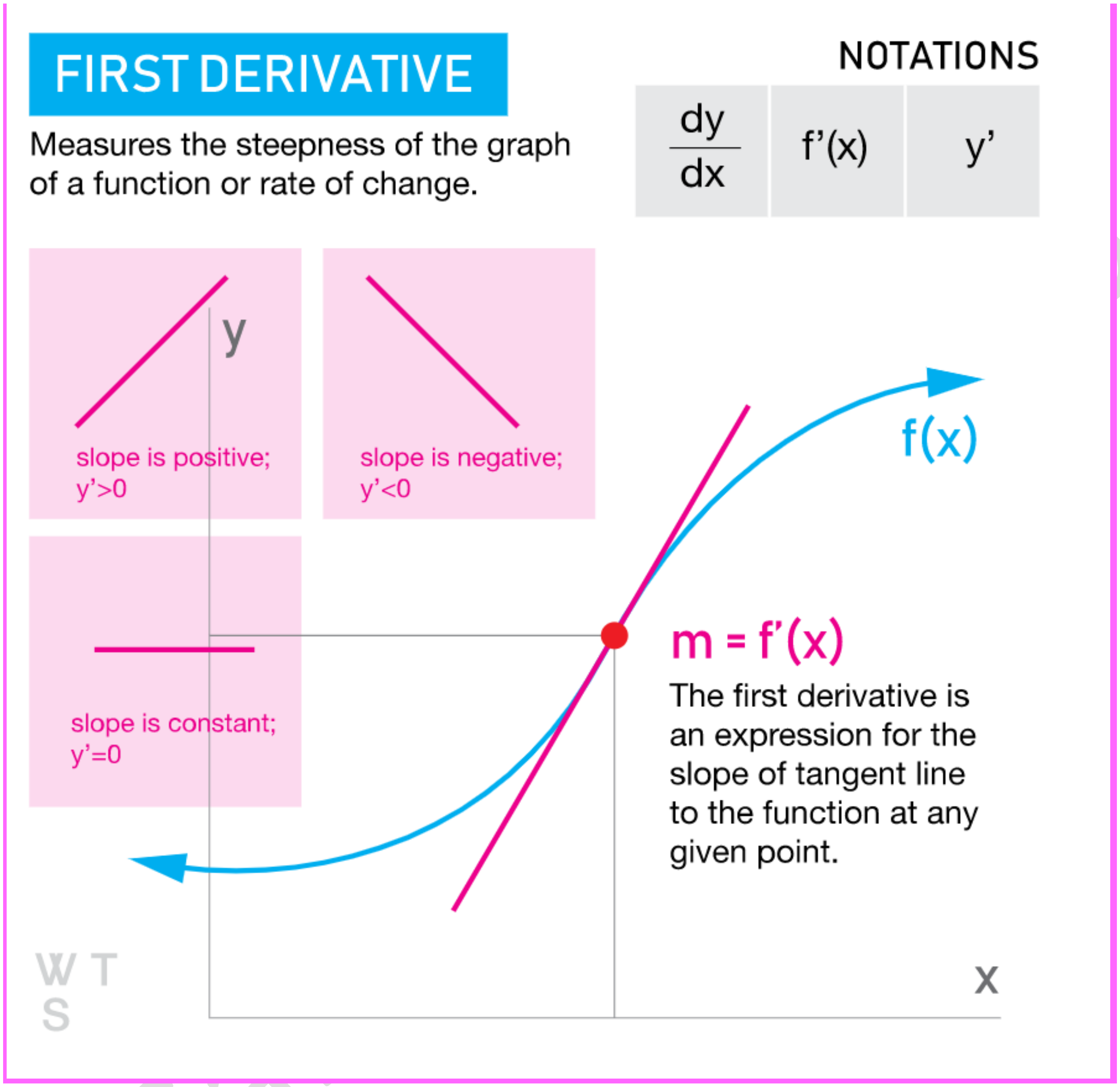

Chapter five - 2024

The derivative

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⇒ Beginning: Explain the geometric meaning of differentiation.

Definition

A function $f: A \to R, A \subseteq R$ is called differentiable at c if and only if

$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. The value of this limit is called the derivative of f at c and denoted by $f'(c)$

Theorem (1)

If f is differentiable at the point c, then f is cont. at c.

Proof: Let $\varepsilon > 0$, to prove that $|f(x) - f(c)| < \varepsilon$?

Since f is diff. $\Rightarrow f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists

Now

$$\lim_{x \to c} \big(f(x) - f(c)\big) = \lim_{x \to c} \left(f(x) - f(c) \cdot \frac{x-c}{x-c}\right)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) = f'(c) \cdot 0 = 0$$

$\therefore \lim_{x \to c} \big(f(x) - f(c)\big) = 0 \Rightarrow |f(x) - f(c)| < \varepsilon \ \ (by\ def.of \lim)$

$\Rightarrow f\ conts.at\ c.$

Remark: The converse of Theorem (1) is not true, see the example below:

Examples

1. $f: R \to R, f(x) = |x|$ conts.but not diff.at $c = 0$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - 0}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \pm \frac{x}{x} = \pm 1 \text{ not exists}$$

$\Rightarrow f'(0)$ is not exists $\Rightarrow$f is not diff.at $c = 0$

2. $f; R \to R, f(x) = \begin{cases} 1 + \sqrt[3]{x} & if\ x \geq 0 \\ \sqrt[3]{x} & if\ x < 0 \end{cases}$ cont. but not diff. at $c = 0$.

3. $f: R \to R, f(x) = \begin{cases} \sqrt{-x}\ if\ x\ < 0 \\ \sqrt{x}\ if\ x\ \geq 0 \end{cases}$ cont. but not diff. at c $= 0$.

Exercise. Give an example for conts. function but not diff. at each point.

Theorem (2) Rules of derivatives: (without proof)

Let f, g be two diff. functions at c

1. $(f + g)\ is\ diff. at\ c\ and\ (f + g)'(c) = f'(c) + g'(c)$

2. $(f \cdot g) is\ diff. at\ c\ and\ (f \cdot g)'(c) = f'(c) \cdot g(c) + g'(c) \cdot f(c)$

3. $if\ f(c) \neq 0, then\ \dfrac{1}{f(c)}\ is\ diff. at\ c\ and\ \left(\dfrac{1}{f}\right)'(c) = \dfrac{-f'(c)}{\left(f(c)\right)^2}$

4. $if\ g(c) \neq 0, then\ \dfrac{f}{g}\ is\ diff. at\ c\ and\ \left(\dfrac{f}{g}\right)'(c) = \dfrac{f'(c)g(c) - g'(c)f(c)}{\left(g(c)\right)^2}$

5. $if\ \forall n \in N, f(x) = x^n, then\ f'(x) = nx^{n-1}$

Theorem (3) (Chain Rule)

Let f be diff. function at c and g diff. at $b = f(c), then\ g \circ f$ is diff. at c and
$(g \circ f)'(c) = g'\big(f(c)\big)f'(c)$

Proof

$Define\ F_c(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$

$\Rightarrow F_c\ is\ conts. function\ at\ c\ since$:

$$\lim_{x\to c}(F_c(x)-F_c(c)) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} - \lim_{x\to c} f'(c) = f'(c)-f'(c) = 0$$
$$\Rightarrow |F_c(x)-F_c(c)| < \varepsilon \quad \forall \varepsilon > 0$$

Now,

$$Define\ G_b(y) = \begin{cases} \dfrac{g(y)-g(b)}{y-b} & y \neq b \\ g'(b) & y = b \end{cases}$$

$\Rightarrow G_b$ *is conts. function at b*

$$\forall x, F_c(x) = \frac{f(x)-f(c)}{x-c} \Rightarrow f(x)-f(c) = (x-c)F_c(x) \dots\dots (*)$$

$$\forall y, G_b(y) = \frac{g(y)-g(b)}{y-b} \Rightarrow g(y) = g(b) + (y-b)G_b(y)$$

$$\Rightarrow (g\circ f)(x) = g\big(f(x)\big) = g(y) = g(b) + (y-b)G_b(y)$$

$$= g\big(f(c)\big) + \big(f(x)-f(c)\big)G_b(y)$$

$$\therefore (g\circ f)(x) = (g\circ f)(c) + (x-c)F_c(x)G_b\big(f(x)\big) \ \big(by(*)\big)$$

$$\Rightarrow \frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} = F_c(x)G_b\big(f(x)\big)$$

Since $F_c(x)$ *is cont. at c and* $f(x)$ *is diff. at c*
$$\Rightarrow f(x)\ cont.\ at\ c\ and\ G_b\ cont.\ at\ b = f(c)$$

$\Rightarrow G_b \circ f(x)$ *conts. at c*

$$\Rightarrow F_c\big(G_b\circ f(x)\big) conts.\ at\ c \Rightarrow \frac{(g\circ f)(x)-(g\circ f)(c)}{x-c}\ exists\ and\ then$$

$$\lim_{x\to c}\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} = F_c(c)\cdot G_b\big(f(c)\big) = f'(c)\cdot G_b(b) = f'(c)g'(b)$$

$$= f'(c)g'\big(f(c)\big)$$

$\therefore\ g\circ f$ *diff. at c*

Theorem (4) (Inverse Function Theorem)

Suppose that $f: A \to B$ is bijective function, where A and B are intervals. If f is diff. $a \in A$ and $f'(a) \neq 0$ then f^{-1} is diff.at $f(a) = b$

$and\ (f^{-1})' = \frac{1}{f'(a)}$ **(Proof: exercise)**

Examples

Explain Th. (4) by the following example

1. $f: [0, \infty) \to [0, \infty) \ni f(x) = x^2$?

Solution: since $f\ is\ (1-1) and\ onto, then\ f^{-1}\ exists$ and

$y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow f^{-1}: [0, \infty) \to [0, \infty) \ni f^{-1}(y) = \sqrt{y}$

$\forall y, \left(f^{-1}(y)\right)' = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$

2. $f: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to (-1,1) \ni f(x) = \sin(x)$?

3. $f: R \to [0, \infty) \ni f(x) = e^x$?

Definition:

1. $\lim\limits_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ is called the left-hand derivative of f at c and denoted by $f_-{}'(c)$.

2. $\lim\limits_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ is called the right-hand derivative of f at c and denoted by $f_+{}'(c)$.

If $f_+{}'(c) = f_-{}'(c)$ exists, then f is diff. at c.

Exercises: Check $f_-{}'(c)$ andby $f_+{}'(c)$ at $x = 0$, if $f: R \to R$ such that

1. $f(x) = \begin{cases} x^2, & x \leq 0 \\ 2x + 1, & x > 0 \end{cases}$

2. $f(x) = \begin{cases} \sqrt{-x}, & x \leq 0 \\ x, & x > 0 \end{cases}$

3. $f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \leq 0 \\ x, & x > 0 \end{cases}$

Definition:

1. A function f has a local maximum value at $c, if\ \exists J$(J is an open interval) $c \in J$ such that $f(x) \leq f(c) \forall x \in J$

2.A function f has a local minimum value at $c, if\ \exists J$(J is an open interval) $c \in J$ such that $f(c) \leq f(x) \forall x \in J$.

Theorem (5) (Local Extreme Value)

If f is diff. at c and f has a local max.(or min.) value at c, then $f'(c) = 0$

Proof

$Suppose\ that\ f\ has\ min.value\ at\ c \Rightarrow by\ def.\exists J(J\ is\ open\ interval)c \in J$

$such\ that\ f(c) \leq f(x)\ \forall x \in J \Rightarrow f(x) - f(c) \geq 0$

There are two cases:

$1. If\ x > c \Rightarrow x - c > 0 \Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \Rightarrow f_+{}'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$

$2. If\ x < c \Rightarrow x - c < 0 \Rightarrow \frac{f(x) - f(c)}{x - c} \leq 0 \Rightarrow\ f_-{}'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$

$Since\ f\ is\ diff.at\ c, then\ 0 \leq f_+{}'(c) = f'(c) = f_-{}'(c) \leq 0 \Rightarrow f'(c) = 0$

Remark

The converse of Th. (5) is not necessarily true. As example

$f: R \to R\ \ni f(x) = x^3\ diff.on\ R, f'(x) = 3x^2 \Rightarrow x = 0$

Exercise

Determine the max. and min. value of $f(x) = x(x-1)(x-2) on$ [0,3].

Theorem (6) (Roll's Theorem)

If f is diff.on $(a, b)$and conts.on $[a, b]$and $f(a) = f(b)$, then there exists c

$a < c < b \ni f'(c) = 0$.

Proof: Since $[a, b] is\ compact\ and\ f\ conts. on [a, b], then$:

$f\ has\ max. and\ min. \Rightarrow \exists c_1, c_2 \in [a, b] \ni f(c_1) max. value, f(c_2) min. value$.

$* If\ f(c_1) = f(c_2) \Rightarrow f is\ constant\ \forall x \in [a, b] \Rightarrow f'(x) = 0\ \forall x \in [a, b]$

$* If\ f(c_1) \neq f(c_2) \Rightarrow at\ least\ one\ of\ c_1\ and\ c_2\ is\ not\ equal\ to\ a\ or\ b$

$\Rightarrow f$ has a local max.or min.(or both) in (a, b).

By Th.(5) $f'(c_1) = 0$or $f'(c_2) = 0$ (or both=0).

Example

Find the value of$c$, which satisfy Roll's Th., $f: [-1,1] \to R \ni f(x) = \sqrt{1 - x^2}$

Solution: $1 - x^2 \geq 0 \Rightarrow -x^2 \geq -1 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$\Rightarrow f\ is\ diff. on\ (-1,1) and\ conts. on [-1,1]$

$a = -1, b = 1 \Rightarrow f(-1) = f(1) = 0$

by Roll^' s Th.$\Rightarrow \exists c. -1 < c < 1 \ni f'(c) = 0$

Since $f'(x) = \frac{-2x}{2 \cdot \sqrt{1-x^2}}\ and\ f'(c) = 0 \Rightarrow c = 0$

Theorem (7) (Mean Value Theorem) [M.V.T]

If f is diff. on $(a, b)$and conts. on$[a, b]$,then there exists $c, a < c < b$ such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof: Define $g(x) = f(x) - \boldsymbol{\lambda} x$, where $\boldsymbol{\lambda} = \frac{f(b)-f(a)}{b-a}$

Since f conts.on $[a, b]$ and $\boldsymbol{\lambda} x$ conts. on $[a, b] \Rightarrow g$ is conts.on $[a, b]$

$\because f\ diff.on\ (a,b)\ and\ \boldsymbol{\lambda}x\ diff.on\ (a,b) \Rightarrow gis\ diff.on\ (a,b).$

To prove $g(a) = g(b)$.

$$g(a) = f(a) - \boldsymbol{\lambda}a = f(a) - \frac{f(b)-f(a)}{b-a}a = \frac{f(a)b-f(a)a-f(b)a+f(a)a}{b-a} = \frac{f(a)b-f(b)a}{b-a}$$

$$g(b) = f(b) - \boldsymbol{\lambda}b = f(b) - \frac{f(b)-f(a)}{b-a}b = \frac{f(b)b-f(b)a-f(b)b+f(a)b}{b-a} = \frac{f(a)b-f(b)a}{b-a}$$

Therefore $gis\ diff.on\ (a,b)\ and\ g(a) = g(b) \Rightarrow \exists c \in (a,b) \ni g'(c) = 0$.

Example

$f:[0,1] \to R, f(x) = \sqrt{x}.find\ the\ value\ of\ c, which\ satisfy\ M.V.T.$

Solution: $f\ is\ conts.on\ [0,1] and\ diff.on\ (0,1), f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(b)-f(a)}{b-a} = \frac{1-0}{1-0} = 1 \Rightarrow f'(c) = 1 \Rightarrow \frac{1}{2\sqrt{c}} = 1 \Rightarrow \frac{1}{\sqrt{c}} = 2 \Rightarrow c = \frac{1}{4}$$

Collorary (1)

$If\ f\ is\ conts.on\ , diff.on\ (a,b) and\ f'(x) = 0\ \forall x \in (a,b), then\ f\ is\ constant$

Proof

$Let\ x_1, x_2 \in [a,b]\ and\ x_1 < x_2, Since\ f\ is\ conts.on\ [a,b] \Rightarrow f\ conts.on[x_1, x_2]$

$and\ f\ diff.on\ (a,b)$

$\Rightarrow f\ diff.on\ (x_1, x_2).$

$$By\ M.V.T. \Rightarrow \exists c; c \in (x_1, x_2) \ni f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$\Rightarrow f(x_2) = f(x_1)$, for any $x_1, x_2\ in\ [a,b] \Rightarrow f$is constant.

Definition: Let $f: S \subseteq R \to R$ be a function, f is called

1. increasing if $(x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2))$.
2. decreasing if $(x_1 \leq x_2 \Rightarrow f(x_2) \leq f(x_1))$.

Examples:

Collorary (2) $If\ f\ is\ conts.on\ [a,b], diff.on\ (a,b) and\ f'(x) \geq 0 \forall x \in (a,b), then\ f\ is\ an$

$increasing$

Collorary (3) $If\ f\ is\ conts.on\ [a,b], diff.on\ (a,b) and\ f'(x) \leq 0\ \forall x \in (a,b), then\ f\ is$

$decreasing$