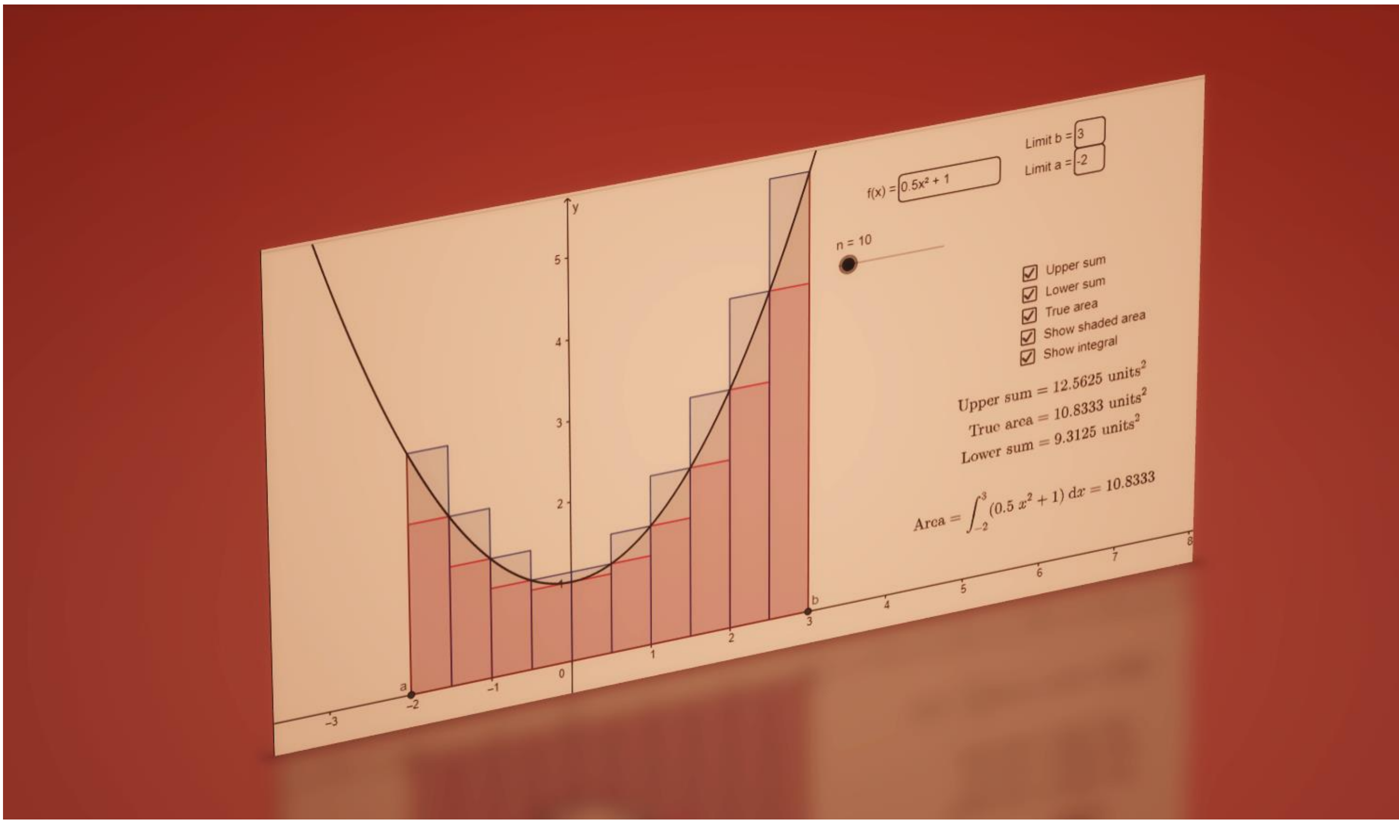
University of Baghdad\College of Education for Pure Science, Ibn Al Haitham

Department of Mathematics



Chapter -7

Riemann's Integral

Dr. Salwa Salman Abed

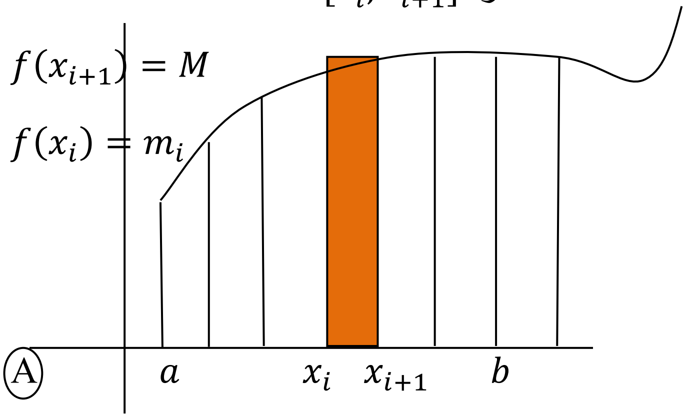
Chapter Seven: Riemann's Integral

اشتقاق تكامل ريمان:-

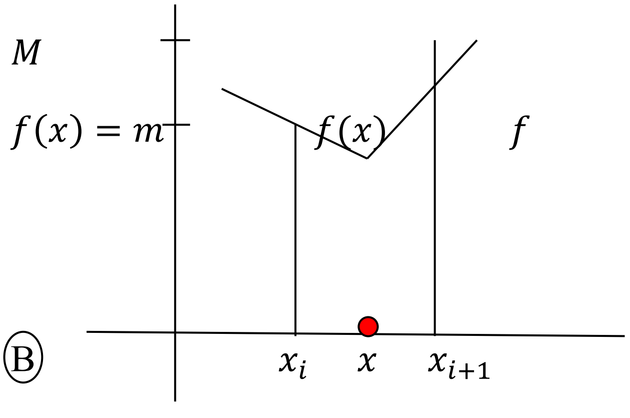
أن فكرة التكامل مبنية على فكرة حساب مساحة المنطقة اسفل منحني الدالة f و هذا ما سنتبعه في حساب تكامل ريمان:-

فعلى سبيل تقريب فكرة التكامل لذهن الطالب سنحسب مساحة المستطيل الموجود في الشكل A:-

1)اذا كان العرض= $x_{i+1}-x_i$
والطول=$m_i = f(x_i)$ اقل قيمة للدالة داخل الفترة $[x_i, x_{i+1}]$



2)اذا كان العرض=$x_{i+1}-x_i$
والطول = $M_i = f(x_{i+1})$ اعلى قيمة للدالة داخل الفترة $[x_i, x_{i+1}]$



$$M = f(x_{i+1})$$

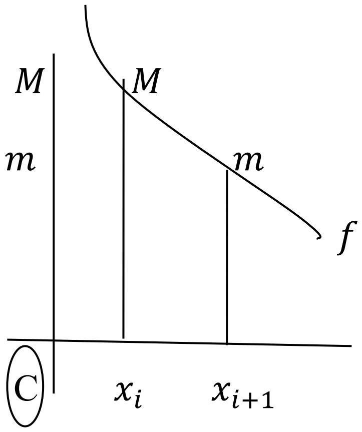
$$m = f(x)$$

فان مجموع كل المساحات من النوع الاول سيسمى لاحقا بمجموع ريمان الادنى.

ومجموع كل المساحات من النوع الثاني سيسمى مجموع ريمان الاعلى.

ملاحظة :*كلما كانت المستطيلات (التجزئة) انعم فان قيمة المجموع (1) سيقترب من قيمة المجموع (2)

*في بعض الحالات قد لا تكون اعلى قيمة (او اقل قيمة) للدالة هي صورة احد رؤوس الفترة $[x_i, x_{i+1}]$
كما في الشكلين C,B



$$M_i = f(x_i)$$

$$m_i = f(x_{i+1})$$

The Definition of Riemann Integral

Let f be a bounded function defined on $[a, b] = J$ where $|J| = b - a$ is the length of J if $\pi = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ is an ordered subset of J where the first element is a and the last one is b, then π is called the partition of J.

And $J_i = [x_i, x_{i+1}], i = 0,1,2, \dots, n-1$ are called the subintervals of J

Remark

We can write the elements of π as $\pi: x_0 < x_1 < x_2 < \cdots < x_n$

Now, since f is a bounded function so that we can talk about:

$M = \sup\{f(x): x \in J\}$, $M_i = \sup\{f(x): x \in J_i\}$

$m = \inf\{f(x): x \in J\}$, $m_i = \inf\{f(x): x \in J_i\}$

Define the following sums:

1- Lower sum of f relative to π المجموع الادنى ل f بالنسبة ل π

$$\underline{R}(f, \pi) = \sum_{i=0}^{n-1} m_i |J_i| = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

2- Upper sum of f relative to π المجموع الاعلى ل f بالنسبة ل π

$$\overline{R}(f, \pi) = \sum_{i=0}^{n-1} M_i |J_i| = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$

Remark: $\forall i\,, m \leq m_i \leq M_i \leq M$

Proof

If $A = \{f(x) : x \in J\}\,, B = \{f(x) : x \in J_i\}\,, \forall i \Rightarrow B \subset A$

By the properties of $\left\{\begin{matrix} inf \\ sup \end{matrix}\right. \Rightarrow \inf(A) \leq \inf(B) \Rightarrow m \leq m_i, \forall i$

$$and\ \sup(B) \leq \sup(A) \Rightarrow M_i \leq M\,, \forall i$$

And since $\inf(B) \leq \sup(B)\,, then\ m \leq m_i \leq M_i \leq M$

Remark: by the above remark:

If f is a bounded function defined on $[a, b]$ then

$$m(b-a) \leq \underline{R}(f,\pi) \leq \overline{\mathrm{R}}(f,\pi) \leq M(b-a)$$

Proof

Since $m \leq m_i \leq M_i \leq M \Rightarrow m|J_i| \leq m_i|J_i| \leq M_i|J_i| \leq M|J_i|, \forall i$

$$\Rightarrow \sum_{i=0}^{n-1} m|J_i| \leq \sum_{i=0}^{n-1} m_i|J_i| \leq \sum_{i=0}^{n-1} M_i|J_i| \leq \sum_{i=0}^{n-1} M|J_i|$$

$$\Rightarrow m \sum_{i=0}^{n-1} |J_i| \leq \underline{R}(f,\pi) \leq \overline{\mathrm{R}}(f,\pi) \leq \mathrm{M} \sum_{i=0}^{n-1} |J_i|$$

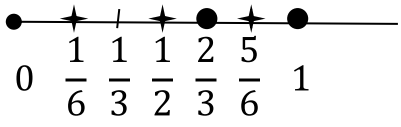
$$\Rightarrow m(x_n - x_0) \leq \underline{R}(f,\pi) \leq \overline{\mathrm{R}}(f,\pi) \leq \mathrm{M}(x_n - x_0)$$

Definition

Let π, π^* be two partitions of $[a, b]$, we say that π^* is the refinement of π if $\pi \subseteq \pi^*$

Example

$\pi = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ $partition\ of$ $[0,1]$



$\pi^* = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\} also\ partition\ of$ $[0,1] \Rightarrow \pi \subset \pi^*$

Remark (without proof)

If f is a bounded function on $[a,b]$ and π^* is a refinement of π then:

1)$\underline{R}(f,\pi) \leq \underline{R}(f,\pi^*)$

2)$\overline{\mathrm{R}}(f,\pi^*) \leq \overline{\mathrm{R}}(f,\pi)$

3)$\underline{R}(f,\pi) \leq \overline{\mathrm{R}}(f,\pi^*)$

By this remark, we define the following two nonempty sets:-

$\overline{\mathrm{R}}(f) = \{\overline{\mathrm{R}}(f,\pi): \pi \text{ is partition}\} \neq \emptyset$ غير خالية لانه توجد على الاقل تجزئه
$\underline{R}(f) = \{\underline{\mathrm{R}}(f,\pi):\pi \text{ is a partition}\} \neq \emptyset$ واحده $J =$
$[a,b]$ وهي الفتره نفسها

وعليه يوجد على الاقل مجموع اعلى واحد وعلى االاقل مجموع ادنى واحد

Now, since $\underline{R}(f,\pi) \leq \overline{\mathrm{R}}(f,\pi^*)$, then any element in $\overline{\mathrm{R}}(f)$ will be the upper bound of $\underline{R}(f)$, also, any element in $\underline{R}(f)$ will be lower of $\overline{\mathrm{R}}(f)$.

So, the upper Riemann integral of f is

$\overline{\int} f = \int \overline{R} f = \inf(\overline{\mathrm{R}}(f))$ تكامل ريمان الاعلى ل f

$\underline{\int} f = \int \underline{R}(f) = \sup\left(\underline{R}(f)\right)$ تكامل ريمان الادنى ل f

And it is clear that. $\underline{\int} f \leq \overline{\int} f$

Definition

Let f be a bounded function defined on the interval$[a,b]$. f is the Riemann integral $(R-I)$ iff $\underline{\int} f = \overline{\int} f$. And denoted by $\int Rf$ or $\int_a^b f$.

Example: find $\int_0^2 f(x), f(x) = 3x$

$[a,b] = [0,2] = J$, $|J| = b - a = 2$

طريقة عامه لعمل اي تجزئه

$$\pi: a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < a + n\frac{b-a}{n} = b$$

$$\Rightarrow 0 < \frac{2}{n} < 2 \cdot \frac{2}{n} < \cdots < 2$$

$$\Rightarrow 0 < \frac{2}{n} < \frac{4}{n} < \frac{6}{n} < \cdots < 2$$

$$x_0 \quad x_1 \quad x_2 \quad x_3 \qquad\qquad x_n$$

∴The sub-intervals are

$$J_0 = \left[0, \frac{2}{n}\right], J_1 = \left[\frac{2}{n}, \frac{4}{n}\right], J_2 = \left[\frac{4}{n}, \frac{6}{n}\right] \ldots J_{n-1} = \left[\frac{2(n-1)}{n}, 2\right], \forall i, |J_i| = \frac{2}{n}$$

$$\underline{R}(f, \pi) = \sum_{i=0}^{n-1} m_i |J_i| = m_0(x_1 - x_0) + m_1(x_2 - x_1) + \cdots + m_{n-1}(x_n - x_{n-1})$$

$$= f(x_0)\frac{2}{n} + f(x_1)\frac{2}{n} + \cdots + f(x_{n-1})\frac{2}{n}$$

$$= 0\,\frac{2}{n} + \frac{6}{n} \cdot \frac{2}{n} + \frac{12}{n} \cdot \frac{2}{n} + \cdots + \frac{6(n-1)}{n} \cdot \frac{2}{n}$$

$$= 0 + \frac{12}{n^2} + \frac{24}{n^2} + \frac{12(n-1)}{n^2} = \frac{12}{n^2}(1 + 2 + \cdots + n - 1)$$

ملاحظه:متسلسلة منتهيه

مجموعها $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

$$= \frac{12}{n^2}\left(\frac{1}{2} n(n-1)\right) = \frac{6(n-1)}{n} = 6 - \frac{6}{n} \quad , \forall n$$

$$\therefore \underline{\int} f = \sup\{\underline{R}(f, \pi) : \pi\} = \sup\left\{6 - \frac{6}{n} : n \in N\right\} = 6$$

$$\overline{\mathrm{R}}(f, \pi) = \sum_{i=0}^{n-1} M_i |J_i|$$

$$= M_0(x_1 - x_0) + M_1(x_2 - x_1) + \cdots + M_{n-1}(x_{n-1} - x_{n-2})$$

$$= f(x_1)\frac{2}{n} + f(x_2)\frac{2}{n} + \cdots + f(x_n)\frac{2}{n}$$

$$= \frac{6}{n} \cdot \frac{2}{n} + \frac{12}{n} \cdot \frac{2}{n} + \cdots + \frac{6n}{n} \cdot \frac{2}{n} = \frac{12}{n^2} + \frac{24}{n^2} + \cdots + \frac{12}{n^2}$$

$$= \frac{12}{n^2}(1 + 2 + \cdots + n) = \frac{12}{n^2}\frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}, \forall n$$

$\therefore \overline{\int} f = \inf\{\overline{\mathrm{R}}(f,\pi):\pi\} = \inf\left\{6+\frac{6}{n} : n \in N\right\} = 6$

$\therefore \overline{\int} f = \underline{\int} f = 6 = \int_0^2 f(x)$

Does there exist a bounded function that is not R-I?

Yes, for example: -

Let $f:[a,b] \to R, f(x) = \begin{cases} 0 & , if\ x \in Q \\ 1 & , if\ x \in Q' \end{cases}$

Suppose that $\pi: x_0 < x_1 < \cdots < x_n$ be a partition of $[a,b]$ by density of

$R \Rightarrow \forall i, J_i$ has infinitely rational and irrationals, so

$\forall i, m_i = \inf\{f(x): x \in J_i\} = 0 \Rightarrow \underline{R}(f,\pi) = \Sigma m_i |J_i| = 0$

$\forall i, M_i = \sup\{f(x): x \in J_i\} = b \Rightarrow \overline{\mathrm{R}}(f,\pi) = \Sigma M_i |J_i| = \Sigma |J_i| = b - a \neq 0$

$\therefore \underline{\int} f = 0 \neq \overline{\int} f = 1 \Rightarrow f$ is not R-I

Example: Let $f:[a,b] \to R, f(x) = c, c$ is constant. Show that

$\int_a^b f(x)dx = c(b-a)$.

$\underline{R}(f,\pi) = \sum_{i=0}^{n-1} m_i |J_i| = c \sum_{i=0}^{n-1} |J_i|$

$= c[(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})] = c(x_n - x_0) = c(b-a)$

Which mean $\underline{R}(f,\pi)$ is constant $\Rightarrow\Rightarrow \underline{\int} f = \sup\{\underline{R}(f,\pi):\pi\} = c(b-a)$.

Also, $\overline{\int} f = \int \overline{R} f = \inf(\overline{\mathrm{R}}(f)) = c(b-a)$. Then (by definition) f is R-I.

H.W. Find $\int Rf$

1)$f:[0,2] \to R, f(x) = x^2$

2)$f:[0,1] \to R, f(x) = x$

$\sum_{x=0}^{n} x^2$

Theorem (1)

If f is a bounded function defined on $[a,b]$ then

f is $RI \Leftrightarrow \forall \varepsilon > 0, \left|\overline{\mathrm{R}}(f, \pi) - \underline{R}(f, \pi)\right| < \varepsilon$ For any partition π

Proof: $\Rightarrow$)Suppose f is $R - I$. Mean that $\overline{\int} f = \underline{\int} f = \int f$

Let $\varepsilon > 0, since\ f\ is\ R - I \Rightarrow \overline{\mathrm{R}}(f, \pi_1) - \overline{\int} f < \frac{\varepsilon}{2}$ for some π_1 (by properties of inf)

And $\pi_2 \ni \underline{\int} f - \underline{R}(f, \pi_2) < \frac{\varepsilon}{2}$ (by properties of sup)

Let $\pi = \pi_1 \cup \pi_2$ refinement

$$\Rightarrow \overline{\mathrm{R}}(f, \pi_1) \leq \overline{\mathrm{R}}(f, \pi) \Rightarrow \overline{\mathrm{R}}(f, \pi) - \overline{\int} f < \frac{\varepsilon}{2} \qquad \text{.... (1)}$$

And since $\underline{R}(f, \pi) \leq \underline{R}(f, \pi_2) \Rightarrow \underline{\int} f - \underline{R}(f, \pi) < \frac{\varepsilon}{2}$ ....(2)

By (1) and (2)$\Rightarrow \overline{\mathrm{R}}(f, \pi) - \overline{\int} f + \underline{\int} f - \underline{R}(f, \pi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow \overline{\mathrm{R}}(f, \pi) - \underline{R}(f, \pi) < \varepsilon$

$\Leftarrow$)For the converse, if $\forall \varepsilon > 0, \exists \pi$ such that $\overline{\mathrm{R}}(f, \pi) - \underline{R}(f, \pi) < \varepsilon$,

To prove f is $R - I$?

Since $\underline{R}(f, \pi) \leq \underline{\int} f \leq \overline{\int} f \leq \overline{\mathrm{R}}(f, \pi)$, $\forall \pi$

Hence $0 \leq \overline{\int} f - \underline{\int} f \leq \overline{\mathrm{R}}(f, \pi) - \underline{R}(f, \pi) < \varepsilon$, $\forall \varepsilon > 0$

Then $\overline{\int} f - \underline{\int} f = 0 \Rightarrow \overline{\int} f = \underline{\int} f \Rightarrow f$ is R-I.

Theorem (2)

Let f be a continuous function on $[a, b]$ then f is a Riemann integral.

Proof

Let $\varepsilon > 0$, since f is cont. on $[a, b]$ then $f$is uniformly continuous.

$\Rightarrow \exists \delta > 0 \ni$ for any x, y in $[a, b], |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

We will construct a partition for $[a, b]$ as follows by Arch-property

$$\exists n \in N \ni n\delta > b-a \Rightarrow \frac{b-a}{n} < \delta$$

$\Rightarrow \pi = \{a = x_1, x_2 \dots, x_n = b\}$ be a partition of $[a,b]$

$$\ni |J_i| = \frac{b-a}{n}, J_i = [x_i, x_{i+1}]$$

$\Rightarrow |J_i| = |x_{i+1} - x_i| < \delta \,, \forall i$

Since f cont. on $[a,b] \Rightarrow f$ is bounded and has min and max values in $[a,b]$ and in $J_i, \forall i$

$\Rightarrow \exists t_i, t_i{}'$ in $J_i \ni M_i = \sup\{f(x) : x \in J_i\} = f(t_i)$

$$m_i = \inf\{f(x) : x \in J_i\} = f(t_i{}')$$

Since $|t_i - t_i{}'| < \delta \Rightarrow |f(t_i) - f(t_i{}')| < \frac{\varepsilon}{b-a}$

Then $\overline{\mathrm{R}}(f,\pi) - \underline{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i| - \sum_{i=1}^{n} m_i|J_i|$

$$= \Sigma|f(t_i) - f(t_i{}')||J_i| < \sum_{i=1}^{n} \frac{\varepsilon}{b-a}|J_i| = \frac{\varepsilon}{b-a}\sum_{i=1}^{n}|J_i| = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

Thus f is R-I.

Remark: The converse is not necessarily true

Example: $f : [-5,5] \to R, f(x) = \begin{cases} 3 & , if\ x < 0 \\ 7 & , if\ x > 0 \end{cases}$ f is Riemann integral but not continuous.

Solution:-$\forall n, \pi = \left\{-5, \frac{-1}{n}, \frac{1}{n}, 5\right\}$

$$\Rightarrow J_0 = \left[-5, \frac{-1}{n}\right], J_1 = \left[\frac{-1}{n}, \frac{1}{n}\right], J_2 = \left[\frac{1}{n}, 5\right]$$

$$\underline{R}(f,\pi) = \sum_{i=0}^{n-1} m_i|J_i| = m_0(x_1 - x_0) + m_1(x_2 - x_1) + m_2(x_3 - x_2)$$

$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$$

$$= 3\left(\frac{-1}{n} - (-5)\right) + 3\left(\frac{1}{n} - \left(\frac{-1}{n}\right)\right) + 7\left(5 - \frac{1}{n}\right)$$

$$= \frac{-3}{n} + 15 + \frac{3}{n} + \frac{3}{n} + 35 - \frac{7}{n} = 50 - \frac{4}{n} \quad , \forall n$$

$$\underline{\int} f = \sup\left(\underline{R}(f)\right) = \sup\left\{50 - \frac{4}{n} : n \in N\right\} = 50$$

$$\overline{\mathrm{R}}(f, \pi) = \sum_{i=0}^{n-1} M_i |J_i| = M_0|J_0| + M_1|J_1| + M_2|J_2|$$

$$= 3\left(\frac{-1}{n} - (-5)\right) + 7\left(\frac{1}{n} - \left(\frac{-1}{n}\right)\right) + 7\left(5 - \frac{1}{n}\right)$$

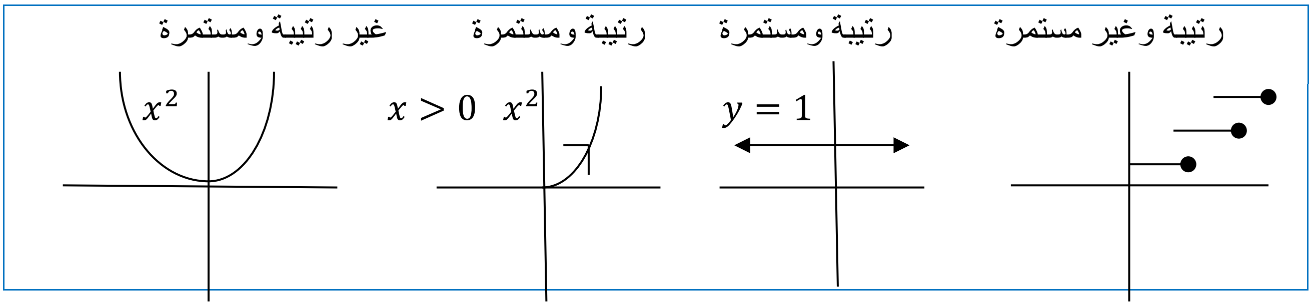
$$= \frac{-3}{n} + 12 + \frac{7}{n} + \frac{7}{n} + 35 - \frac{7}{n} = 50 + \frac{4}{n}$$

$$\overline{\int} f = \inf\left(\overline{R}(f)\right) = \inf\left\{50 + \frac{4}{n} : n \in N\right\} = 50$$

$$\therefore \overline{\int} f = \underline{\int} f \Rightarrow \int_{-5}^{5} f = 50$$

Theorem (3)

If f is bounded monotonically (non-decreasing) on $[a, b]$ then f is the Riemann integral



Proof

Let $\varepsilon > 0, \pi = \{x_1 = a, x_2, \dots, x_n = b\}$ be a partition on $[a, b]$ such that $J_i = [x_i, x_{i+1}], |J_i| = \frac{b-a}{n}$

Since f is non-decreasing $\begin{cases} M_i = f(x_{i+1}) \\ m_i = f(x_i) \end{cases} \forall i$

$$\Rightarrow \overline{\mathrm{R}}(f,\pi) - \underline{R}(f,\pi) = \sum_{i=1}^{n} [f(x_{i+1}) - f(x_i)]\,|J_i| = \big(f(b) - f(a)\big)\frac{b-a}{n}$$

By arch-property, $\exists n \in N \ni n\varepsilon > \big(f(b) - f(a)\big)\frac{b-a}{n}$

$$\Rightarrow \big(f(b) - f(a)\big)\frac{b-a}{n} < \varepsilon$$

$\therefore \overline{\mathrm{R}}(f,\pi) - \underline{R}(f,\pi) < \varepsilon \quad \Rightarrow f \ is$ R-I

Negligible set المجموعة المهملة

Definition: Let $S \subset R, S$ is called a negligible set if there exists a countable family of open intervals $\{I_k\}$ such that:(1)$S \subseteq \bigcup_{k=1}^{\infty} I_k$,(2)$\forall \varepsilon > 0, \sum_{k=1}^{\infty} |I_k| < \varepsilon$

Sمجموعة مهملة اذا امكن تغطية S بطائفة $\{I_k\}$ من الفترات المفتوحة والتي يقترب مجموع اطوالها من الصفر

Examples

1-Any finite set of real numbers is negligible:

Let $S = \{x_1, x_2, \ldots, x_n\}, \varepsilon > 0, \forall k, 1 \le k \le n$

Let I_k open interval with center x_k and $|I_k| = \frac{\varepsilon}{2n} \Rightarrow S \subset \bigcup_{k=1}^{n} I_k$

And$\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} \frac{\varepsilon}{2n} = \frac{\varepsilon}{2n} \cdot n = \frac{\varepsilon}{2} < \varepsilon$. Then S is neg.

2- Any countable set of real numbers is negligible:

Let $\varepsilon > 0, \forall k, I_k$ open interval with center x_k and $|I_k| = \frac{\varepsilon}{2^{k+1}}$

$\Rightarrow S \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon$

3- Any subset of negligible in negligible

4- The union of a family of negligible sets is negligible.

5- Q is neg.

6- Any interval $[a,b],(a,b),[a,b)$ is not neg. since its length$=b-a$

7- R is not neg. since its length goes to ∞

8- Q' is not neg. since $R=Q\cup Q'$, if Q neg.⇒ R is neg. ⇒C!

9- There is an uncountable neg. set, which is the Cantor set.

Theorem (4) (Lebesgue's Theorem for R-I)

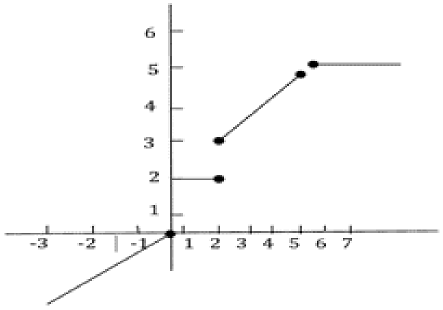
Let f be a bounded function on$[a,b]$, then:

f is the Riemann integral⟺ the set of points of discontinuity of$f$ is neg. set.

f قابلة للتكامل الريماني اذا وفقط اذا كانت مجموعة نقاط عدم الاستمرارية لf مجموعة مهملة

Example

Use Lebesgue's Theorem to show that f is R-I.$f:[-4,7]\to R$,

$f(x)=\begin{cases} x\,,if-3\le x<0 \\ 2\quad,if0\le x<2 \\ x+1\,,if2\le x\le 5 \\ 5\quad,if5<x\le 7\end{cases}$	

Clearly, f is bounded since $|f(x)|\ \le 6\,,\forall x\in[-4,7]$. The set of discontinuous points is $\{0,2,5\}$ finite ⇒ $\{0,2,5\}$ is neg.
By Theorem (4) ⇒ f is R-I

Some properties of Riemann integration

1-If f *and* g are Riemann integrals on $[a,b]$ $c \in R$ then $f+g, cf$ are R-I

Proof: To prove that $f+g, cf$ are R-I we use Lebesgue's theorem.$(Th.4)$

Since f is R-I⇒ the collection of points where there is a discontinuity of f is neg.

And g is R-I⇒ the collection of points where there is a discontinuity of g is neg.

Then the collection of points where there is a discontinuity of $f+g$ is neg. ⇒R-I.

2-**1-If f *and* g are Riemann integrals on $[a,b]$ $c \in R$ then $f+g, cf$ are R-I and**

$$\text{i} - \int_a^b (f+g)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$ii - \int_a^b (cf)(x)dx = c\int_a^b f(x)dx$$

Proof:(***i***) To prove $f+g$ R-I, let $\varepsilon > 0$ and π_1, π_2 be a partition of $[a,b]$ then

$\left|\overline{\mathrm{R}}(f,\pi_1) - \underline{R}(f,\pi_1)\right| < \frac{\varepsilon}{2}$ and $\left|\overline{\mathrm{R}}(g,\pi_2) - \underline{R}(g,\pi_2)\right| < \frac{\varepsilon}{2}$

Let $\pi = \pi_1 \cup \pi_2 \Rightarrow \pi$ is a refinement of π_1 and π_2

Let $m_i = \inf\{f(x): x \in J_i\}, J_i$ interval in π

$m_i' = \inf\{g(x): x \in J_i\}, J_i$ interval in π

Similarly, we can prove that.

$\overline{R}(f,\pi) - \overline{R}(g,\pi) \geq \overline{R}(f+g,\pi)$

Now $\overline{R}(f+g,\pi) - \underline{R}(f+g,\pi) \leq \overline{R}(f,\pi) + \overline{R}(g,\pi) - \left[\underline{R}(f,\pi) + \underline{R}(g,\pi)\right]$

$= \overline{R}(f,\pi) - \underline{R}(f,\pi) + \overline{R}(g,\pi) - \underline{R}(g,\pi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Then $f+g$ is R-I by Th.(1)

To prove that $\int f+g=\int f+\int g$. $f+g$ R-I

For any $\varepsilon>0$, $\int_{-a}^{b} f-\varepsilon<\underline{R}(f,\pi_1)\,,\overline{\mathrm{R}}(f,\pi_1)<\int_{a}^{-b} f+\varepsilon$ and

$$\int_{-a}^{b} g-\varepsilon<\underline{R}(g,\pi_2)\,,\ \overline{\mathrm{R}}(g,\pi_2)<\int_{a}^{-b} g+\varepsilon$$

$\Rightarrow \int_{-a}^{b} f+\int_{-a}^{b} g-2\varepsilon<\underline{R}(f,\pi_1)+\underline{R}(g,\pi_2)<\underline{R}(f+g,\pi)$ by (1)

And $\overline{R}(f+g,\pi)\leq\overline{\mathrm{R}}(f,\pi_1)+\overline{\mathrm{R}}(g,\pi_2)\leq\int_{a}^{-b} f+\int_{a}^{-b} g+2\varepsilon$ by (2)

This is true for all $\varepsilon>0\Rightarrow$ It's true at $\varepsilon=0$

Since (by def. of $\overline{\int}\,,\underline{\int}$):

$$\underline{R}(f+g,\pi)\leq\int_{-a}^{b} f+g\ \ ,\int_{a}^{-b} f+g\leq\overline{R}(f+g,\pi)$$

Then we have

$$\int_{-a}^{b}\boldsymbol{f}+\int_{-a}^{b}\boldsymbol{g}\leq\underline{R}(f+g,\pi)\leq\int_{-a}^{b} f+g\leq\int_{a}^{-b} f+g\leq\overline{R}(f+g,\pi)$$
$$\leq\int_{a}^{-b} f+\int_{a}^{-b} g=\int_{-a}^{b}\boldsymbol{f}+\int_{-a}^{b}\boldsymbol{g}$$

$\underline{\int} f+g=\overline{\int} f+g\Rightarrow f+g$ is R-I.

Remark: **The integral $\int$ is a linear transformation**

Let $\boldsymbol{X}=\boldsymbol{RI}[\boldsymbol{a},\boldsymbol{b}]$ =the set of all Riemann integral functions on$[a,b]$. Then, by $2-(i$ and $ii)$, we get the transformation

$\mathcal{T}:\boldsymbol{X}\ \rightarrow R$ such that $\mathcal{T}(f)=\int_a^b f\,(x)dx$, is a linear.

2- If $\boldsymbol{f}$ is the Riemann integral on $[\boldsymbol{a},\boldsymbol{b}],\boldsymbol{f}(\boldsymbol{x})\geq\mathbf{0},\forall\boldsymbol{x}$ then $\int\boldsymbol{f}\geq\mathbf{0}$

Proof: Since $f(x)\geq 0,\forall x\in[a,b]\Rightarrow\underline{R}(f,\pi)\geq 0\ and\ \overline{R}(f,\pi)\geq 0$

$\underline{\int} f \geq 0 \; and \; \overline{\int} f \geq 0$ ⇒since f is R-I ⇒$\underline{\int} f = \overline{\int} f = \int f \geq 0$.

3- If $f_1$and f_2 are R-I on $[a,b]$ and if $f_1 \geq f_2$then $\int f_1 \geq \int f_2$

Proof: let $g = f_1 - f_2$

$\because f_1(x) \geq f_2(x) \Rightarrow f_1(x) - f_2(x) \geq 0, \forall x \in [a,b]$

$\therefore g(x) \geq 0, \forall x \in [a,b]$

By (2)$\int g\,(x) \geq 0$

$$\Rightarrow \int_a^b g(x) = \int_a^b f_1 - f_2 \geq 0 \Rightarrow \int_a^b f_1 + (-f_2) \geq 0$$

$$\Rightarrow \int_a^b f_1 - \int -f_2 \geq 0, by\ (1), i\ , ii \Rightarrow \int f_1 \geq f_2$$

4- If f is R-I, then $|f|$ is R-I and $|\int f| \leq \int |f|$ (H.W)

5- If f is R-I, then f^2 is R-I (H.W)

5'- If f and g are R-I then the product fg is R-I (H.W)

6- $\int : RI[a,b] \rightarrow R$ is not a (one-to-one) function, we must prove that for some f, g and $\int f = \int g$

Proof (6): let $f, g \in RI[-1,1] \Rightarrow \begin{cases} f: [-1,1] \rightarrow R \\ g: [-1,1] \rightarrow R \end{cases}$ we define f, g as follows

$$f(x) = x\ , \forall x\ \ , g(x) = \begin{cases} 0 & , if\ x \neq 1 \\ 5 & , if\ x = 1 \end{cases}$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = 0 \quad and \quad \int_{-1}^{1} g(x) = 0 \qquad but\ f \neq g$$

7- If f cont. on $[a,b], f(x) \geq 0, \forall x$ and $\int f(x) = 0$ then $f = 0$

Proof: Suppose that $f \neq 0 \Rightarrow \exists x \in [a,b] \ni f(x) \neq 0 \Rightarrow y = f(x) > 0$

Let $\left(\frac{y}{2}, \frac{3y}{2}\right)$ be an open interval contains y

Since f is cont.⇒ $\exists V$ open interval in $R \ni x \in V$ and $f(V \cap [a,b]) \subseteq \left(\frac{y}{2}, \frac{3y}{2}\right)$

$\Rightarrow V \cap [a,b] \subseteq f^{-1}\left(\frac{y}{2}, \frac{3y}{2}\right)$

$\Rightarrow \exists I$ closed interval $\ni x \in I \subseteq [a,b]$ and $f(x) > 0, \forall x$ since I bounded and closed$\Rightarrow$ I compact

And since f cont $\Rightarrow$ f has min. value in I say m

$\Rightarrow m = \min\{f(x): x \in I\} \Rightarrow m > 0 \Rightarrow f(x) \geq 0$

$\Rightarrow \int_a^b \geq \int_I f \geq m|I| > a \Rightarrow C!$ since $\int f = 0$

8- If f, g are R-I on $[a,b]$ then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, a \leq c \leq b$ (H.W).

Theorem (5): If f is R-I on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ then F is cont. on $[a,b]$

Proof: Since f is R-I$\Rightarrow$ f is bounded on $[a,b]$

$\Rightarrow \exists M > 0 \ni |f(t)| \leq M, \forall t \in [a,b]$

Let $c \in [a,b]$ and for any $x \in [a,b]$

$$|F(x) - F(c)| = \left|\int_a^x f(t)dt - \int_a^c f(t)dt\right|$$

$= \left|\int_c^x f(t)dt\right| \leq \int_c^x |f(t)|\,dt \leq \int_0^x M dt \leq M|x-c|$

Now, for given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$ $\Rightarrow$if $|x-c| < \delta$

Then $|F(x) - F(c)| \leq M|x-c| < M \cdot \delta < \varepsilon$

$\Rightarrow F$ is cont. at $c, \forall c \in [a,b] \Rightarrow F$ is cont. on $[a,b]$.

Theorem (6) (Fundamental Theorem of Calculus)

If f is R-I on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ and f is conts. on $[a,b]$ then F is diff. on (a,b) and $F' = f$

Proof: Let $c \in [a,b]$ for any $x \in [a,b]$

To prove F is diff. at c

$$\left|\frac{F(x)-F(c)}{x-c}-f(c)\right| = \left|\frac{\int_a^x f(t)\,dt - \int_a^c f(t)dt}{x-c}-f(c)\right|$$

$$= \left|\frac{\int_c^x f(t)dt}{x-c}-f(c)\right| = \left|\frac{\int_a^x f(t)dt}{x-c}-f(c)\frac{x-c}{x-c}\right|$$

$$= \left|\frac{\int_c^x f(t)dt}{x-c}-\frac{\int_c^x f(c)}{x-c}\right| = \left|\frac{\int_c^x \big(f(t)-f(c)\big)\,dt}{x-c}\right| \leq \frac{\int_c^x |f(t)-f(c)|dt}{|x-c|}$$

Then f is continuous $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni |x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$

Thus $\left|\frac{F(x)-F(c)}{x-c}-f(c)\right| \leq \frac{\int_c^x |f(t)-f(c)|dt}{|x-c|} \leq \left|\frac{\int_c^x \varepsilon dt}{x-c}\right| = \frac{\varepsilon|x-c|}{|x-c|} = \varepsilon.$

So, F is differentiable and $F' = f$.

Exercise Let f be R-I on $[a,b]$, if $F(x) = \int_a^x f(t)\,dt$ and F is diff. on

$(a,b) \ni F' = f$. Show that $\int_a^b f(t)dt = F(b)-F(a)$.

Theorem (7) (The integral Mean Value Theorem)

Let f *and* g be cont. on $[a,b]$ with $g(x) \geq 0$ *for* $x \in [a,b]$ then there is $c, c \in (a,b)$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)\,dx$

Proof: Since f is cont. on $[a,b] \Rightarrow f$ is bounded

$\Rightarrow \exists m, M > 0$ *such that* $m \leq f(x) \leq M, \forall x \in [a,b]$

since $g(x) \geq 0 \Rightarrow mg(x) \leq f(x)g(x) \leq \mathrm{M}g(x)\,, \forall x$

$$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

$-$If $\int_a^b g(x)\,dx = 0 \Rightarrow \int_a^b f(x)g(x) = 0 = f(c)\int_a^b g(x)\,dx$

$-$If $\int_a^b f(x)g(x) \neq 0 \Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$

$$\Rightarrow m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

Apply the intermediate value property on f and $[a,b]$ and $k = \frac{\int_a^b fg}{\int g}$

$\Rightarrow \exists c \in (a,b) \ni f(c) = k$

$\Rightarrow f(c) = \frac{\int_a^b fgdx}{\int gdx} \Rightarrow \int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$

Corollary: If f is cont. on $[a,b]$ then $\exists c \in [a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$

Proof: In Theorem (7) take $g(x) = 1$, I.M.V.T.