CHAPTER 0

PRELIMINARIES

0-0 Introduction

Throughout this chapter, some basic known concepts and facts are given. Which has been divided in to six sections: basic elements in normed spaces, normalized duality mapping, hausdorff distance, multi-valued mappings, types of iterative schemes and important fixed point theorems.

It is worth mentioning that, section (0.4) includes interesting considerations and examples to explain the relationships between some known contractive conditions.

0-1 Basic Elements in Normed Spaces

A vector space over a field $L = \mathbb{R}$ or $\mathbb{C}$ and wedding with a nonnegative real valued function $\|\,.\,\|$ is called normed linear space (shortly, normed space) if it is satisfied $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x$ is zero vector, $\|\lambda x\| = |\lambda|\|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in X$ and is scalar $\lambda \in L$. The ordered pair $(X, \|\,.\,\|)$ denote to normed linear space. Also, X is called real normed space (or complex normed space) if $L = \mathbb{R}$ field (or, respectively $L = \mathbb{C}$). It is known that every normed space is metric space by defining a function distance $d: X \times X \rightarrow \mathbb{R}^+$ with $d(x, y) = \|x - y\|$, for all x, y in X [16].

The following definition are needed later.

Definition (0.1.1): [16] [27]

In a normed space $(X, \| . \|)$ a sequence $\langle x_n \rangle$ is called *Cauchy sequence* if for every $\varepsilon > 0$, there exists a positive integer k such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > k$ or denoted by $\|x_m - x_n\| \rightarrow 0$ $as\ n, m \rightarrow \infty$.

Definition (0.1.2): [16] [31]

In a normed space X a sequence $\langle x_n \rangle$ in X *converges (or converges strongly)* to $x \in X$ if for every $\varepsilon > 0$, there exists a positive integer k such that $\|x_n - x\| < \varepsilon$ for all $n \geq k$, denoted by $x_n \rightarrow x \ \ as\ n \rightarrow \infty$ or $\lim\limits_{n\rightarrow\infty} x_n = x$. The point x is called the limit of $\langle x_n \rangle$.

Definition (0.1.3): [31] [27]

The normed space X is said to be *complete* if every Cauchy sequence in X converges. However, a complete normed space is called *Banach space*.

Definition (0.1.4) [16]:

A Banach space X is said to satisfy *Opial's condition* if for any sequence $\langle x_n \rangle$ in X, $x_n \rightarrow x$ implies that $\lim\limits_{n\rightarrow\infty} sup\|x_n - x\| < \lim\limits_{n\rightarrow\infty} sup\|x_n - y\|$ for all $y \in X$ with $y \neq x$.

Definition (0.1.5): [16]

Let X be a normed space, a mapping $f: X \to L$ is called *linear functional* if for all x, y in X and α, β in L,

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

Definition (0.1.6): [16] [27]

Let X be a normed space. The set of all continuous linear functional on X together with the norm $\|f\| = sup_{x\in X} |f(x)|$ is called *dual space* of X. and denoted by X^*.

Remark (0.1.7): [1]

Let X be a normed space and $f \in X^*$, for given $x \in X$, the equation $\varphi_x(f) = \langle x, f\rangle$ define a functional on $X^{`}$ and:

i. φ_x is linear

ii. $|\varphi_x(f)| = |\langle x, f\rangle| \leq \|x\|\|f\|$ hence φ_x is bounded.

iii. $|\langle x, f\rangle| = \|x\|\|f\|$ and $\|f\| = \|x\|$ hence $\|f_x\| = \|x\|$.

Definition (0.1.8):[16]

Let X is a normed space and the sequence $\langle x_n\rangle \subset X$ is said to be *weakly convergent* to $x \in X$, written $x_n \overset{w}{\to} x$, if $\forall\, f \in X^{`}$,

$\lim\limits_{n\to\infty} \langle x_n, f\rangle = \langle x, f\rangle, i.e,\ \lim\limits_{n\to\infty} f(x_n) = f(x)$.

Note that every strongly convergence is weakly but the converse is not true [16].

0-2 Normalized Duality Mapping

Recall the following:

Definition (0.2.1): [32].

Let X and Y be two nonempty sets. F is said to be *a multi-valued mapping* defined by

$$F: M \subseteq X \to 2^{Y} \qquad \dots (0.2.1)$$

We mean a mapping assigns to each point $x \in M$ a subset $F(x) \subseteq Y$. Every single valued mapping $F_0: M \to Y$, can be identified with a multi-valued mapping like (0.4.1) by setting

$$F(x) = \{F_0(x)\}, \text{ for all } x \in X.$$

Thus $F(x)$ is a *singleton*.

Definition (0.2.2):[30] [34]

Let X be a normed space and let $X^{`}$ be the dual space of $X$ and $2^{X}$ is the set of all nonempty subsets of $X^{`}$, the multivalued mapping $J: X \to 2^{X^{`}}$ is said to be the *normalized duality* if

$$J(x) = \{f \in X^{`}: \langle x, f\rangle = \|f\|\|x\|, \|f\| = \|x\|\}, \forall\, x \in X$$

The following example for a single value normalized duality mapping which is depend on the nature of the normed linear space l_p

Example (0.2.3): [16] [27]

Consider the normed space $X = l_p(\mathbb{R})$ with norm $\|x\|_p = (\sum_{i=1}^{\infty}|x_i|^{p})^{\frac{1}{p}}$, when $X = (x_1, x_2, \dots.)$. It is dual space $X^{`}$with $\frac{1}{p} + \frac{1}{q} = 1$.

Defined duality mapping $J: X \to X^{`}$ by

$J(x) = (sgn\, x_1|x_1|^{p-1}, sgn\, x_2|x_2|^{p-1}, \dots \dots)$, since

$$\left(\sum_{i=1}^{\infty} |sgn\, x_i |x_i|^{p-1}|^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty} |x_i|^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{q}},$$

$J_x \in X^*$ and $\|J_x\| = \|x\|^{\frac{p}{q}} \Longrightarrow \|J_x\| = \|x\|^{p-1}$,

$$\langle x, J_x \rangle = \sum_{i=1}^{\infty} x_i\, sgn\, x_i |x_i|^{p-1} = \sum_{i=1}^{\infty} |x_i|^p = \|x\|^p = \|x\|\|x\|^{p-1}$$
$$= \|x\|\|J_x\|.$$

Definition (0.2.4): [30] [13]

Let X be a normed space and $U(X) = \{x \in X: \|x\| = 1\}$, then X is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \qquad \dots (0.2.1)$$

exists for each $x, y \in U(X)$. Also X is said to be *uniformly smooth* if the limit (0.2.1) is attained uniformly for (x, y) in $U(X) \times U(X)$.

Definition (0.2.5): [32]

A normed space X is said to be *uniformly convex* if and only if for every $\varepsilon \in (0,2]$ there is a $\delta(\varepsilon) \in (0,1]$ such that

$\|x\| \leq k,\ \|y\| \leq k,\ \|x - y\| \geq \varepsilon k, \quad x, y \in X, k > 0,$

implies that

$$\|(x + y)/2\| \leq \big(1 - \delta(\varepsilon)\big)k.$$

Example (0.2.6): [32]

Consider $X = \mathbb{R}^2$ with Euclidian norm $\|x\| = (x_1^2 + x_2^2)^{\frac{1}{2}}$ where $x = (x_1, x_2)$ is uniformly convex.

But $\mathbb{R}^2$ with maximal norm $\|x\| = \max\{|x_1|, |x_2|\}$ where $x = (x_1, x_2)$ is not uniformly convex on $\mathbb{R}^2$.

Proposition (0.2.7): [13]

A normed space X is uniformly smooth if and only if $X^{`}$ is uniformly convex.

We illustrate proposition (0.2.7) by the following two example

Example (0.2.8): [32]

i- Consider $X = l_2(\mathbb{R})$ with $\|x\| = \sqrt{\sum_{i=1}^{\infty} {x_i}^2}$ is uniformly smooth since the dual space of $l_2(\mathbb{R})$ isomorphic to $l_2(\mathbb{R})$ and $l_2(\mathbb{R})$ is uniformly convex.

ii- $\mathbb{R}^2$ with $\|x\| = |x\ |$ is uniformly smooth since $X^{`}$ of $\mathbb{R}^2$ isomorphic to $\mathbb{R}^2$ and $\mathbb{R}^2$ with same norm is uniformly convex, but $\mathbb{R}^2$ with

$\|x\| = \max\{|x_1|\,, |x_2|\}$, where $x = (x_1, x_2)$ is not uniformly smooth also the dual space of $\mathbb{R}^2$ is not uniformly convex with the same norm.

Proposition (0.2.9): [30] [3]

Let X be a Banach space and let $J: X \to 2^{X^{`}}$ be the normalized duality mapping, then we have the following:

1. For each $x \in X, J(x)$ is a nonempty bounded closed and convex subset of X^*;
2. $J(0) = \{0\}$;
3. Let α be a real numbed and $x \in X$ then $J(\alpha x) = \alpha J(x)$;
4. X is smooth Banach space if and only if J is single valued;
5. X is uniformly smooth Banach space if and only if J is single valued and uniformly continuous on any bounded sub set of X, (where J is uniformly continuous if and only if $\forall \varepsilon > 0, \exists\ \delta > 0$ such that $\forall\ x, y \in X, \|J(x) - J(y)\| < \varepsilon$ if $\|x - y\| < \delta$).

0-2 Hausdorff Distance

Recall the following important classes of subsets of a normed space X:

2^X the family of all nonempty subsets of X.

$CB(X)$ the family of all nonempty closed and bounded subsets of X.

$K(X)$ be the collection of all nonempty compact subsets of X.

Definition (0.3.1): [50]

Let X be a normed space, a subset M of X is called *Proximinal* if $\forall\, x \in X, \exists\, m \in M$ such that

$$\|x-m\| = d(x,M) = \inf\,\{\|x-y\| : y \in M\}.$$

We denote the class of all bounded proximinal subset of M in X by $P(M)$.

Definition (0.3.2): [32]

Let $A, B \in CB(X)$, the *Hausdorff distance* between A and B denoted by $D\,(A,B)$ which is defined by

$$D\,(A,B) = \max\{sup_{a\in A} d(a,B), sup_{b\in B}\, d(b,A)\}$$

Where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from a point a to the set B and d induced by the norm.

Example (0.3.3):

Consider $X = \mathbb{R}$, $A = [2,3]$, $B = [4,6]$ then we find $H\,(A,B)$

$d(A,B) = 2$ and $d(B,A) = 3$

hence $d(A,B) \neq d(B,A)$ and $D\,(A,B) = \max\,\{d(A,B), d(B,A)\} = 3.$

Proposition (0.3.4): [A]

Let A and B be a nonempty sub set of metric space (X,d).

1. If A and B are both bounded then $D\,(A,B) < \infty$.

2. If A is bounded and $D\ (A,B) < \infty$ then B is bounded.
3. If A is bounded and B is unbounded then $D\ (A,B) = \infty$.
4. If A is unbounded and $D\ (A,B) < \infty$ then B is unbounded.

In the converse of proposition (i) above It is possible that $D\ (A,B) < \infty$ if both A and B are unbounded, for example of this, if A and B are parallel lines in $\mathbb{R}^2$. [A]

Theorem (0.3.5): [B]

If (X,d) be a metric space then $(K(X),D)$ is metric space, morovere, if X is complete then $K(X)$ is complete.

Proposition (0.3.6): [A]

If (X,d) be a metric space, and let A and B be a nonempty subset of X

1. If B is compact, then for each $a \in A$ there exists some $b \in B$ satisfying $d(a,b) \leq D(A,B)$.
2. If A and B are both compact, then there exists $a \in A$ and $b \in B$ such that $d(a,b) = D(A,B)$.

If $D\ (A,B) = d(a,b)$ where $a \in A$ and $b \in B$ it is not expected true if A and B are not closed.

In following example we show that A and B are closed and not compact but $D\ (A,B) \neq d(a,b)$.

Example (0.3.7): [30]

Consider the space $l_2(\mathbb{R}) = \{x : x = (x_1, x_2, \dots, x_n, \dots);$

$x_i \in \mathbb{R}\ \forall i\ and\ \sum_{i=1}^{\infty} x_i < \infty\}$ with usual norm $\|x\| = \sqrt{\sum_{i=1}^{\infty} {x_i}^2}$, so $d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$, for $x, y\ \in l_2(\mathbb{R})$, l_2 is the Banach space [16] the set $B = \{e_1, e_2, \dots.\}$ of unit coordinate vectors it is easy to show that B is closed set.

Let $x = (-1, -\frac{1}{2}, \dots\dots, -\frac{1}{n}, \dots)$ and $A = B \cup \{x\}$.

Clearly, A is closed, then $sup_{b\in B}\, d(b, A) = 0$, so

$D(A, B) = d(x, B) = \inf d(x, e_n)$. Now,

$$d(x, e_n) = \|x - e_n\| = \sqrt{\left(-\frac{1}{n} - 1\right)^2 + \sum_{i=1}^{\infty}\left(\frac{1}{i}\right)^2}$$

$$= ((1 + \frac{1}{n})^2 + \sum_{\substack{i=1 \\ i\neq n}}^{\infty} \frac{1}{i^2})^{\frac{1}{2}}$$

$$= (1 + \frac{2}{n} + \frac{1}{n^2} + \sum_{\substack{i=1 \\ i\neq n}}^{\infty} \frac{1}{i^2})^{\frac{1}{2}}$$

$$= (1 + \frac{2}{n} + \sum_{\substack{i=1 \\ i\neq n}}^{\infty} \frac{1}{i^2})^{\frac{1}{2}}$$

$$= (1 + \frac{2}{n} + \sum_{i=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}}$$

$$= (1 + \frac{\pi 2}{6} + \frac{2}{n})^{\frac{1}{2}}$$

So, $D(A, B) = \inf d(x, e_n) = \inf\{(1 + \frac{\pi 2}{6} + \frac{2}{n})^{\frac{1}{2}}\} = (1 + \frac{\pi 2}{6})^{\frac{1}{2}}$

Where $D(A, B) = (1 + \frac{\pi 2}{6})^{\frac{1}{2}} <\ d(x, e_n)$.

Lemma (0.3.8): [20] [13]

Let X be a normed space. If $A, B \in CB(X)$ and $a \in A$, then given $\varepsilon > 0$ there exists a point $b \in B$ such that $\|a - b\| \leq D(A, B) + \varepsilon$.

Lemma (0.3.9): [13]

Let X be a normed space and A_n, B_n be two sequence in $CB(X)$. Then therefore $a_n \in A_n, b_n \in B_n\ \ \forall\ n \geq 0$ such that

$\|a_n - b_n\| \leq D(A_n, B_n) + \varepsilon_n$ with $\lim_{n\to\infty} \varepsilon_n = 0$ ... (0.3.1)

0-4 Types of Multi-Valued Contractive Conditions

Recall, some important definition of multi-valued contraction mappings

Definition (0.4.1):

Let X be a normed space, $F: M \to 2^M$ be a multi-valued mapping. We introduce some types of contraction conditions as following: for all $x, y \in X$

Banach's or (Nadler's) multi-valued contraction condition [20]is

$D(Fx, Fy) \leq ad(x, y)$ where $0 \leq a < 1$ ... (0.4.1)

Kannan's multi-valued contraction condition [10] is

$D(Fx, Fy) \leq b[d(x, Fx) + d(y, Fy)]$ where $0 \leq b \leq 0.5$... (0.4.2)

Chatterjea's multi-valued contraction condition [11] is

$D(Fx, Fy) \leq c[d(x, Fy) + d(y, Fx)]$ where $0 \leq c \leq 0.5$ ... (0.4.3)

Zamfirescu's-multi-valued contraction condition [18] (z-operator)

(z1) $D(Fx, Fy) \leq ad(x, y)$

(z2) $D(Fx, Fy) \leq b[d(x, Fx) + d(y, Fy)]$

(z3) $D(Fx, Fy) \leq c[d(x, Fy) + d(y, Fx)]$

where $0 \leq a < 1, 0 \leq b < 0.5,\ 0 \leq c < 0.5$ ... (0.4.4)

Remark (0.4.2):

By [Berinde 12], z-operator lead to the following conclusions for all $x, y \in X$:

i) $D(Fx, Fy) \leq \delta d(x, y) + 2\delta d(x, Fx)$ by using condition (0.4.2) and

ii) $D(Fx, Fy) \leq \delta d(x, y) + 2\delta d(x, Fy)$ by using condition (0.4.3)

where $\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$,and $\delta \in [0,1) \forall x, y \in X$.

Definition (0.4.3): [9]

Let $F: X \rightarrow X$ be a mapping on metric space X is said to be *quasi-contraction* mapping iff there exists a number $0 \leq q < 1$, such that

$$D(Fx, Fy) \leq qmax\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\} \quad \dots (0.4.5)$$

Dung and el.at gave the following generalization of (0.4.5)

Definition (0.4.4) [20]:

Let $F: X \rightarrow X$ be a mapping on metric space X. The mapping F is said to be a *general quasi-contraction (g.q.c.-mappings)* iff there exists $q \in [0,1)$ such that for all $x, y \in X$,

$$D(Fx, Fy) \leq qmax\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx) \\ d(F^2x, x), d(F^2x, Fx), d(F^2x, y), d(F^2x, Fy)\} \quad \dots (0.4.6)$$

The following two examples to show that the contraction a g.q.c-mappings is a generalization of (0.4.5) and (0.4.1), (0.4.2) and (0.4.3) independent.

Example (0.4.5): [8]

Consider $X = [0, 1]$ with usual norm and $F: [0, 1] \rightarrow [0, 1], F(x) = \frac{x}{3}$,

$\forall\, x \in [0, 1]$. Since F satisfies (0.4.1), but dose not satisfy (0.4.2) when

$x = 0$ and $y = \frac{1}{3}$.

Now, let $F(x) = \frac{1}{2}, 0 \le x < 1$ and $F(x) = \frac{1}{4}, x = 1$ since F satisfies (0.5.3), but $\|f(x) - f(1)\| = \frac{1}{4} > |x - 1|$, for any $x \in \left(\frac{3}{4}, 1\right)$, so that F dose not satisfy (0.4.1).

Consider $X = \mathbb{R}$ and $F: \mathbb{R} \to \mathbb{R}, F(x) = \frac{-x}{2}$, then F satisfies (0.4.2), but dose not satisfy (0.4.3) when $x = 2, y = -2$.

Let X be unit interval, define $F: X \to X$ by $F(x) = \frac{x}{2}$, if $0 \le x < 1$ and $F(x) = 0$, if $x = 1$, then F holds (0.4.3), but F dose not hold (0.4.2) with $x = \frac{1}{2}, y = 0$.

Example (0.4.6): [20]

Let X = {1,2,3,4,5} with d defined as:

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 2 & if\ (x,y) \in \{(1,4),(1,5),(4,1),(5,1)\} \\ 1 & otherwise \end{cases}$$

Let $F: X \to X$ be defined by

$$F1 = F2 = F3 = 1, F4 = 2, F5 = 3.$$

F is not quasi-contraction for x =4 and y=5 because there is no a nonnegative number $q < 1$ satisfying the equation (0.4.5). However,F is generalized quasi-contraction since the (0.4.6) hold for some $q \in [0.5,1)$, for all $x, y \in X$.

Definition (0.4.7): [13]

Let $F: X \to X$ be a mapping on metric space X is said to be *like contraction (or weak contraction)* if $q \in (0,1)$ and $L \ge 0$ exist, such that

$$D(Fx, Fy) \le q\|x - y\| + L\|x - Fx\| \quad \forall x, y \in X \qquad \dots (0.4.7)$$

This definition is the general of condition (0.4.7)

Definition (0.4.8) [12, 14]:

A mapping F is called *general like contraction (g.l.c)* if there exist q∈ $(0,1)$ and defined continuous strictly increasing function $\emptyset: [0,\infty) \to [0,\infty)$ with $\emptyset(0) = 0$ such that

$D(Fx, Fy) \leq q\|x - y\| + \emptyset(d(x, Fx)) \quad \forall x, y \in X$ … (0.4.8)

The following example show that F satisfies condition (0.4.8)

Example (0.4.9):

Consider X= [0, ∞) and $F: X \to X$ defined by

$$Fx = \begin{cases} 1.0 & 0 \leq x \leq 0.6 \\ 0.4 & 0.6 < x \end{cases}$$

assume that $x < y. Then$, for $0 \leq x < y \leq 0.6$ or

$0.6 < x < y$, $\|Fx - Fy\| = 0$, and (0.4.8) is automatically satisfied

If $0 \leq x \leq 0.6 < y$,then $\|Fx - Fy\| = 0.6$

defined $\emptyset$ by $\emptyset(t) = Lt$ for any $L \geq 2$. Then $\emptyset$ is increasing continuous , and $\emptyset(0) = 0$, Also,

$\|x - Fx\| = 1 - x$, so that

$\emptyset(\|x - Fx\|) = L(1 - x) \geq 0.4L \geq 0.6$ therefore

$0.6 = \|Fx - Fy\| \leq L\|x - Fx\| \leq \delta\|x - y\| + L\|x - Fx\|$

for any 0≤ $\delta < 1$, and (0.4.8) is satisfied for 0≤ $x \leq 0.6 < y$

Proposition (0.4.10):[88]

Any mapping satisfying condition (0.4.5) with $0 < q < \frac{1}{2}$ is also, satisfying condition (0.4.7).

In the following example the mapping satisfied condition (0.4.7) but not satisfied (0.4.5).

Example (0.4.11):

Let $X = [0,1]$ be unit interval with usual norm when $n = 1$ define $F: [0,1] \to [0,1]$ by $F(x) = \frac{x}{4}$ for all $x \in [0,1)$ and $F(x) = \frac{1}{4}$, if $x = 1$. Then F satisfies condition (0.4.7), since $\|Fx - Fy\| \leq q\|x - y\| + L\|x - Fx\|$, then is true if we take $q = \frac{1}{4}$ and $L \geq 4$. If $x = q, y = 0$ then F is not satisfy condition (0.4.5).

Definition (0.4.13): [12] [19] [5]

Let X be a normed space and $\emptyset \neq M \subseteq X$. A multi-valued mapping $F: M \to 2^X$ is said to be *k-strictly pseudo contractive* if for each $x, y \in M$ and for each $\mu \in Fx, \xi \in Fy$ such that

$$D^2(Fx, Fy) \leq \|x - y\|^2 + k\|x - \mu - (y - \xi)\|^2 \qquad \ldots (0.4.10)$$

The inequalities (0.4.10)

If there exists $j(x - y) \in J(x - y)$ such that

$$\langle \mu - \xi, j(x - y) \rangle \leq k\|x - y\|^2 \qquad \ldots (0.4.11)$$

or respectively $\langle Fx - Fy, j(x - y) \rangle \leq k\|x - y\|^2$ … (0.4.12)

In [50] condition (0.4.10) is said to be *k-strictly pseudo-contraction* of proximinal set if there exists k$\in (0,1)$ such that given any $x, y \in M$ and $\mu \in Fx$, there exists $\xi \in Fy$ satisfying $\|\mu - \xi\| \leq D(Fx, Fy)$ and

$$D^2(Fx, Fy) \leq \|x - y\|^2 + k\|x - \mu - (y - \xi)\|^2 \qquad \ldots (0.4.13)$$

Remark (0.4.14):

i. From the definition (0.4.13), if $k = 1$ then F is said to be a *pseudo contractive* mapping $(i.e. D^2(Fx, Fy) \leq \|x - y\|^2 + \|x - \mu - (y - \xi)\|^2$ for all $x, y \in M)$ … (0.4.14)

or respectively

$$\|Fx - Fy\|^2 \leq \|x - y\|^2 + \|x - Fx - (y - Fy)\|^2 \quad \dots (0.4.15)$$

if $k = 0$ F is called *nonexpansive* mapping

$(i.e.\ D^2(Fx, Fy) \leq \|x - y\|^2$ for all $x, y \in M)$ … (0.4.16)

(or respectively $\|Fx, Fy\|^2 \leq \|x - y\|^2)$ … (0.4.17)

Also every multivalued nonexpansive mapping is k-strictly pseudo contractive and every k-strictly pseudo contractive mapping is pseudo contractive

ii. Every k-strictly pseudo contractive mapping is L-*Lipschitzian* mapping $(i.e. D(Fx, Fy) \leq L\|x - y\|$ for all $x, y \in M)$ … (0.4.18) (or respectively $\|Fx, Fy\| \leq L\|x - y\|$ for all $x, y \in M)$ …(0.4.19) [proposition 2,50].

iii. L-Lipschitzian is Banach contraction, if $L < 1$ then F is contraction and if $L = 1$ then F is said to be a multi-valued nonexpansive mapping [28, 2].

iv. Every nonexpansive mapping with nonempty fixed point set is *quasi-nonexpansive*. But there exist quasi-nonexpansive mappings that are not nonexpansive [50].

In the following two examples to illustrate that k-pseudo contraction mapping is not necessarily nonexpansive mapping or pseudo contraction.

Example (0.4.15):

Let $F: X \to 2^X$ defined by $Fx = \left[-\frac{3x}{2}, -x\right]$.

Then $P_{Fx} = \{-2x\}$ for all $x \in [0, \infty)$, F is not nonexpansive since:

$$D^2(Fx, Fy) = max\left\{|(x - y)|^2, \left|\frac{3}{2}(x - y)\right|^2\right\}$$

$$= \frac{9}{4}|x - y|^2 = |x - y|^2 + \frac{5}{4}|x - y|^2.$$

By definition (0.4.5) for each $\mu \in Fx, \mu = -\alpha x, 1 \leq \alpha \leq \frac{3}{2}$,

choose $\xi = -\alpha y$. Then

$$|\mu-\xi| = |-\alpha x-(-\alpha y)| = \alpha|x-y| \le \tfrac{3}{2}|x-y| = D(Fx,Fy)$$

$$\begin{aligned}|x-\mu-(y-\xi)|^2 &= |x-(-\alpha x)-(y-(-\alpha y))|^2\\ &= |(1+\alpha)x-(1+\alpha)y|^2\\ &= (1+\alpha)^2|x-y|^2.\end{aligned}$$

$$\begin{aligned}D^2(Fx,Fy) &= |x-y|^2+\tfrac{5}{4}|x-y|^2\\ &= |x-y|^2+\frac{5}{4(1+\alpha)^2}(1+\alpha)^2|x-y|^2\\ &= |x-y|^2+\frac{5}{4(1+\alpha)^2}|x-\mu-(y-\xi)|^2\\ &\le |x-y|^2+\frac{5}{4(1+1)^2}|x-\mu-(y-\xi)|^2\\ &= |x-y|^2+\frac{5}{16}|x-\mu-(y-\xi)|^2.\end{aligned}$$

F is k-strictly pseudo contraction with $k=\frac{5}{16}$. And follows F is pseudo contraction but is not nonexpansive mapping.

Example (0.4.16):

Let X=ℝ (the reals with usual metric). Defined $F: X \to 2^X$ by

$$Fx = [-\sqrt{5}x, -x].$$

Then $P_{Fx} = \{-x\}$ which is not nonexpansive.

$$\begin{aligned}D^2(Fx,Fy) &= max\left\{\left|\sqrt{5}(x-y)\right|^2, |(x-y)|^2\right\} = 5|x-y|^2\\ &= |x-y|^2+4|x-y|^2.\end{aligned}$$

Furthermore, for each $\mu \in Fx, \mu=-\alpha x, 1 \le \alpha \le \sqrt{5}$, choose $\xi=-\alpha y$. Then

$$|\mu-\xi| = |-\alpha x-(-\alpha y)| = \alpha|x-y| \le \sqrt{5}|x-y| = D(Fx,Fy)$$

$$\begin{aligned}|x-\mu-(y-\xi)|^2 &= |x-(-\alpha x)-(y-(-\alpha y))|^2\\ &= |(1+\alpha)x-(1+\alpha)y|^2 = (1+\alpha)^2|x-y|^2.\end{aligned}$$

It then follows that

$$D^2(Fx,Fy) = |x-y|^2+4|x-y|^2$$

$$= |x-y|^2 + (1+1)^2|x-y|^2$$
$$\leq |x-y|^2 + (1+\alpha)^2|x-y|^2$$
$$= |x-y|^2 + |x-\mu-(y-\xi)|^2.$$

So, F is pseudo contraction.

For any pair $x, 0 \in [0,\infty), F0 = \{0\}$. It then follows that for any $\mu \in Fx$ the corresponding $\xi = 0$. In particular, for $\mu = -x$,

$$D^2(Fx, F0) = 5|x-y|^2$$
$$= |x-0|^2 + 4|x-0|^2$$
$$= |x-0|^2 + |x-(-x)|^2$$
$$> |x-y|^2 + k|x-\mu-(y-\xi)|^2 \qquad \forall k \in (0,1).$$

This shows that F is not k-strictly pseudo contraction mapping.

Example (0.4.17): [49]

Let $M = [0,\infty)$ with the usual metric and $F: M \to CB(M)$ be defined by
$$Fx = \begin{cases} \{0\}, & if\ x \leq 1, \\ \left[x-\frac{3}{4}, x-\frac{1}{3}\right], & if\ x > 1. \end{cases}$$

Indeed, it is clear that $Fix_F = \{0\}$ and for any x we have $D(Fx, F0) \leq |x-0|$, hence, F is quasi-nonexpansive. However, if $x = 2, y = 1$ we get $D(Fx, Fy) > |x-y| = 1$, and hence, F is not nonexpansive.

Definition (0.4.18): [48]

Let X be a normed space, M be a nonempty subset of X. A mapping $F: M \to P(M)$ is said to be multivalued *generalized nonexpansive* mapping if

$$D(Fx, Fy) \leq a\|x-y\| + b\{d(x, Fx) + d(y, Fy)\} + c\{d(x, Fy) + d(y, Fx)\} \qquad \ldots (0.4.20)$$

for all $x, y \in X$, where $a + 2b + 2c \leq 1$.

Definition (0.4.19): [8]

Let X be a normed space and $\emptyset \neq M \subseteq X$. A mapping $F: M \to 2^X$ is said to be multi-valued *monotone* if there exist a constant $k \in (0,1)$ and $z \in M$ such that for each $x \in M$ and for each $\mu \in Fx$ there exists $j(x-z) \in J(x-z)$ with

$$\langle \mu - z, j(x-z) \rangle \leq k\|x-z\|^2 \qquad \dots (0.4.21)$$

(or respectively $\langle Fx - z, j(x-z) \rangle \leq k\|x-z\|^2$) $\qquad \dots$ (0.4.22)

Proposition (0.4.20): [89]

Every k-strictly pseudo contraction mapping F with a fixed point is monotone mapping.

Definition (0.4.21): [5] [21]

Let X be a normed space and $\emptyset \neq M \subseteq X$. A mapping $F: M \to 2^X$ is said to be multi-valued *strongly accretive* if there is a constant $k > 0$ such that for each $x, y \in M$ and for each $\mu \in Fx, \xi \in Fy$ there exist $j(x-y) \in J(x-y)$ with

$$\langle \mu - \xi, j(x-y) \rangle \geq k\|x-y\|^2 \qquad \dots (0.4.23)$$

(or respectively $\langle Fx - Fy, j(x-y) \rangle \geq k\|x-y\|^2$) $\qquad \dots$ (0.4.24)

Now, since without loss the generality we assume that $k \in (0,1)$. And if $k = 0$ in (0.4.23) or in (0.4.24), then F is said to be multi-valued accretive.

Definition (0.4.22): [21]

Let X be a real Banach space, $\emptyset \neq M \subseteq X$ and $F: M \to 2^M$ be a multi-valued *ϕ-accretive* mapping, if for any $x, y \in M$ and for any $\mu \in Fx, \xi \in$

Fy, there exists a $j\,(x,y) \in J(x,y)$ and a strictly increasing function $\phi\colon [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle \mu - \xi, j(x-y)\rangle \geq \phi(\|x-y\|) \qquad \dots (0.4.25)$$

0-5 Types of Iterative Schemes

In the following, some types of iterative schemes for multi-valued mappings:

Let X be a normed space, $\emptyset \neq M \subseteq X$ and $F\colon M \to 2^X$ be a multi-valued mapping. For $x_0 \in M$ if the sequence $\langle x_n\rangle \subset M$ with $\langle \alpha_n\rangle, \langle \beta_n\rangle$ sequence in (0,1). Now we started with the *Picard iteration* method [1].

$$x_0 \in M \quad , x_{n+1} = \mu_n \qquad \dots (0.5.1)$$

where $\mu_n \in Fx_n$

when F is single valued mapping, (0.5.1) will be $x_{n+1} = Fx_n$

The *Mann iterations* [2] defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\mu_n \quad \text{for } n \geq 0 \qquad \dots (0.5.2)$$

where $\mu_n \in Fx_n$,

when F is single valued mapping, (0.5.2) will be

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Fx_n \quad \text{for } n \geq 0$$

Another iteration method is *Ishikawa iteration* [3] defined by

$$\begin{cases} x_0 \in M \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\,\xi_n & \text{for } n \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n\mu_n \end{cases} \qquad \dots (0.5.3)$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$

when F is single valued mapping, (0.5.3) will be

$$\begin{cases} x_0 \in M \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\,Fy_n & \text{for } n \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n Fx_n \end{cases}$$

The *s-iteration* process [4] is

$$\begin{cases} x_0 \in M \\ x_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \xi_n \\ y_n = (1-\beta_n)x_n + \beta_n \mu_n \end{cases} \quad \text{for } n \geq 0 \qquad \dots (0.5.4)$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$

when F is single valued mapping, (0.5.4) will be

$$\begin{cases} x_0 \in M \\ x_{n+1} = (1-\alpha_n)Fx_n + \alpha_n Fy_n \\ y_n = (1-\beta_n)x_n + \beta_n Fx_n \end{cases} \quad \text{for } n \geq 0$$

The *2step-Mann iteration* [5] is defined by

$$\begin{cases} x_0 \in X \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n \xi_n \\ y_n = (1-\beta_n)x_n + \beta_n \mu_n \end{cases} \quad \text{for } n \geq 0 \qquad \dots (0.5.5)$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$

when F is single valued mapping, (0.5.5) will be

$$\begin{cases} x_0 \in X \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Fy_n \\ y_n = (1-\beta_n)x_n + \beta_n Fx_n \end{cases} \quad \text{for } n \geq 0$$

Also, *Picard-Mann iteration* process [6] is

$$\begin{cases} x_{n+1} = \xi_n \\ y_n = (1-\alpha_n)x_n + \alpha_n \mu_n \end{cases} \quad \text{for } n \geq 0 \qquad \dots (0.5.6)$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$

when F is single valued mapping, (0.5.6) will be

$$\begin{cases} x_{n+1} = Fy_n \\ y_n = (1-\alpha_n)x_n + \alpha_n Fx_n \end{cases} \quad \text{for } n \geq 0$$

Remark (0.5.1):

We have some special cases for iterative schemes

1. If $\beta_n = 0$ in (0.5.3) then reduced Mann iteration
2. If $\alpha_n = 1$ in (0.5.2) then reduced Picard iteration
3. If $\alpha_n = 1$ in (0.5.5) or in (0.5.4) then reduced Picard-Mann iteration.

0-6 Important Fixed Point Theorems

In this section, recall some definition and some important theorems of fixed point.

Definition (0.6.1): [1]:

Let $x_0 \in X$, an orbit of Fat x_0 is a sequence $O_F(x_0) = \{x_n : x_n \in Fx_{n-1}, n \in \mathbb{N}\}$. A space X is called to be F-*orbitally complete* if every Cauchy sequence in $O_F(x_0)$ converge in X.

Definition (0.6.2): [28] [32]

Let X be a normed space and $\emptyset \neq M \subseteq X$, the point $x \in M$ is called *fixed point* of the multi-valued mapping $F: M \to 2^M$ (or, single valued mapping $F: M \to M$) if $x \in Fx$ (respectively $x = Fx$). The set of all fixed points of F denoted by Fix_F

If $G: M \to 2^M$ be another multi-valued mapping a point x is called *common fixed point* of F and G if $x \in Fx \cap Gx$.

Theorem (0.6.3): [15] (Banach's Fixed Point Theorem)

Let (x, d) be a complete metric space and $F: X \to X$ be a Banach's contraction (0.4.1) with $a \in [0, 1)$ fixed. Then

i) F has a unique fixed point, that is $Fix_F = \{z\}$;

ii) The Picard iteration associated to $F, i.e.$, the sequence $\langle x_n \rangle_{n=0}^{\infty}$, defined by

$$x_n = F(x_{n-1}) = F^n(x_0), \qquad n = 1, 2, \dots,$$

converges to z, for any initial guess $x_0 \in X$;

iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, z) \leq \frac{a^n}{1-a} . d(x_0, x_1), \ \ n = 0,1,2, \dots.,$$

$$d(x_n, z) \leq \frac{a}{1-a} . d(x_{n-1}, x_n), \ \ n = 0,1,2, \dots,$$

iv) The rat of convergence is given by

$$d(x_n, z) \leq a . d(x_{n-1}, z) \leq a^n . d(x_0, z), \ \ n = 1,2, \dots,$$

Theorem (0.6.4): [13] (Nadler's Fixed Point Theorem (1941))

Let M be a nonempty closed subset of a Banach space X and $F: M \to CB(M)$ be a multi-valued contraction mapping (0.4.1). Then F has a fixed point.

Theorem (0.6.5): [Theorem 3, 4] [9]

Let (X, d) be a metric space and $F: X \to CB(X)$ be a quasi contraction mapping (0.4.5). If X is F- orbitally complete. Then

i) F has a unique fixed point z in X and $Fz = \{z\}$

ii) for each $x_0 \in X$ there exists an orbit $\langle x_n \rangle$ of F at x_0 such that $\lim_{n\to\infty} x_n = z$ for all $x \in X$ and

$d(x_n, z) \leq \frac{(q^{1-a})^n}{1-q^{1-a}} d(x_0, x_1)$ for all $n \in N$, where $a < 1$ is any fixed positive number.

Theorem (0.6.6): [Theorem (3.4), 20]

Let (X, d) be a metric space and $F: X \to CB(X)$ be a g.q.m.c-mapping (0.4.6). If X is F- orbitally complete. Then

i) F has a unique fixed point z in X and $Fz = \{z\}$

ii) for each $x_0 \in X$ there exists an orbit $\langle x_n \rangle$ of F at x_0 such that $\lim_{n\to\infty} x_n = z$ for all $x \in X$ and

$d(x_n, z) \leq \frac{(q^{1-a})^n}{1-q^{1-a}} d(x_0, x_1)$ for all $n \in N$, where $a < 1$ is any fixed positive number.

Theorem (0.6.7): [15]

Let (X, d) be a complete metric space and $F: X \to X$ be a weak contraction (0.4.7) with $q \in (0,1)$ and $L \geq 0$. Then

i) $Fix_F \neq \emptyset$;

ii) For any $x_0 \in X$, the Picard iteration $\langle x_n \rangle$ (0.5.1) converges to some $z \in Fix_F$;

iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, z) \leq \frac{q^n}{1-q} . d(x_0, x_1), \quad n = 0,1,2, \ldots..,$$

$$d(x_n, z) \leq \frac{q}{1-q} . d(x_{n-1}, x_n), \quad n = 0,1,2, \ldots.,$$

Theorem (0.6.8): [15]

Let (X, d) be a complete metric space and $F: X \to X$ be a weak contraction for which there exist $q \in (0,1)$ and some $L_1 \geq 0$ such that

$$d(Fx, Fy) \leq q\, d(x, y) + L_1\, d(x, Fx), \quad \forall\, x, y \in X$$

Then

i) F has a unique fixed point, $Fix_F \neq \emptyset$;

ii) The Picard iteration $\langle x_n \rangle$ (0.5.1) converges to $z \in Fix_F$ for any $x_0 \in X$;

iii) The a priori and a posteriori error estimates

$$d(x_n, z) \leq \frac{q^n}{1-q} . d(x_0, x_1), \quad n = 0,1,2, \ldots..,$$

$$d(x_n, z) \leq \frac{q}{1-q} . d(x_{n-1}, x_n), \quad n = 0,1,2, \dots.,$$

iv) The rat of convergence is given by

$$d(x_n, z) \leq q \, . d(x_{n-1}, z), \quad n = 1,2, \dots.,$$

Remark (0.6.9): [15]

i) If F satisfied condition (0.4.7), then it is not ensuring that F has a fixed point. But if F has a fixed point, the it is unique.

ii) The theorem (0.6.8) unifies and generalizes the fixed point theorems of (0.4.1), (0.4.2), (0.4.3) and (0.4.4) and convex theorem (0.6.5).