\therefore \ by \ Th. (1) \ f$ is not conts. at $p = 0$

2. Let $f: [a,b] \to R \ni f(x) = \begin{cases} 1 \ if \ x \in Q' \\ 2 \ if \ x \in Q \end{cases}$

Use Th. (1) to prove that f is not cont.?

Solution: let $p \in [a,b] \Rightarrow p \in Q \ or \ p \in Q'$

$\boldsymbol{i}. if \ p \in Q \ s$ince(between any two reals there are infinetly rationals and irrationals)

and (for every real p other is rational(or irrational) a irrational (or rational) sequence converges to p)

$\Rightarrow$there is an irrational seq. $\langle x_n \rangle$and $x_n \to p$

$$\Rightarrow f(x_n) = 2 \Rightarrow \langle f(x_n) \rangle = 2,2,2,\ldots conv. to \ 2 \neq f(p) = 1$$

So, by Th. (2) $\Rightarrow f$is not cont. at $p, \forall p \ni p \in Q$.

ii. $if\ p \in Q'$

by similar way we can show that f is not conts. at $p, \forall p \ni p \in Q'$

Then f is not cont. at $p, \forall p \ni p \in [a, b]$.

<u>Theorem (2)</u>

Let (X, d_1) and (Y, d_2) are two metric spaces $f: X \to Y$ be a function, then:-

f continuous on $X \Leftrightarrow f^{-1}(V)$ is open set in $X, \forall V$ is open set in Y

Proof($\Rightarrow$)Suppose that f is cont. on X. Let V be open set in Y. To prove that $f^{-1}(V)$ is open in X?

let $x \in f^{-1}(V) \Rightarrow f(x) \in V$. Since V is open set [by hypothesis]

$\Rightarrow \exists \varepsilon > 0 \ni B(f(x), \varepsilon) \subset V$

Since f is cont. on $X \Rightarrow \exists \delta > 0 \ni f\big(B(x,\delta)\big) \subset B(f(x), \varepsilon) \subset V$

$\Rightarrow f^{-1}(f(B(x,\delta))) \subset f^{-1}(V)$

$\Rightarrow B(x,\delta) \subset f^{-1}(V)$.This is true for all $x \in f^{-1}(V) \Rightarrow f^{-1}(V)$ is open set

($\Leftarrow$)Suppose that $f^{-1}(V)$is open set in X, for each V be open set in Y.

To prove that f is cont. on X?

Let $\varepsilon > 0, x \in X \Rightarrow B(f(x), \varepsilon)$ is open set in Y (every ball is open set)

$\Rightarrow f^{-1}\big(B(f(x), \varepsilon)\big)$is open set in X[by hyp.]

$\Rightarrow \forall x \in f^{-1}\big(B(f(x), \varepsilon)\big) \Rightarrow \exists \delta > 0 \ni B(x,\delta) \subset f^{-1}\big(B(f(x), \varepsilon)\big)$By def. of open set

$\Rightarrow \exists \delta > 0 \ni f\big(B(x,\delta)\big) \subset B(f(x), \varepsilon)$

$\Rightarrow f\big(B(x,\delta)\big) \subset B(f(x), \varepsilon) \Rightarrow f$ is cont.at x and this true for all $x \in X$ (by Remark)

$\Rightarrow f$ is conts.on X.

Examples

1. Let $f: R \to R \ni f(x) = x^2, use\ Th. (2)\ to\ prove\ that\ f\ is\ cont. on\ R$?

Solution

$let\ V\ open\ set\ in\ R\ \Rightarrow V = (a, b), there\ are\ three\ cases$:

$i. if\ a, b > 0 \Rightarrow f^{-1}(V) = (\sqrt{a}, \sqrt{b}) \cup (-\sqrt{b}, -\sqrt{a}) open \Rightarrow f^{-1}(V) is\ open\ in\ X$

$ii. if\ a < 0, b > 0 \Rightarrow f^{-1}(V) = (-\sqrt{b}, \sqrt{b}) open \Rightarrow f^{-1}(V)\ is\ open\ in\ X$

$iii. a, b < 0 \Rightarrow f^{-1}(V) = \emptyset\ open \Rightarrow f^{-1}(V)\ is\ open\ in\ X$

$\therefore \forall\ V\ open\ set\ in\ Y\ \Rightarrow f^{-1}(V)\ is\ open\ in\ X \Rightarrow f\ is\ cont. on\ X\ by\ Th. (2)$.

2. $Let\ f: R \to R \ni f(x) = |x|\ use\ Th. (3) to\ prove\ that\ f\ is\ conts.$?

3. $Let\ f: R \to R \ni f(x) = x + 1\ use\ Th. (3) to\ prove\ that\ f\ is\ conts.$?

4. $Let\ f: R \to R \ni f(x) = \begin{cases} 1\ \ if\ x > 0 \\ 0\ \ if\ x = 0 \\ -1\ \ if\ x < 0 \end{cases}$

$Use\ Theorem\ (2)\ to\ prove\ f is\ not\ cont. at 0$?

Solution

Let $V = \left(-\frac{1}{2}, \frac{1}{2}\right)\ is\ open\ set\ in\ Y \Rightarrow f^{-1}(V) = \{0\} is\ closed\ set\ in\ X$

$\therefore\ by\ Th. (2)\ f\ is\ not\ cont. at\ p = 0$.

Theorem (3)

Let (X, d) *be a metric space and* f, g *are two real valued functions*

if f *and* g *are cont. then*:

$$f \pm g, \quad f \cdot g, \quad rf, r \in R \ \text{ and } \frac{f}{g}, g \neq 0 \quad \text{ are cont.}$$

Corollary

Every polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ is cont., where a_i are constants, $i = 1,2, \ldots, n$

Proof: *Since* $f(x) = x^n$ *is conts. function* $\forall n$ *and* rf *is conts. (by Th. (3))*

$\Rightarrow$ *The sum of conts. function is conts. (by Th. (3))*

Definition

$(V, +, \cdot)$ *is a vector space over the field* R, *if*

$V \neq \emptyset$, $(V, +)$ commutative group and • is scalar product such that

$(r + s)v = rv + sv \quad r, s \in R, v \in V$

$r(v + w) = rv + rw \quad r \in R, v, w \in V$

$(rs)v = r(sv) \qquad r, s \in R, v \in V$

$1. v = v$

Examples about vector spaces

1. $(R^2, +, \cdot)$ is vector space over R, where if $x = (x_1, x_2), y = (y_1, y_2)$

$x + y = (x_1 + y_1, x_2 + y_2), \qquad rx = (rx_1, rx_2)$

$2.\ (M,+,\cdot) is\ vector\ space\ over\ R, where\ if\ A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}, rA = \begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem (4)

Let (X, d) be a metric space, define the following set

$C(X) = \left\{ {}^{f}/_{f} : X \to R\ cont. \right\}$Then C(X) is vector space over R, where
$(f + g)(x) = f(x) + g(x),\ \ (rf)(x) = rf(x)$.

The Properties of Real Valued Function Defined on a Compact Spaces

Theorem (5)

Let $(X, d_1), (Y, d_2)$ are two metric spaces and $f: X \to Y$ continuous function. If X is compact, then $f(X)$ is compact.

Proof: define $f(X)$ as $f(X) = \{f(x): x \in X\}$.

$Let\ \{V_\lambda : V_\lambda open\ subset\ of\ Y, \lambda \in \Lambda\}\ be\ an\ open\ cover\ for\ f(X)$

$$\Rightarrow f(X) \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$$

$Since\ V_\lambda\ is\ open\ set\ in\ Y, \forall \boldsymbol{\lambda} \in \Lambda, and\ f\ is\ cont., then\ f^{-1}(V_\lambda) is\ open\ in\ X$

$\forall \lambda \in \Lambda\ (by\ Th. (2))$

$$Since\ f(X) \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda , then\ X \subseteq f^{-1}\left(\bigcup_{\lambda \in \Lambda} V_\lambda\right) = \bigcup_{\lambda \in \Lambda}\left(f^{-1}(V_\lambda)\right)$$

$\Rightarrow \{f^{-1}(V_\lambda): \lambda \in \Lambda\}$ is open cover for X

⇒there is finite subcover $f^{-1}\{f^{-1}(V_1), f^{-1}(V_2), \dots, f^{-1}(V_n)\}$for X(X is compact)

$$\Rightarrow X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i) = f^{-1}\left(\bigcup_{i=1}^{n} V_i\right) \Rightarrow X \subseteq f^{-1}\left(\bigcup_{i=1}^{n} V_i\right)$$

$\Rightarrow f(X) \subseteq \cup_{i=1}^{n} V_i \Rightarrow \{V_1, V_2, \dots, V_n\}$ is finite subcover for $f(X)$.

$\Rightarrow f(X)$is compact

Definition

Let (X, d) be a metric space and $f: X \to R$. we say that f is bounded if $\exists M > 0 \ni |f(x)| \leq M, \forall x \in X$.

On other hand $f(X) = \{f(x): x \in X\}$ is bounded set if it has upper and lower bounds.$i.e., f\ is\ bounded\ \Leftrightarrow f(X)\ is\ bounded\ set.$

Theorem (6)

Let (X, d) be a metric spaces and $f: X \to R$. continuous function if X is compact, then f is bounded

Proof

$Since\ f\ is\ continuous\ and\ X\ compact\ \Rightarrow f(X) is\ compact (by\ Th. (5))$

$\Rightarrow f(X) is\ closed\ and\ bounded\ \Rightarrow f\ is\ bounded.$

Exercise: Give example for bounded function and its domain not compact.

Solution let $X = (0,1)\ isnot\ compact, f: X \to R, f(x) = 3x$

$if\ M = 3 \Rightarrow |f(x)| \leq 3 \Rightarrow f\ bounded$

Remark: The condition of compactness in Th. (6) is necessary. To explain this we give the following example:

Let $X = (0, \infty)$ $isnot\ compact, f(x) = \frac{1}{x}\ cont., x > 0$. (check)

$f\ is\ not\ bounded\ since\ \forall m > 0, \exists k \in N \ni M < \dfrac{1}{k} = f(k)(by\ Arch. prop.)$

Definition

Let (X, d) be a metric space and $f: X \to R$ bounded function. A point a is called:

-Maximum extreme point of $f\ if\ f(x) \leq f(a) \forall x \in X$

-Minimum extreme point of $f\ if\ f(a) \geq f(x) \forall x \in X$

Example Let $f: R \to R, f(x) = \sin(x).\ Find\ Max. Min. extreme\ points$?

Solution

$\sin(x)\ is\ bounded\ function, -1 \leq \sin(x) \leq 1, \forall x$

$if\ a = \dfrac{\pi}{2} + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$

$then\ f(a) = \sin(a) = 1$
$\Rightarrow a\ is\ Max. extreme\ point\ and\ f\ has\ infinite\ max. extreme\ points$

$if\ a = \dfrac{-\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$then\ f(a) = \sin(a) = -1$

$\Rightarrow a\ is\ min. extreme\ point\ and\ f\ has\ infinite\ min. extreme\ points$

Example

Give example for a function f which has an unique max. and min. extreme points.

Theorem (7)

Let $f: X \to R$ be a cont. function. If X is compact, then $\exists a, bX \text{ such that}$

$f(a) \le f(x) \le f(b), \forall x \in X.$

$i.e. f \text{ has min. extreme point at } a \text{ and } f \text{ has max. extreme point at } b$

Proof: To prove that $\exists b \in X \ni f(x) \le f(b)$?

$\text{Since } f \text{ is cont. function on compact space } X$

$\Rightarrow f(X) \text{ is compact. } (\text{by } Th(5))$

$\Rightarrow f(X) \text{ is closed and bounded}$

$\text{Since } f(X) \text{ is bounded in } R, \text{ then } f(X) \text{ is bounded above}$

$\Rightarrow (\text{by completeness axiom}) f(X) \text{ has supremum, say } sup\big(f(X)\big) = M$

$\Rightarrow f(x) \le M, \forall x \in X.$

To prove that $M \text{ is an accumlation point of } f(X)$?

$i.e., \text{to prove that } \forall \varepsilon > 0, (M - \varepsilon, M + \varepsilon) \backslash \{M\} \cap f(X) \ne \emptyset$?

$if \text{ not} \Rightarrow \exists \varepsilon > 0, (M - \varepsilon, M + \varepsilon) \backslash \{M\} \cap f(X) = \emptyset$

$\Rightarrow M - \varepsilon \text{ is upper bound of } f(X) \Rightarrow C!$

$\Rightarrow M \text{ is accumlation point for } f(X)$

$\text{Since} f(X) \text{ is closed} \Rightarrow M \in f(X) \Rightarrow \exists b \in X \ni f(b) = M \text{ and } f(x) \le M = f(b)$

$\Rightarrow f \text{ has a max. extreme point at } b.$

$\text{By similar way prove that } \exists a \in X \ni f(a) \le f(x), \forall x \in X$

Uniformly continuous

Definition

Let (X, d) be a metric space and $f: X \to R$. f is called uniformly continuous if:-

$\forall \varepsilon > 0, \exists \delta > 0 \ni |f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta, \forall x, y \in X$

Remark: The chosen of δ in the definition of uniform continuity is depending on ε only.

Theorem (8) Every uniformly continuous function is continuous.

Proof

Let f be uniformly cont. on X

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni |f(x) - f(y)| < \varepsilon \text{ whenever } d(x, y) < \delta, \forall x, y \in X$

$take y = p \Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni |f(x) - f(p)| < \varepsilon \text{ whenever}$

$d(x, y) < \delta, \forall x, p \in X$

$\Rightarrow f \text{ is conts. at } p, \forall p.$

Remark: The converse of theorem (8) is not true, for example:

$Let\ f: R \to R, f(x) = x^2 \text{ is cont function}$. (Check) to show it is not uniformly.

$$let\ x = n, y = n + \frac{1}{n}, n \in N$$

$$d(x, y) = |x - y| = \left|n - n - \frac{1}{n}\right| = \left|-\frac{1}{n}\right| = \frac{1}{n} < \delta$$

$$by\ Arch.prop.any\ real\ \delta, \exists n \in N \ni \frac{1}{n} < \delta$$

$take\ \varepsilon = 1\ \Rightarrow |f(x) - f(y)| = \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| = \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right|$

$= 2 + \frac{1}{n^2} > \varepsilon = 1\ \Rightarrow f\ is\ not\ uniformly\ conts.$

Theorem (9)

$Let\ f: X \to R$ be a cont. $function$ If X is compact, then f is uniformly cont.

Example

$Let\ f: [a, b] \to R, f(x) = 3x\ cont. \Rightarrow f\ is\ uniformly\ conts.$

- Prove f is uniformly by definition.

Theorem (10) (Intermediate Value property (I.V.P.))

$Let\ f\ be\ a\ cont.\ function\ on\ [a, b]$. If $f(a) = \alpha, f(b) = \beta\ then\ for\ all\ \gamma, \alpha < \gamma < \beta, \exists c, a < c < b\ and\ f(c) = \gamma$

Proof : Let $S = \{x: x \in [a, b], f(x) \leq \gamma\}.\ S \neq \emptyset\ since\ a \in S.$

$S\ is\ bounded\ above\ (since\ b\ is\ upper\ bound\ of\ S)$

$\Rightarrow by\ completeness\ axiom\ , S\ has\ a\ sup.\ say\ \sup(S) = c$

There are three cases$f(c) = \gamma\ or\ f(c) < \gamma or\ f(c) > \gamma$

$if\ f(c) < \gamma$

$Since\ f\ is\ conts.\ at\ c\ and f(c) < \gamma$

$\Rightarrow \forall \varepsilon > 0 \ni f(x) < \gamma, \forall x \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$

$For\ this\ x, if\ c < x < b \Rightarrow x \in S\ and\ c < x \Rightarrow C!$

$\Rightarrow f(c) \nless \gamma (Since\ sup(S) = c)$

$if\ f(c) > \gamma$

$by\ similar\ way\ f(c) \ngtr \gamma \Rightarrow f(c) = \gamma \Rightarrow \exists c, a < c < b \ni f(c) = \gamma.$

Exercise Give example for discontinuous function for all elements of its domain?

<u>Theorem (11)</u> (Interval Theorem)

$If\ f\ is\ continuous\ on\ I = [a,b], then\ f(I) is\ closed\ bounded\ interval.$

Proof: $Since\ I\ is\ closed\ and\ bounded\ \Rightarrow I\ compact$

$Since\ f\ is\ conts. on\ I, then\ f\ has\ max. extreme\ point$

$\Rightarrow \exists c, d, f(c)m, f(d) = M\ such\ that\ m \leq f(x) \leq M, \forall x \in$

$There\ are\ two\ cases : c < d\ or\ d < c$

$if\ c < d. Apply\ (I.V.P.) on\ f\ and\ [c,d]$

$\Rightarrow \forall y, y \in (m,M), \exists x \in (c,d) \ni f(x) = y \Rightarrow f(I) = [m,M].$

Theorem (12) (fixed point theorem)

$Let\ f: [0,1] \to [0,1]\ conts., then\ there\ is\ at\ least\ one\ number\ c\ [c\ is\ called$

$fixed\ point\ of\ f] \ni f(c) = c$

Proof: $suppose\ that\ g: [0,1] \to R \ni g(x) = f(x) - x\ conts. on\ [0,1]$

$(Since\ the\ identity\ function\ i: X \to X\ is\ conts. and\ f\ conts.)$

$\Rightarrow g\ is\ conts. on [0,1]$

$if\ f(0) = 0\ or\ f(1) = 1 \Rightarrow\ the\ proof\ is\ complete$

$Suppose\ that\ f(0) \neq 0 and\ f(1) \neq 1$

$since\ f\ is\ onto\ (f: [0,1] \to [0,1]) \Rightarrow g(0) = f(0) - 0 = f(0) > 0 and$

$g(1) = f(1) - 1 < 0 \Rightarrow g(1) < 0 < g(0)$

$\Rightarrow by (I.V.P.) \exists c, 0 < c < 1 \ni g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

$\Rightarrow f$ *has a fixed point*.

Theorem (13)

Suppose that $f: I \to J$ *is bijective, whereI, J are closed intervals . if* f *cont. then* f^{-1} *conts*.

Theorem (14)

Any polynomial of odd degree has at least one real root.

Exercise

1. *show that* f *discont. and bounded and no max. min., where* $f: [0,2] \to R$,

$f(x) = x - [x]$

2. *show that* f *disconts. and does not satisfy (I.V.P.),* $f: [-1,1] \to R$,

$$f(x) = \begin{cases} -2 & if\ x < 0 \\ 2 & if\ x \geq 0 \end{cases}$$