THE METRIC SPACES

Chapter Two

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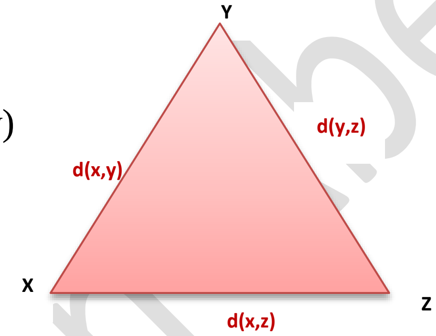
The Metric Spaces

Definition: Let X be a non-empty set and $d: X \times X \longrightarrow \mathbb{R}^+$ be a mapping. We say that, the ordered pair (X,d) is **metric space** if it is satisfying for all x, y and z in X:

(1) $d(x,y) \geq 0$
(2) $d(x,y) = d(y,x)$
(3) $d(x,y) \leq d(x,z) + d(z,y)$ (triangular inequality)
(4) $d(x,y) = 0 \Leftrightarrow x = y$



Remarks:-

(1) d is called **metric mapping**.
(2) d(x,y) = distance between x and y.

Definition:-

A mapping $d: X \times X \longrightarrow \mathbb{R}^+$ is called **a pseudo metric** on X iff d satisfies the conditions (1-3) in the above definition and **(4')** $d(x,y) = 0$, for different $x, y \in X$.

Remark:

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two n-triple of complex numbers then:

(1) Cauchy-Shwarz inequality

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}}$$

(2) Minkowskis inequality

$$\left(\sum_{i=1}^{n} |a_i + b_i|^{p}\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}, \; p \geq 1.$$

Examples:

(1) If $X = \mathbb{R}$ with $d(x,y) = |x - y|$, prove that (X,d) is metric space?

Proof: Let x, y and $z \in \mathbb{R}$

(1) $d(x,y) = |x - y| \geq 0$ (by absolute value)

(2) $d(x,y) = |x-y| = |-(y-x)| = |y-x| = d(y,x) \Rightarrow d(x,y) = d(y,x)$

(3) $d(x,y) = |x-y| = |x-z+z-y|$

$\leq |x-z| + |z-y|$

$= d(x,z) + d(z,y)$

$\therefore\ d(x,y) \leq d(x,z) + d(z,y)$

(4) $d(x,y) = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x-y = 0 \Leftrightarrow x = y$

$\therefore\ d(x,y) = 0 \Leftrightarrow x = y$

By (1), (2), (3) and (4) $\Rightarrow$ (X,d) is metric space.

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(2) Describe metric space:

Let $X \neq \phi$ and $d: X \times X \longrightarrow \mathbb{R} \ni d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ for all x, y in X.

Show that (X,d) metric space?

Proof:

(1) $d(x,y) = |x-y| \geq 0$ for all $x, y \in X$ (by def. of d)

(2) $d(x,y) = d(y,x)$?

$$\left.\begin{array}{l} \text{If } x = y \Rightarrow d(x,y) = 0 = d(y,x) \\ \text{If } x \neq y \Rightarrow d(x,y) = 1 = d(y,x) \end{array}\right\} \Rightarrow d(x,y) = d(y,x)$$

(3) Let x, y and $z \in X$. to prove $d(x,y) \leq d(x,z) + d(z,y)$

If $x = y \Rightarrow d(x,y) = 0$. Since $d(x,z) \geq 0$ and $d(z,y) \geq 0 \Rightarrow d(x,y) \leq d(x,z)+d(z,y)$

If $x \neq y \Rightarrow d(x,y) = 1$ and either $x \neq y \neq z$ or $x \neq y$, $y = z$

either $d(x,y) = d(x,z) = d(z,y) = 1$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow 1 \leq 1 + 1$

or $d(x,y) = d(x,z) = 1$ and $d(z,y) = 0$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow 1 \leq 1 + 0$

Then condition (3) holds $\forall x, y, z \in X$

(4) $d(x,y) = 0 \Leftrightarrow x = y$

then (X,d) is metric space

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(3) If $X = \mathbb{R}^2$, $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$,where $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $z = (z_1,z_2)$. Prove that (X,d) is metric space?

Proof: Let x, y and z $\in \mathbb{R}$

(1) $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \geq 0$ (by root function)

(2) $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

$=\sqrt{(-(y_1-x_1))^2+(-(y_2-x_2))^2}$

$=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$

$=d(y,x)$

$\Rightarrow$ d(x,y) = d(y,x)

(3) $d(x,z)=\sqrt{(x_1-z_1)^2+(x_2-z_2)^2}$

$=\sqrt{(x_1-y_1+y_1-z_1)^2+(x_2-y_2+y_2-z_2)^2}$

$\leq\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}+\sqrt{(y_1-z_1)^2+(y_2-z_2)^2}$) (by Minkowski's inequality)

= d(x,z) + d(z,y)

(4) d(x,y) = 0 $\Leftrightarrow \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}=0$

$\Leftrightarrow (x_1-y_1)^2+(x_2-y_2)^2=0$

$\Leftrightarrow (x_1-y_1)^2=0 \ \& \ (x_2-y_2)^2=0$

$\Leftrightarrow x_1-y_1=0 \ \& \ x_2-y_2=0$

$\Leftrightarrow x_1=y_1 \ \& \ x_2=y_2$

$\Leftrightarrow x=(x_1,x_2)=(y_1,y_2)=y$

$\therefore$ d(x,y) = 0 $\Leftrightarrow$ x = y

By (1), (2), (3) and (4) $\Rightarrow$ (X,d) is metric space.

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(4) Pseudo metric not metric space:

Let $d:X \times X \longrightarrow \mathbb{R} \ni d(x,y) = |x^2 - y^2| \ \forall \ x, y$ in $\mathbb{R}$. Show that $(\mathbb{R},d)$ pseudo metric space but not metric?

Proof: Let x, y and $z \in \mathbb{R}$

(1) $d(x,y) = |x^2 - y^2| \geq 0$ (by def. of absolute value القيمة المطلقة)
(2) $d(x,y) = |x^2 - y^2| = |y^2 - x^2| = d(y,x) \Rightarrow d(x,y) = d(y,x)$
(3) $d(x,y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2|$
$\leq |x^2 - z^2| + |z^2 - y^2|$

$= d(x,z) + d(z,y)$

(4) $d(x,y) = |x^2 - y^2| = 0 \ \forall \ x \in \mathbb{R}$

$\therefore$ $(\mathbb{R},d)$ pseudo metric space but not metric since:-

If $d(x,y) = 0 \Rightarrow |x^2 - y^2| = 0 \Rightarrow x^2 = y^2 \not\Rightarrow x = y \ \forall \ x, y.$

" إذا كانت $x^2 = y^2$ فهناك احتمالين الاول قد $x = y$ والاخر قد يكون $x \neq y$"

i.e. when $d(x,y) = 0$ does not always implies $x = y$ for example:-
$d(1,-1) = |1^2 - (-1)^2| = 0$ but $1 \neq -1$.

$(\mathbb{R},d)$ not metric

Exercises:- Show the following are metric space?

(1) $d:\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{R}$, $d(z,w) = \sqrt{(x-u)^2 + (y-v)^2}$ (usual distance), where $z = x + iy$, $w = u + iv$.

(2) d: $l_2 \longrightarrow l_2 \ni d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$,
$x = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_1, y_2, \ldots, y_n, \ldots)$, $x_i, y_i \in R$,
$i = 1,2, \ldots, n, \ldots$.

(3) $d:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$; $d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$ where $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3)$.

(4) $d:\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{R}$; $d(z,w) = \max\{|x - u|, |y - v|\}$.

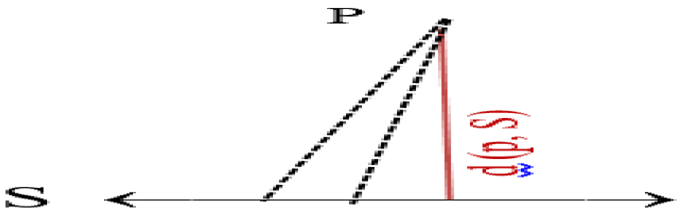
(5) If $X \neq \phi$ and $\forall$ x, y, z in X, $\exists$ $d:X\times X \longrightarrow \mathbb{R} \ni d(x,y) = 0 \Leftrightarrow x = y$ and $d(x,y) \leq d(x,z) + d(z,y)$. Show that (X,d) metric space.

(6) If (X,d) metric space, $\forall$ x, y, z $\in$ X. Show that $|d(x,z) - d(y,z)| \leq d(x,y)$.

(Needed in chapter 5 to prove d is continuous function)

Definition:

Let (X,d) be a metric space, S,T be two subset of X and $p \in X$, then



(1) The **distance between p and S** is
$d(p,S) = \inf\{d(p,x), x \in S\}$

(2) The **distance between S and T** is $d(S,T) = \inf\{d(x,y), x \in S, y \in T\}$.

(3) **Diameter of S**, $\delta(S) = \sup\{d(x,y): x,y \in S\}$. Note that, $\delta(\phi) = -\infty$.

(4) S is called **bounded** if $\exists\ \mu > 0$ such that $d(x,y) \leq M, \forall x, y \in S$.
i.e. $\delta(s) \leq M$. If S is not bounded then it is called **unbounded**.

(5) The distance between two subsets *S* and *T* of metric space X is

$$d(S,T) = inf\{d(a,b), a \in S \text{ and } b \in T\}$$

But this d is pseudo metric not metric. If we take

$S = (0,1)$ and $T = (-1,0)$ then $d(S,T) = 0$ even though $S \cap T = \emptyset$.

Properties: Let (X,d) be a metric space, $S, T \subseteq X$, show that:

(1) $\delta(S) = 0 \Leftrightarrow$ S contains at the most one point.

(2) $S \subseteq T \Rightarrow \delta(S) \leq \delta(T)$.

(3) If $S \cap T \neq \phi \Rightarrow \delta(S \cup T) \leq \delta(S) + \delta(T)$.

Without proof (1-3)

(4) $|d(x,S) - d(y,S)| \leq d(x,y)$ (Homework)

(5) **Which is bounded, find δ(S):-**

(1) S = {-1,-2, 3, 5} **(2)** $S = \left\{3 - \frac{1}{n} : n \in N\right\}$ **(3)** $S = \{x \in \mathbb{R}: x \text{ is odd}\}$

(4) $S = \{(x,y): 1 \le x \le 2, 0 \le y \le 3\}$ **(5)** $S = \{3n : n \in N\}$

(6) $S = \{(x,y,z): (x + 1)^2 + (y + 2)^2 + z^2 = 10\}$ **(7)** $S = \{(x,y): x < 0\}$.

Definition: Let (X,d) be a metric space, $S \subseteq X$, S is called a metric subspace of X if (S, d) satisfies the conditions (1 – 4) in the definition of metric space.

Examples:

(1) Q is a subspace of $\mathbb{R}$ with the usual distance.

(2) $S = \{(x, y): x \ge 0\}$ subspace of $\mathbb{R}^2$ with usual distance.

Definition:

Let (X,d) be a metric space and $x \in X$, $r > 0$, then

(1) The set $B(x,r) = \{y \in X: d(y,x) < r\}$ is called **ball** with center x and radius r.

(2) The set $D(x,r) = \{y \in X: d(y,x) \le r\}$ is called **disk** with center x and radius r.

Examples: Describe the following set:

(1) In $\mathbb{R}$ with usual distance (i.e. $d(x,y) = |x - y|$, find $B(x,r)$ & $D(x,r)$.

Solution: $B(x,r) = \{y \in X: d(y,x) < r\}$

$= \{y \in \mathbb{R}: |y - r| < r\} = \{y \in \mathbb{R}: - r < y < r\}$

$x= \{y \in \mathbb{R}: x - r < y < x + r\} = (x - r,x + r)$ open interval.

$D(x,r) = \{y \in \mathbb{R}:d(y,x) \le r\} = [x - r,x + r]$ closed interval.

(2) In $\mathbb{R}$ with usual distance (i.e. $d(x,y) = |x - y|$, find $B(- 2,4)$ & $D(0,10)$.

Solution: $B(- 2,4) = \{y \in \mathbb{R}: d(y,x) < r\}$

$= \{y \in \mathbb{R}: d(y,- 2) < 4\}$

$= \{y \in \mathbb{R}: |y + 2| < 4\} = \{y \in \mathbb{R}: -4 < y + 2 < 4\}$

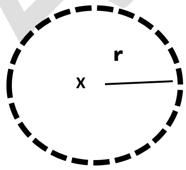
$= \{y \in \mathbb{R}: -4 - 2 < y < 4 - 2\} = \{y \in \mathbb{R}: -6 < y < 2\}$

$= (-6,2)$

$D(0,10) = \{y \in \mathbb{R}: |y - 0| < 10\} = \{y \in \mathbb{R}: -10 \le y \le 10\} = [-10,10]$.

(3) In $\mathbb{R}^2$ with usual distance, find B(x,r), D(x,r).

Solution: $B(x,r) = \{y \in \mathbb{R}^2: d(y,x) < r\}$



$$= \left\{(y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r\right\}$$

$$= \left\{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\right\}$$

= inside the circle with center $x=(x_1,x_2)$ and radius =r

$$D(x,r) = \left\{(y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \le r\right\}$$

$$= \left\{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 \le r^2\right\}$$

= on and inside the circle with center x and radius r

Exercise:- (Home Work) In $\mathbb{R}^2$ with usual distance, find

$B(x,3)$; $x = (-2, -1)$ and $D(0,4)$, $O = (0,0)$.

Examples:

(1) In $\mathbb{R}^2$, $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, find $B(0,1)$, $O = (0,0)$.

Solution: $B(0,1) = \{y \in \mathbb{R}^2: d(y,0) < 1\}$

$= \{(y_1,y_2) \in \mathbb{R}^2: |y_1 - 0| + |y_2 - 0| < 1\}$

$= \{(y_1,y_2) \in \mathbb{R}^2: |y_1| + |y_2| < 1\}$

But $|y_1| + |y_2| = 1$ this implies to the following four equations:

$\Rightarrow\ y_1 + y_2 = 1$

$y_1 - y_2 = 1$

$-y_1 + y_2 = 1$

$-y_1 - y_2 = 1$

(a) $y_1 + y_2 = 1$

If $y_1 = 0 \Rightarrow y_2 = 1 \Rightarrow (0,1)$

If $y_2 = 0 \Rightarrow y_1 = 1 \Rightarrow (1, 0)$

(b) $y_1 - y_2 = 1$

If $y_1 = 0 \Rightarrow y_2 = -1 \Rightarrow (0,-1)$

If $y_2 = 0 \Rightarrow y_1 = 1 \Rightarrow (1,0)$

(c) $-y_1 + y_2 = 1$

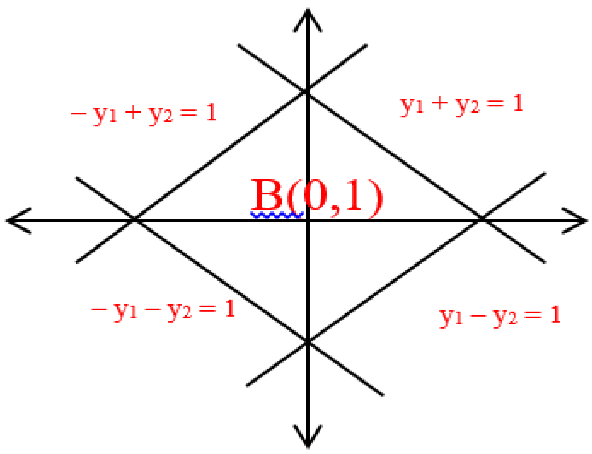
If $y_1 = 0 \Rightarrow y_2 = 1 \Rightarrow (0,1)$

If $y_2 = 0 \Rightarrow y_1 = -1 \Rightarrow (-1,0)$

(d) $-y_1 - y_2 = 1$

If $y_1 = 0 \Rightarrow y_2 = -1 \Rightarrow (0,-1)$

If $y_2 = 0 \Rightarrow y_1 = -1 \Rightarrow (-1,0)$



(2) In $\mathbb{R}^2$, $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$, find B(0,1), B(0,2) and D(0,1).

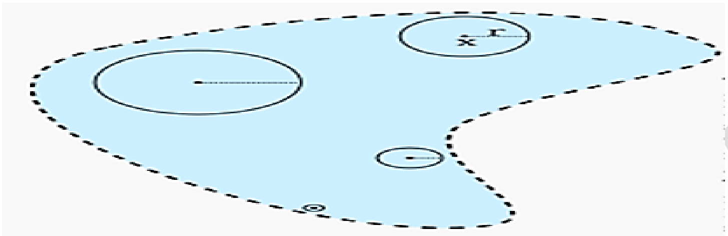
Solution: $B(0,1) = \{y \in \mathbb{R}^2 : d(y,x) < 1\} = \{y \in \mathbb{R}^2 : d(y,x) = 0\} = \{(0,0)\}$.

$B(0,2) = \{y \in \mathbb{R}^2 : d(y,x) < 2\} = \mathbb{R}^2$.

$D(0,1) = \{y \in \mathbb{R}^2 : d(y,x) \leq 1\} = \mathbb{R}^2$.

Open Sets and Closed Sets

Definition: Let (X,d) be a metric space and $S \subseteq X$, S is called **open set** if $\forall$ $x \in S$, $\exists\, r > 0 \ni B(x,r) \subset S$.



Examples:

(1) $S = \phi$ open set.

Solution: Since if $x \in S \Rightarrow \exists\, r > 0 \ni B(x,r) \subset S$

$$F \Rightarrow F \quad \text{or} \quad T$$

$$T$$

(2) $S = X$ open set.

Solution: Since all balls contains in X.

(3) Any open interval is open set.

Proof: Let $x \in S \Rightarrow x \in (a,b) \subseteq (a,b) = S$ $\qquad r = \min\{|x-b|,|x-a|\}$

$\Rightarrow$ S is open set $\qquad (x-r,x+r) \subset (a,b)$

Note that an open set in R is not necessarily an open interval.

Example: Let $S = (-2,-1) \cup (1,2)$

Let $x \in S \Rightarrow x \in (-2,-1)$ or $x \in (1,2) \Rightarrow x \in (-2,-1) \subset S$ or $x \in (1,2) \subset S$

$\Rightarrow$ S is open set. But S is not open interval.

(4) In general, any ball is open set.

Proof: Let $B(x,r)$ be any ball to prove $B(x,r)$ open set.

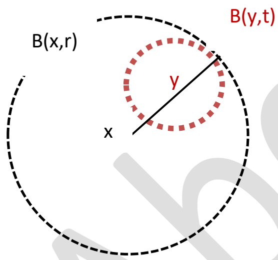
i.e. To prove $\forall\, y \in B(x,r)$, $\exists\, \varepsilon > 0 \ni B(y,t) \subset B(x,r)$

Let $t = r - d(x,y) > 0$ and $z \in B(y,t) \Rightarrow d(z,y) < t$.

By triangular inequality

$d(x,z) \leq d(x,y) + d(y,z)$

$\quad < d(x,y) + t \qquad$ (since $z \in B(y,t) \to d(y,z) < t$)



$\Rightarrow z \in B(x,r) \Rightarrow B(y,t) \subset B(x,r)$

This is true for all y in B(x,r) by def. B(x,r) is open set.

(5) S = {x}, x ∈ ℝ not open set.

Proof: Since there is not open interval in S containing x and contained in S.

i.e., $\forall r > 0, \nexists B(x,r) = (x - r, x + r) \subset S$.

(6) [*a*,*b*], [*a*,*b*), [*a*,∞) and (– ∞,*b*] are not open sets.

Proof: If $S = [a,b]$, then S is not open set, since

if $x = a \Rightarrow \forall r > 0, B(a,r) = (a - r, a + r) \not\subset [a,b]$.

(7) The intersection of any two open sets is open set.

"In general, the intersection of any finite family of open set is open set"

Proof: Let $A = \{S_k : S_k$ open set, $k = 1,2,\ldots,k\}$ to prove $\bigcap_{k=1}^{\infty} S_k$ is open set.

Let $x \in \bigcap_{k=1}^{\infty} S_k \Rightarrow x \in S_k, \forall k$ but S_k is open set $\forall k \Rightarrow \exists r_k > 0 \ni B(x,r_k) \subset S_k$.

Let $r = \min\{r_1, r_2, \ldots, r_n\} \Rightarrow B(x,r) \subset S_k \ \forall k$

$\Rightarrow B(x,r) \subset \bigcap_{k=1}^{\infty} S_k$, so by def. $\Rightarrow \bigcap_{k=1}^{\infty} S_k$ is open set.

If this r is not suitable to make $B(x,r) \subset \bigcap_{k=1}^{\infty} S_k$, The fact that all sets S_k are open is always allowed to have an appropriate radius to make B(x,r) within the intersection.

(8) Is the intersection of two balls also ball? (Homework)

(9)

(10) The infinity intersection of open sets is not necessary open set.

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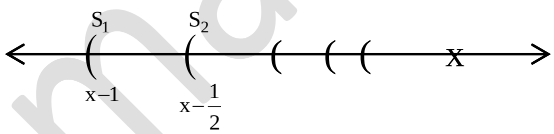
Example: Let $x \in \mathbb{R}$, $S_n = (x - \frac{1}{n}, x + \frac{1}{n})$ open interval, $\forall$ n

$n = 1 \Rightarrow S_1 = (x - 1, x + 1)$

$n = 2 \Rightarrow S_2 = (x - \frac{1}{2}, x + \frac{1}{2})$



$n = 3 \Rightarrow S_3 = (x - \frac{1}{3}, x + \frac{1}{3})$

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When $n \to \infty \Rightarrow \cap_{n=1}^{\infty} S_n = \{x\}$ is not open set.

(11) The union of any family (finite or infinite) (countable or uncountable) of open set is open set

Proof: Let (X, d) be a metric space

A = $\{S_\lambda : S_\lambda$ is open subset of $X, \lambda \in \Lambda\}$, to prove $\bigcup_{\lambda \in \Lambda} S_\lambda$ is open set.

Let $x \in \bigcup_{\lambda \in \Lambda} S_\lambda \Rightarrow \exists\, \lambda \in \Lambda \ni x \in S_\lambda$.

Since S_λ is open set $\Rightarrow \exists\, r_\lambda > 0 \ni B(x,r_\lambda) \subset S_\lambda$

$\Rightarrow x \in B(x,r_\lambda) \subset S_\lambda \subset \bigcup_{\lambda \in \Lambda} S_\lambda$ this is true for all $x \in \bigcup_{\lambda \in \Lambda} S_\lambda$

$\Rightarrow \bigcup_{\lambda \in \Lambda} S_\lambda$ is open set.

(12) Prove that: S is open $\Leftrightarrow$ S = union of all balls. (Homework)

(13) The set of rationals is not open. (Homework)

Definition: Let X be a non-empty set and τ = family of subset of X. If T satisfy the following

(1) $X, \phi \in \tau$.

(2) If $A, B \in \tau \Rightarrow A \cap B \in \tau$.

(3) If $\{A_\lambda : A_\lambda \in \tau\} \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau$.

Then the ordered pair (X,τ) is called **topological space**.

Theorem (1): Every metric space is topological space.

Proof: Let (X,d) be a metric space and τ = the family of all open subsets of X, then:

(1) X, ϕ open sets $\Rightarrow X, \phi \in \tau$.

(2) $S_1, S_2 \in \tau \Rightarrow S_1, S_2$ are open sets $\Rightarrow S_1 \cap S_2$ open set $\Rightarrow S_1 \cap S_2 \in \tau$.

(3) If $x \in S_\lambda, \lambda \in \Lambda \Rightarrow \forall\lambda, S_\lambda$ open subset of X $\Rightarrow \bigcup_{\lambda\in\Lambda} S_\lambda$ open subset of X

$\Rightarrow \bigcup_{\lambda\in\Lambda} S_\lambda \in \tau$.

$\therefore$ (X,τ) is topological space.

Remark: The converse of Theorem (1) is not true. [Go To Stage-4]

Definition:

Let d_1 and d_2 be two metric mappings on the set X. Then d_1, d_2 are called **equivalent** if every open set in (X,d_1) is open in (X,d_2) and vice versa.

Example:

If $X = \mathbb{R}^2$, d_1 = usual distance, $d_2 = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ then d_1, d_2 are equivalent.

Definition: Let (X,d) be a metric space and $S \subseteq X$, S is called closed set if S^c is open set where $S^c = X \backslash S$ (**complement of S**).

Examples:

(1) S = X is closed set.

Proof: Since $S^c = X^c = \phi$ open set.

(2) $S = \phi$ is closed set.

Proof: Since $S^c = \phi^c = X$ open set.

(3) $S = [a,b]$ or $S = [a,\infty)$ or $S = (-\infty,b]$ are closed sets in $\mathbb{R}$.

Proof: If $S = [a,b] \Rightarrow S^c = (-\infty,a) \cup (b,\infty)$ open set $\Rightarrow$ S is closed set.

(4) In $\mathbb{R}$, $S = \{x\}$ is closed set.

Proof: Since $S^c = \underbrace{(-\infty,x)}_{\text{open}} \cup \underbrace{(x,\infty)}_{\text{open}}$

$\Rightarrow S^c$ is open $\Rightarrow$ S is closed set.

(5) In general, any finite set in $\mathbb{R}$ is closed set.

Proof: Let $S = \{x_1,x_2,\ldots,x_n\} \subset \mathbb{R}$, to prove S is closed, i.e. to prove S^c open set.

$$S^c = \underbrace{(-\infty,x)}_{\text{open}} \cup \underbrace{(x_1,x_2)}_{\text{open}} \cup \ldots \cup (x_{n-1},x_n) \quad \underbrace{(x_n,\infty)}_{\text{open}}$$

$\Rightarrow S^c$ is open $\Rightarrow$ S is closed set.

(6) $S = N$ or $S = \mathbb{Z}$ closed set.

Proof: $S = N$

$$S^c = (-\infty,1) \cup (1,2) \cup \bigcup_{n=2}^{\infty} (n,n+1)$$

S^c open $\Rightarrow$ S closed set.

$S = \mathbb{Z}$ (Home Work)

(7) $S = Q$ or $S = Q'$ not closed sets.

Proof: $S = Q \Rightarrow S^c = Q'$ not open. Similarly, when $S = Q'$.

(8) In any metric space X, if S is finite set, then S is closed set.

Proof: Let $S = \{x_1,x_2, \ldots, x_n\} \subset X$ T.p. S is closed set.

i.e. T.p. S^c is open set

Let $y \in S^c \Rightarrow y \neq x_i \;\; \forall\, i \Rightarrow \exists\, r_i = d(y,r_i) > 0, \forall\, i.$

Let $r = \min\{r_1, r_2, \ldots, r_n\} \Rightarrow B(y,r) \cap S = \phi$

$\Rightarrow B(y,r) \subset S^c$ this is true for all $y \in S^c$

$\Rightarrow S^c$ is open $\Rightarrow$ S closed set.

(9) Every disk is closed set. **(Homework)**

(10) The union of finite number of closed sets is closed.

Proof: Let $A = \{S_i, S_i$ closed set in X, $i = 1,2,\ldots,n\}$

T.p. $\overset{n}{\underset{i=1}{\cup}} S_i$ is closed i.e. T.p. $(\overset{n}{\underset{i=1}{\cup}} S_i)^c$ is open set.

Since S_i is closed, $\forall\, i \Rightarrow (S_i)^c$ is open $\forall\, i$

$\Rightarrow \overset{n}{\underset{i=1}{\cap}} (S_i)^c$ open (تقاطع عدد منته من المجموعات المفتوحه يكون مجموعة مفتوحه)

$\Rightarrow (\overset{n}{\underset{i=1}{\cup}} S_i)^c$ open $[(\overset{n}{\underset{i=1}{\cup}} S_i)^c = \overset{n}{\underset{i=1}{\cap}} S_i]$

$\Rightarrow \overset{n}{\underset{i=1}{\cup}} S_i$ is closed

(11) The infinite union of closed sets is not necessary closed.

For Example: $S_n = \left[\frac{-n}{n+1}, \frac{n}{n+1}\right]$, $n \in N$. S_n closed intervals. Is $\overset{\infty}{\underset{n=1}{\cup}} S_n$ closed?

If $n = 1 \Rightarrow S_1 = \left[\frac{-1}{2}, \frac{1}{2}\right]$,

if $n = 2 \Rightarrow S_2 = \left[\frac{-2}{3}, \frac{2}{3}\right]$,

if $n = 3 \Rightarrow S_3 = \left[\frac{-3}{4}, \frac{3}{4}\right]$, ...

When $n \to \infty \Rightarrow \lim\limits_{n\to\infty} \dfrac{\pm n}{n+1} = \lim\limits_{n\to\infty} \dfrac{\pm \frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \mp 1 \Rightarrow \bigcup\limits_{n=1}^{\infty} S_n = (-1,1)$ open set.

(12) The infinite intersection of closed sets is closed. (Homework)

Accumulation Points

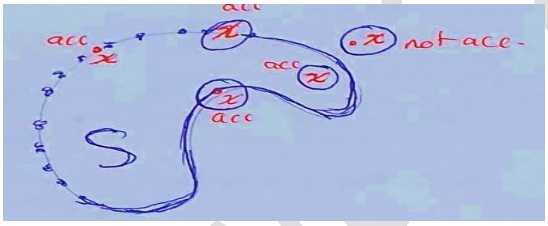
Definition: Let X be a metric space and $S \subseteq X$, $p \in X$, p is called an **accumulation point of S** if every open set contain p, contains another point q $\ni p \neq q.\ q \in S$.

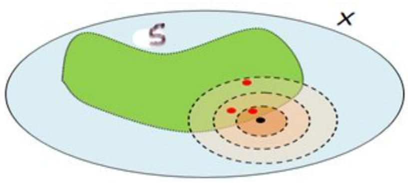
i.e. p is acc. Point of S if $\forall$ U, U open set, $p \in U$ then $U\backslash\{p\} \cap S \neq \phi$.

Remark:

(1) Since every open set = union balls, then we can define acc. Point as following:

p is acc. Point of S if $\forall\, r > 0, B(p,r)\backslash\{p\} \cap S \neq \phi$





(2) S' = the set of all acc. Point of S = derived set.

(3) $\overline{S}$ = the closure of S, $\overline{S} = S \cup S'$.

(4) p is not acc. point, if $\exists$ U, U open set and $p \in U \ni U\backslash\{p\} \cap S = \phi$.

or $\exists\ r > 0 \ni B(p,r)\backslash\{p\} \cap S = \phi$.

Examples:

(1) If S = {1,3}, find S' and $\overline{S}$.

Solution: To find S', there are some cases: x = 1, x = 3, x < 1, x > 3, 1 < x < 3

If x = 1 $\Rightarrow$ x is not acc. point since $\exists\ r > 0 \ni B(x,r)\backslash\{x\} \cap S = \phi$, then r = 1

$\Rightarrow$ $B(1,1)\backslash\{1\}\cap\{1,3\} = (0,2)\backslash\{1\}\cap\{1,3\} = \phi$.

If $x = 3$ $\Rightarrow$ x is not acc. point since $\exists\ r > 0 \ni B(x,r)\backslash\{x\}\cap S = \phi$, then $r = 1$

$\Rightarrow$ $B(3,1)\backslash\{3\}\cap\{1,3\} = (2,4)\backslash\{3\}\cap\{1,3\} = \phi$.

If $x < 1$ $\Rightarrow$ x are not acc. point since $x \in (x-1,1)$ and $(x-1,1) \cap S = \phi$.

If $x > 3$ $\Rightarrow$ x are not acc. points since $x \in (3,x+1)$ and $(3,x+1) \cap S = \phi$.

If $1 < x < 3$ are not acc. point since $x \in (1,3)$ and $(1,3) \cap S = \phi$.

$\therefore$ S has no acc. point $\Rightarrow$ $S' = \phi$.

$\bar{S} = S \cup S' = S \cup \phi = S$.

(2) If $S = \left\{\frac{1}{n}, n = 1, 2, \ldots\right\}$, prove that S' = {0}.

Proof: $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ T.p. $x = 0$ is acc. point.

$\forall\ r > 0,\ 0 \in B(0,r) = (-r,r)$ $\Rightarrow$ $0 \in (-r,r)$ T.p. $(-r,r)\backslash\{0\} \cap S \neq \phi$

$\because$ $r > 0$ $\Rightarrow$ by Arch. Prop. $\exists\ n \in N \ni nr > 1$

$\frac{1}{n} < r \Rightarrow 0 < \frac{1}{n} < r \Rightarrow -r < 0 < \frac{1}{n} < r \Rightarrow \frac{1}{n} \in (-r,r)$

$\Rightarrow$ $(-r,r)\backslash\{0\}\cap S \neq \phi$ $\Rightarrow$ $x = 0$ is acc. point.

Now, T.p. if $x \neq 0$ $\Rightarrow$ x is not acc. point.

Let $x \in S$ $\Rightarrow$ $\exists\ n \in N \ni x = \frac{1}{n}$

$\because$ $n - 1 < n < n + 1$

$\Rightarrow \frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$

$\Rightarrow \frac{1}{n} \in \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$

$$\Rightarrow \left(\frac{1}{n+1},\frac{1}{n-1}\right)\setminus\left\{\frac{1}{n}\right\}\cap S=\phi$$

$\Rightarrow \forall\ n, \frac{1}{n}$ is not acc. point

Let $x \notin S$ $\left\{\begin{array}{l} x, \frac{1}{n+1} < x < \frac{1}{n} \\ x > 1 \\ x < 0 \end{array}\right\}$ are not acc. point.

$\therefore\ S' = \{0\}$.

(3) If S = (*a*,*b*), find S'.

Solution: If $x = a \Rightarrow$ x is acc. point, since $\forall\ r > 0$, $a \in B(a,r) = (a - r, a + r)$ and $B(a,r)\setminus\{a\}\cap S \neq \phi$.

If $x = b \Rightarrow$ x is acc. point since $\forall\ r > 0$, $b \in B(b,r) = (b - r, b + r)$ and $B(b,r)\setminus\{b\}\cap(a,b) \neq \phi$.

If $a < x < b \Rightarrow$ x are acc. points since $\forall\ r > 0$, $x \in B(x,r) = (x - r, x + r)$ and $B(x,r)\setminus\{x\}\cap S \neq \phi$ i.e. $(x - r, x + r)\setminus\{x\} \cap (a,b) \neq \phi$.

If $x < a \Rightarrow$ x are not acc. points since $x \in (x - 1, a)$ and $(x - 1, a) \cap S = \phi$.

If $x > b \Rightarrow$ x are not acc. points since $x \in (b, x + 1)$ and $(b, x + 1) \cap (a,b) = \phi$.

$\therefore\ S' = [a,b] \Rightarrow \overline{S} = S \cup S' = [a,b]$

(4) Prove that S = [*a*,*b*] = $\overline{S}$. **(Homework)**

- **Theorem (2):** Let X be a metric space, $S \subset X$, then:

(1) S is closed set $\Leftrightarrow$ $S' \subset S$.

(2) $\overline{S}$ is closed set.

(3) $\overline{S} = S \Leftrightarrow$ S closed set.

(4) If F is closed set, $S \subset F$, then $\overline{S} \subset F$.

(5) $\overline{S}$ is smallest closed set contains S.

Proof: (1) $\Rightarrow$ let S is closed set and let x is accumulation of S $\Rightarrow x \in S'$. To prove that $x \in S$.

$Suppose\ that\ x \notin S \Rightarrow x \in X \backslash S = S^c$. Since S is closed $\Rightarrow S^c$ is open and $x \in S^c \Rightarrow \exists r > 0 \ni B(x,r) \subset S^c \Rightarrow S \cap B(x,r) \neq \emptyset \Rightarrow C!$ (since x is accumulation point of S). So, $x \in S$.

Conversely,$\Leftarrow$. Suppose that $S' \subset S$ to prove S is closed. We must prove that $S^c$is open set.

For $x \in S^c \Rightarrow \exists r > 0 \ni B(x,r) \subset S^c$ since $S' \subset S \Rightarrow x \notin S' \Rightarrow \exists \varepsilon > 0$ such that $B(x,\varepsilon) \backslash \{x\} \cap S = \emptyset$ or since $x \notin S \Rightarrow B(x,\varepsilon) \cap S = \emptyset \Rightarrow B(x,\varepsilon) \subset S^c$. Then S^c is open set. So, S is closed.

Parts 2-5 exercise.

Separable spaces

Definition: A subset S of a metric space X is called **dense** if $\overline{S} = X$.

Example: Prove that $\bar{Q}$=R (i.e. Q dense set in R).

Proof: for any $p \in R \Rightarrow \forall \varepsilon > 0$, the ball $B(p,\varepsilon) = (p-\varepsilon, p+\varepsilon)$ contains infinitly rationals $\Rightarrow B(p,\varepsilon) \backslash \{p\} \cap Q \neq \emptyset \Rightarrow p$ is acumuldtion. $\Rightarrow p \in \bar{Q} \Rightarrow R = \bar{Q}$.

Examples

1- Let K={$(x_1,x_2) \in R$: $x_2 < 0$ }. Find $\overline{K}$.
2- Let X be a metric space and G be a finite subset of X, prove that G'=ϕ.
3- Let S be a subset of a metric space X and$x \in X$. Show that $d(x,S) = 0 \Leftrightarrow x \in \bar{S}$.

Definition: a metric space X is called **separable** if there is a dense countable sub set S of X.

Examples

1- R is separable metric space. Since Q is countable dense subset of R.

2- Let S= [a,b] with usual metric of on R, then S is separable metric space.
3- Is R^2 separable??
4- Show that $\mathbb{C}$ is separable space. . . Hint: take $S=\{w=p+iq: p, q\in Q\}$.

Note that: Postpone the example of the non-separable space. [Go to MSc].

Compact Spaces

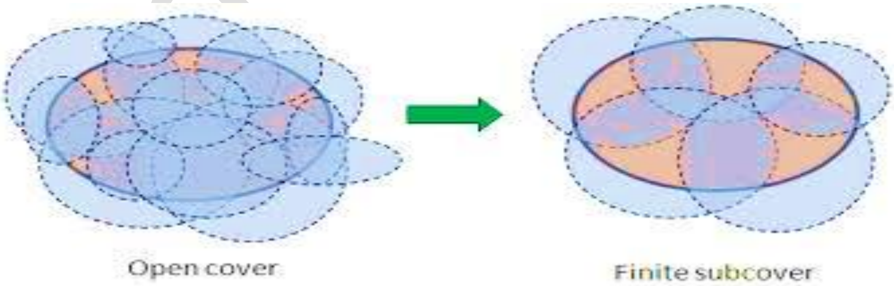
Definition: Let (X,d) be a metric space, $\phi \neq S \subseteq X$, if the set $\{U_\lambda, U_\lambda$ open set, $\lambda \in \Lambda\}$ is a family of open subsets of X such that $S \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$ then the family $\{U_\lambda\}$ is called **open cover for S** in X.

* If the family $\{U_\lambda\}$ is finite and $S \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$, then $\{U_\lambda\}$ is called **finite cover**.

* Let $\{U_\lambda\}$, $\{U_\beta\}$ be two open covers for S and $U_\lambda \in \{U_\beta\}$ $\forall\, \lambda$ then $\{U_\lambda\}$ is called **subcover** for $\{U_\beta\}$.

Definition:

Let S be a subset of a metric space (X,d), S is called **compact set** if every open cover for S in X has a finite subcover.



Examples:

(1) Every finite subset S of a metric space (X,d) is compact set.
Proof: Let $S = \{x_1,x_2,\ldots,x_n\}$ to prove that S is compact set

Let $A= \{ V_\lambda : V_\lambda \subset X, V_\lambda$ is open$\}$ be an open cover for S $\Rightarrow S\subseteq \bigcup V_\lambda$

$\Rightarrow \forall x_i \in S$ we get $x_i \in \bigcup V_i \Rightarrow \exists V_{\lambda i} \in A$ such that $x_i \in V_{\lambda i}$. Then $S\subseteq \bigcup_{i=1}^{n} V_{\lambda i}$

$\Rightarrow$ $\{ V_{\lambda i} : i=1,2,..,n\}$ is finite open subcover for S from A $\Rightarrow$ S is compact.

(2) $\mathbb{R}$ is not compact since there is an open cover for $\mathbb{R}$ which has no finite subcover for $\mathbb{R}$, for example:

$A = \{(n,n+1): n \in \mathbb{Z}\} \cup \{(n - \frac{1}{2}, n + \frac{1}{2}): n \in \mathbb{Z}\}$

$\Rightarrow \mathbb{R} \subseteq [\bigcup_{n\in\mathbb{Z}} (n,n+1)] \cup [\bigcup_{n\in\mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2})]$

$\Rightarrow$ A is open cover for $\mathbb{R}$ and A has no finite subcover since if we cancel any open interval from A, then some reals will be out.

i.e. if we cancel $(-\frac{1}{2}, \frac{1}{2})$ then 0 will be out A.

i.e. $\mathbb{R} \nsubseteq [\cup (n,n+1)] \cup [\cup(n - \frac{1}{2}, n + \frac{1}{2})] \setminus (-\frac{1}{2}, \frac{1}{2})$.

(3) Any open interval $S = (a,b)$ is not compact.

Proof: We prove for a special case when $S = (0,1)$.

Let $A = \{A_n = (\frac{1}{n},2); n \in N\} = \{(1,2), (\frac{1}{2},2), (\frac{1}{3},2),\ldots\}$

$A_1 = (1,2)$, $A_2 = (\frac{1}{2},2)$ … and $A_1 \subset A_2 \subset A_3 \subset \ldots$

To prove that A is open cover for S. i.e. $S \subseteq \bigcup_{n=1}^{\infty} (\frac{1}{n},2)$

Let $r \in S = (0,1) \Rightarrow 0 < r < 1, r > 0 \rightarrow$ by Arch.prop.

$\exists\, k \in N \ni \frac{1}{k} < r \Rightarrow r \in (\frac{1}{k},2) = A_k \subset \bigcup_{n=1}^{\infty} (\frac{1}{n},2) \Rightarrow$ A is open cover for S.

To prove that A has no finite subcover for S

Suppose that A has a finite subcover for S, $\{A_1,A_2,\ldots,A_m\}$

$\Rightarrow S \subseteq \bigcup_{i=1}^{m} A_i = A_m = (\frac{1}{m},2)$ but $\frac{1}{m+1} \in S$ and $\frac{1}{m+1} \in A_m$

$\Rightarrow$ A has no finite subcover for S $\Rightarrow$ S is not compact.

(4) Any closed interval S = [*a*,*b*] is compact. (Exercise)

Theorem (3): (Bolzano-Weierstrass Theorem)

In compact space X, every infinite subset S of X has at least one accumulation point.

Proof: Suppose that S has no acc. point.

$\Rightarrow S' = \phi \Rightarrow S' \subset S \Rightarrow$ S is closed (by Th.(3))

$\Rightarrow X\backslash S = S^c$ is open set.

Since $S' = \phi \Rightarrow \forall x \in S$, x is not acc. point (by def. of acc. point)

$\Rightarrow \forall x \in S, \exists U_x$ open set $\ni x \in U_x$ and $U_x \cap S = \{x\}$

$\Rightarrow X = S^c \cup (\bigcup_{x\in S} U_x) \Rightarrow S^c \cup \{U_x; x \in S\}$ is open cover for X.

But X is compact space $\Rightarrow$ there is a finite subcover for X.

$x_1, x_2, \dots, x_n \in S \ni X = S^c \cup (\bigcup_{i=1}^{n} U_{x_i})$

$\Rightarrow S \cup S^c = S^c \cup (\bigcup_{i=1}^{n} U_{x_i})$ $[X = S \cup S^c]$

$\Rightarrow S \subseteq \bigcup_{i=1}^{n} U_{x_i}$ [since $S \cap S^c = \phi$]

$\Rightarrow S = \{x_1, x_2, \dots, x_n\}$ [since $U_{x_i} \cap S = \{x_i\}\ \forall i$]

$\Rightarrow$ S is finite set C! [since S is infinite set]

$\therefore$ S has at least one acc. point.

Theorem (4): In compact metric space, every closed subset is compact.

Proof: Let X be a compact metric space, and S be subset of X $\Rightarrow S^c$ is open.

T.p. S is compact.

Let $A = \{V_\lambda : V_\lambda$ is open set in X, $\forall \lambda \in \Lambda\}$ be open cover for S $\Rightarrow S \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$

since $X = S \cup S^c \subseteq (\bigcup_{\lambda\in\Lambda} V_\lambda) \cup S^c$, but S^c is open set

$\Rightarrow \bigcup_{\lambda\in\Lambda} V_\lambda \cup S^c$ is open cover for X.

Since X is compact set $\Rightarrow$ there exists a finite number $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $X = S^c \cup (\bigcup_{i=1}^{n} V_{\lambda_i})$ since $S \cap S^c = \phi \Rightarrow S \subseteq \bigcup_{i=1}^{n} V_{\lambda_i} \Rightarrow$ A has a finite subcover $\{V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n}\}$ for S $\Rightarrow$ S is compact.

Theorem (5): Let (X,d) be a metric space, $S \subseteq X$. If S is compact, then S is closed.

Proof: Suppose that S is not closed set

$\Rightarrow S' \not\subset S \quad [S \text{ is closed} \Leftrightarrow S' \subset S]$

$\Rightarrow \exists\, x \in S'$ and $x \notin S$.

$\Rightarrow \forall\, n \in N,\ B(x,\frac{1}{n})\backslash\{x\}\cap S \neq \phi \Rightarrow D(x,\frac{1}{n})\backslash\{x\}\cap S \neq \phi.$

Let $V_n = [D(x,\frac{1}{n})]^c = X \setminus D(x,\frac{1}{n}),\ n \in N$

$\Rightarrow V_n$ is open set $\forall\, n$ [since $D(x,\frac{1}{n})$ is closed set]

Let $D = \bigcap_{n\in N} D(x,\frac{1}{n}) = \{x\}$

Since [if $\exists y \in D$ and $y \neq x \Rightarrow d(x,y) > 0$

by Arch. Prop. $\Rightarrow \exists k \in N \ni k\, d(x,y) > 1 \Rightarrow \frac{1}{k} < d(x,y)$

$\longrightarrow y \notin B(x,\frac{1}{k}) \longrightarrow y \notin D(x,\frac{1}{n}) \longrightarrow D = \{x\}$]

$\therefore\ S \subseteq X\backslash\{x\} = X\backslash D = X\backslash \bigcap_{n\in N} D(x,\frac{1}{n})$

$$= X\backslash\{D(x,1) \cap D(x,\frac{1}{2}) \cap \ldots\}$$

$$= X\backslash D(x,1) \cup X\backslash D(x,\frac{1}{2}) \cup \ldots$$

$$= \bigcup_{n\in N} X \backslash D(x,\frac{1}{n}) = \bigcup_{n\in N} V_n.$$

$\therefore\ S \subseteq \bigcup_{n\in N} V_n \Rightarrow \{V_n : n \in N\}$ is open cover for S.

But S is compact set $\Rightarrow \exists$ finite number $v_1, v_2, \ldots, v_n \ni S \subseteq \bigcup_{i=1}^{n} V_i$

$\Rightarrow S \subseteq \bigcup_{i=1}^{n} (X \backslash D(x,\frac{1}{n})) \Rightarrow S \cap D(x,\frac{1}{n}) = \phi$ C!

$\Rightarrow x \in S \Rightarrow$ S is closed set.

Theorem (6): Let (X,d) be a metric space, $S \subseteq X$. If S is compact, then S is bounded.

Proof: Let $a \in X$, defined $B(a,n)$ open balls, $n \in N$

$\Rightarrow B(a,1) \subseteq B(a,2) \subseteq \ldots$

$\Rightarrow \forall x \in S, \exists n \in N \ni x \in B(a,n) \Rightarrow S \subseteq \bigcup_{n\in N} B(a,n)$

$\Rightarrow \{B(a,n)\}$ open cover for S

But S is compact $\Rightarrow \{B(a,n),\ n \in N\}$ has finite subcover for S $\{B(a,1), B(a,2), \ldots, B(a,n)\} \Rightarrow S \subseteq \bigcup_{k=1}^{n} B(a,k)$

$\because\ B(a,1) \subseteq B(a,2) \subseteq \ldots \subseteq B(a,n) \Rightarrow S \subseteq B(a,n) \Rightarrow$ S is bounded set.

Theorem (7): lf $\{I_n: n\in N\}$ be a family of closed bounded nested intervals such that $I_n=[a_n, b_n]\supset[a_{n+1}, b_{n+1}]=I_{n+1}$ then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof: let $A=\{a_n: n=1, 2, \ldots\}$ and $B=\{b_n: n=1, 2, \ldots\}$.

Since for any $m>n$, $[a_n, b_n]\supset[a_m, b_m] \Rightarrow a_m< b_m< a_n< b_n$, $\forall n, m$.

$\Rightarrow$ for any n.m, we get $a_m< b_n \Rightarrow$ A is bounded above by any element belongs to B $\Rightarrow$ A has sup, say x, (by completeness axiom). $a_n < x$, $\forall n$.

Also, $x < b_n$, $\forall n. \Rightarrow x\in[a_n, b_n]= I_n$, $\forall n \Rightarrow x\in\bigcap_{n=1}^{\infty} I_n$.

Theorem (8): Hien-Borel Theorem

Any closed bounded subset of R^n, $n\geq 1$, is compact set.

Proof: (prove in R)

Let S closed bounded subset of R, to prove S is compact?

Since S is bounded then there is a closed interval [a,b] such that $S\subseteq[a,b]$.

Then (by Theorem 4) S is compact. [any closed subset of compact is compact.

Theorem (9) Hien-Borel Theorem in R^2.

Proof: we need prove that the product of two closed intervals [a,b] X [c,d] is compact set. Complete similar to proof of Theorem (8).