Topological vector spaces :-

Let X be vector space (or linear space) over F and τ be a topology on X. If

$+ : (x,y) \to x+y$, $+ : X \times X \to X$

$\cdot : (\alpha, x) \to \alpha x$, $\cdot : F \times X \to X$

$F = \mathbb{R}$ or $\mathbb{C}$

are continuous functions with respect to τ then (X, τ) is called Topological vector space

Continuity: That is, for any $x, y \in X$, $\forall V_{x+y}$ is neigh of $x+y$, $\exists V_x$ and V_y are neigh of x and y $\ni$ $V_x + V_y \subset V_{x+y}$

Any $\forall x \in X$, $\alpha \in F$, $\forall V_{\alpha x}$ neigh of αx, $\exists V_\alpha, V_x \ni$ $V_\alpha \cdot V_x \subset V_{\alpha x}$.

Examples

1. $X = \mathbb{R}$ or $\mathbb{C}$ are T.V.S with usual space topology.
2. Any normed space is T.V.S. Since:
   - the triangle inequality $\Rightarrow$ continuity of $+$
   - the fact $\|\alpha x\| = |\alpha| \|x\|$ $\Rightarrow$ " " of $\cdot$

For continuity of $+$:-

Let $B(x+y, r)$ be the open ball with center $x+y$ and radius r (neigh of $x+y$) and let $v \in B(x, \frac{r}{2})$ & $w \in B(y, \frac{r}{2})$ then

$\|(v+w) - (x+y)\| \le \|v - x\| + \|w - y\| < r$ Thus

$B(x, \frac{r}{2}) + B(y, \frac{r}{2}) \subset B(x+y, r)$

continuity of $\cdot$ (check).

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Lower Semi continuity of Convex Functions:

X is T.V.S. (or only Top. Sp.)

Def $f : X \to [-\infty, \infty]$

- L.S.C. if $\forall c \in \mathbb{R}$, the set $\{x \in X : f(x) \le c\}$ is closed
 or $\{x \in X : f(x) > c\}$ is open

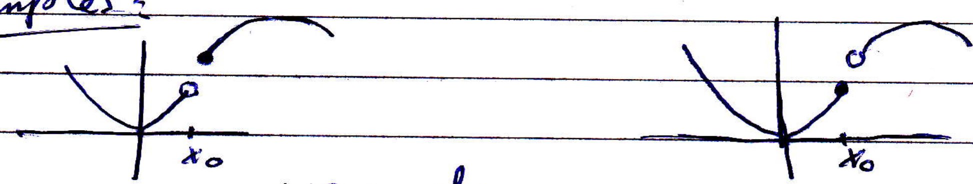
equivalently: [lemma 2.42] page 43
 f is l.s.c. at x if $\forall \langle x_n \rangle$ in X, $x_n \to x$ and $f(x_n) \to r$
 $\Rightarrow r \ge f(x)$.

- u.s.c. if $\forall c \in \mathbb{R}$, the set $\{x \in X : f(x) \ge c\}$ closed
 or $\{x \in X : f(x) < c\}$ open.

equivalently:
 f is u.s.c. at x if $\forall x_n \to x$, $f(x_n) \to r$ then
 $r \le f(x)$.

f is l.s.c. at x, $\forall x$ (u.s.c. at x, $\forall x$) then it is called l.s.c. on X (u.s.c. on X)

Examples:

1. 

u.s.c. at x_0 l.s.c. at x_0

2. $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ is u.s.c. not l.s.c. at 0

3. $f(x) = [x]$ is the greatest integer less than or equal to x
 – floor function – is u.s.c. at x, $\forall x \in \mathbb{Z}$
 $f(x) = [x]$ – ceiling function – is l.s.c.

- f continuous $=$ l.s.c. $+$ u.s.c.
- u.s.c. (or l.s.c.) not meaning left (or right continuity)

Examples:

Def • N_x neigh. of x in T.p. (X,τ) is
$N_x = \{U \in \tau : x \in U\}$

• In top.sp. (X,τ), $\tau' \subset \tau$ (τ collection of open set) is called base for τ if every open set is union of members of τ'.

Ex The family of open balls $\{B(x,r) : x\}$ is base of any metric space.

• Local base: a collection $V_x \subset N_x$ is local base if every neigh. of x contains an element of V_x

• The union of all local bases is a base for

a topology. (check).

Def • let X be T.v.s., $E \subset X$, E is bounded
$\forall V \in N_0$, $\exists s \in \mathbb{R}$, $E \subset tV$, $\forall t > s$

• E is called balanced if $\forall \alpha \in F$, $|\alpha| < 1$ have $\alpha E \subset E$.

• symmetric if $x \in E \Rightarrow -x \in E$

Def $\forall a \in X$, translation operator $T_a : X \to X$,
$T_a(x) = x + a$
$\forall \alpha \neq 0$, def multiplication operator $M_\alpha : X \to X$,
$M_\alpha(x) = \alpha x$.

Th. Both T_a, M_α are homeomorphism on X onto X.

proof: ⊛ bij (from vector space axioms)
⊛ T_{-a}, $M_{1/\alpha}$ their inverses.
⊛ T_a, M_α, T_{-a}, $M_{1/\alpha}$ are cont. (by def of T.V.S.

* Def// A seminorm on vector space X is the function $\rho: X \to \mathbb{R} \ni \rho(x+y) \leq \rho(x)+\rho(y)$
$\rho(\alpha x) = |\alpha| \rho(x)$

- The family P of all seminorms on X is called Separating (iff) $\forall x \neq 0 \in X, \exists \rho \in P \ni \rho(x) \neq 0$

- different between a seminorm and a norm is that the first doesnot separate between points "$\rho(x-y)=0$ does not gurantee $x=y$".
- While single seminorm doesnot separate between point, a family of seminorms might do so.

- $A \subset X$, A is called absorbing (radical) if $\forall x \in X, \exists \alpha_0 > 0 \ni \alpha x \in A$, $\forall |\alpha| \leq \alpha_0$. Some multiple of A containing the line segment between 0 & x

- Is $0 \in$ Balanced or Absorbing? Yes.
- any closed unit ball is Absorbing.

Def// The Minkowski functional of A is
$\rho_A(x) = \inf\{\alpha > 0 : x \in \alpha A\}$, A subset of X
$\inf \phi = \infty$
Lemma 5.49 — pag (192) without proof

Lemma 5.50 =

Def// Star-shaped at zero = if it includes the line segments joining each of its points with zero
i.e. $\forall x \in A$ and $0 \leq \alpha \leq 1$, $\alpha x \in A$.

Star-shaped at P = ??

Albundi 5.5

Cor E open subset of $\mathfrak{X}$ (T.V.S.) $\Leftrightarrow$ $a+E$ open

Seperally: $\mathcal{N}_a = a + \mathcal{N}_0$. (check)

Def let $\mathfrak{X}$ be a T.V.S.

- $\mathfrak{X}$ is locally convex if there exists a local base at 0 whose member are convex.
- $\mathfrak{X}$ local bounded if 0 has a bounded neigh.
- $\mathfrak{X}$ local compact if 0 has a neigh. whose closure is compact.
- $\mathfrak{X}$ is metrizable if it is compatible with τ which generated by open balls. (metric Top.)
- $\mathfrak{X}$ is F-space if its topology is complete metrizable

  Ex every Banach space is F-space.

- $\mathfrak{X}$ is Frechet space if it's locally convex F-space.

Example

let $\mathfrak{X} = \mathbb{R}$ (vectorspace over $\mathbb{R}$)

and the base : $\mathcal{B} = \{[a,b) : a<b\}$

Show that $(\mathbb{R}, \tau)$ is not topological vector sp.

- Every T.V.S is regular Top. sp. & $\forall V$ neigh of 0, $\exists U$ neigh of 0 $\ni$ $\bar{U} \subset V$
- A T.V.S. $\mathfrak{X}$ is T_0 - Top. $\Leftrightarrow$ $\mathfrak{X}$ is Hausdorff $\Leftrightarrow$ $\cap V(0) = \{0\}$

* let $\mathfrak{X}$ be a T.V.S. and A, B subset of $\mathfrak{X}$

Then :

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- $\bar{A} = \prod_{V \in N_0} (A+V)$
- $\bar{A} + \bar{B} \subset \overline{A+B}$
- If A linear subspace then so is $\bar{A}$
- If A convex so is $\bar{A}$ and A°
- If A is balanced so is $\bar{A}$
- If A is balanced & $0 \in B^\circ$ then B° is balanced
- If A is bounded then $\bar{A}$ is bounded.

Exercise 1. let $A = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le |z_2|\}$
Show that A is balanced but A° is not.
→ → حل التمرين حسب الرغبة

2. $B[0,1]$ in $\mathbb{R}^2$, $0 = (0,0)$ is balanced (closed unit ball) + convex

3. in $\mathbb{C}$, $S = \{a+ib : |a| \le 1, |b| < 1\}$, S is convex symmetric but not balanced since
if $\alpha = (1+i) \Rightarrow \alpha \cdot (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) = \sqrt{2}\, i$, $|\sqrt{2}| > 1$
$\Rightarrow \alpha z \notin S$.

4. $A = \{(x,y) \in \mathbb{R}^2 : x=0, y=0\}$ is balanced not convex.
axises only

- Examples (without prove)
  - If A, B bounded subset of T.V.S then $A+B$ is bounded
  - If A, B are compact so is $A+B$
    - If A compact & B closed $\Rightarrow A+B$ closed.
  - Example for A, B are closed but $A+B$ not:

let $A = \{(x,y) : x > 0, y \ge \frac{1}{x}\}$

$B = \{(x,y) : x < 0, y \ge \frac{-1}{x}\}$ (check)

$A+B = \{(x,y) : y > 0\}$

- $f(x) = \begin{cases} 1 & x<1 \\ 2 & x=1 \\ 1/2 & x>1 \end{cases}$ is u.s.c. at $x=1$ but not left or right.

There $\lim\limits_{x\to 1^-} f(x) = 1 \neq 2$ but $\lim\limits_{x\to 1^+} f(x) = 1/2 \neq 2$

- $f(x) = \begin{cases} \sin(1/x) & x\neq 0 \\ 1 & x=0 \end{cases}$ is u.s.c. at $x=0$

$\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$ not exists.

- f is l.s.c. $\Leftrightarrow$ $-f$ is u.s.c.
- $\mathcal{X}$ is vector space C convex subset of $\mathcal{X}$

$f: C \to [-\infty, \infty]$ convex, strictly, concave, strictly concave if:

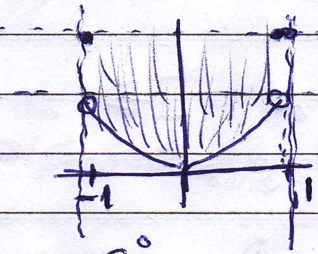
$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

$f(\quad) <$ " "

$-f$ is convex , $-f$ strictly convex.

Ex// • let $C = \{x : -1 \leq x \leq 1\}$

$f(x): C \to \mathbb{R}$ $f(x) = \begin{cases} x^2 , & |x|<1 \\ 2 , & |x|=1 \end{cases}$



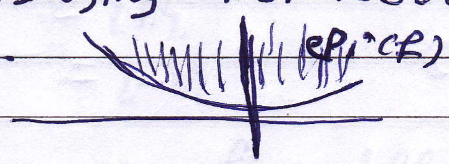
f is convex over C but continuous only on C°

- $f(x) = e^x$ convex function ⊕ l.s.c. on $\mathbb{R}$?
- $f(x) = \begin{cases} x^2 , & x>0 \\ \infty , & x\leq 0 \end{cases}$ convex not l.s.c. on $\mathbb{R}$?

Def// $f: \mathcal{X} \to [-\infty, \infty]$, $epi(f) \subset \mathcal{X} \times \mathbb{R}$

$epi(f) = \{(x, \alpha) : x\in \mathcal{X}, \alpha \geq f(x)\}$

= the set of all points lying on or above the graph of f.



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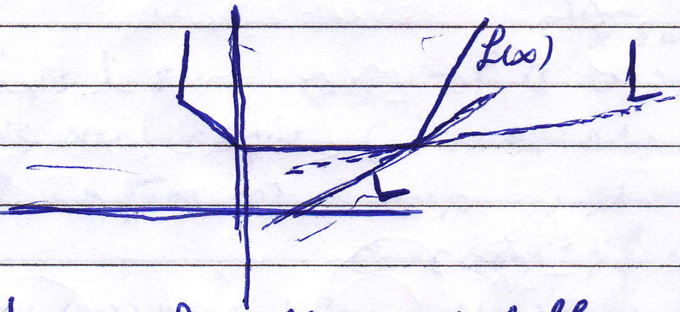
→ Subderivative, Subgradient, Subdifferential

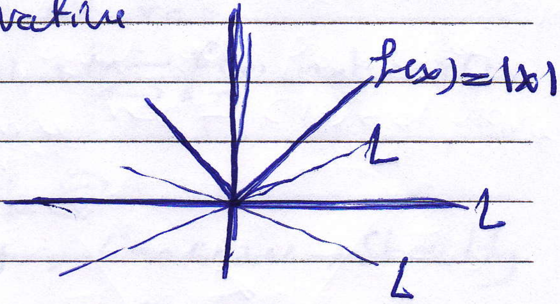
These concepts generlize the derivative functions which are not differentiable such as,

I open interval, $f: I \to \mathbb{R}$, $f(x) = |x|$

f is not diff at $x_0 = 0 \Rightarrow L$ is touching $f(x)$ at x_0 and is under the graph

$c =$ the slop of L is called subderivative





→ note that the subdifferential of a function is set-valued.

Def معرف العالم التالي:

A subderivative of $f: I \to \mathbb{R}$ at $x_0 \in I$ is a real number $c \ni \frac{f(x) - f(x_0)}{x - x_0} \geq c$ (or $f(x) - f(x_0) \geq c(x - x_0)$)

$\forall x \in I$. the set of all c is called subdifferenal $= \partial f$

Remark (1) Let f convex function on then the set of subderivative at $x_0 \in [a,b] \neq \emptyset$,
and $= [a,b]$, a,b are one sided limits at x_0

Example $f(x) = |x|$ – convex $x_0 = 0$, $[-1,1]$

The subdifferenal at $x_0 = [-1,1]$

and, at $x > x_0 = \{1\}$

at $x < x_0 = \{-1\}$ check.

(2) if $\partial f|_{x_0} =$ singleton then f is differentable at x_0

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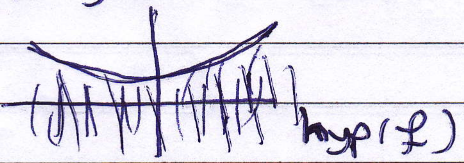
$hyp(f) \subset \mathcal{X} \times \mathbb{R}$

$hyp(f) = \{(x,\alpha) : x \in \mathcal{X},\ \alpha \in \mathbb{R},\ \alpha \le f(x)\}$ is convex

= the set of all points lying on or below the graph of f



→ Check :-

- if f convex then αf convex, $\forall \alpha > 0$
- f_i convex $i = 1,2 \dots n$, $\sum f_i$ convex
- $f_i \to f$ pointwise & f_i convex, $\forall i$ then f convex
- $f(x) = \sup f_i(x)$, f_i convex then f is convex.

Lemma 5.41 ✓ ! ✓ pag 187

- Theorem 5.42 (local cont. of convex function)
  proof ✓
- Th. 5.43 Global Cont of convex function
  proof ✓
- Th. 5.44 (Cont convex at interior → local Lip. cont.)
  proof ✓

The Geometrically Consequence of Hahn-Banach Th. is having the following sets:

$\mathcal{X}$ v.s. ⊕ $f \neq 0$ linear functional ∃ $\alpha \in \mathbb{R}$

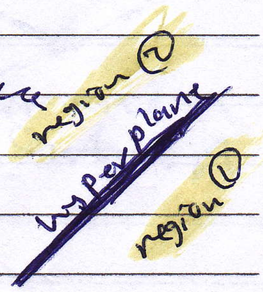
$[f = \alpha] = \{x : f(x) = \alpha\}$ (hyperplane)

$[f > \alpha] = \{x : f(x) > \alpha\}$ strictly half space

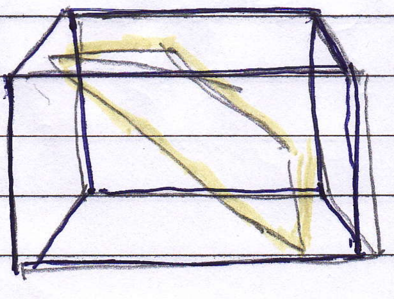
$[f < \alpha] = \{x : \quad < \alpha\}$ " "

$[f \le \alpha]$ weak halfspace

$[f \ge \alpha]$ " "



- polyhedron $S = \bigcap$ finite weak halfspace.



the topology generated by $\{P_x : x \in \mathfrak{X}\}$ on $\mathfrak{X}'$
where $P_x(f) = |f(x)|$, $f \in \mathfrak{X}'$.

Support Point

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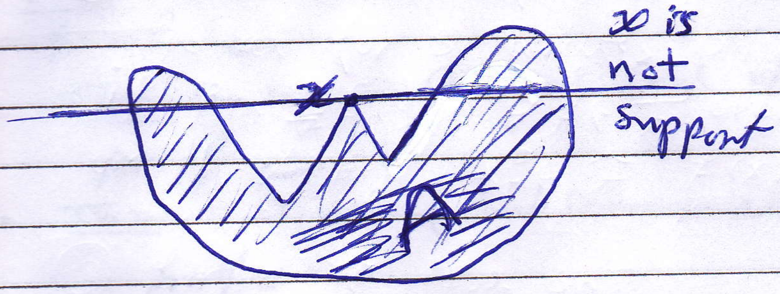
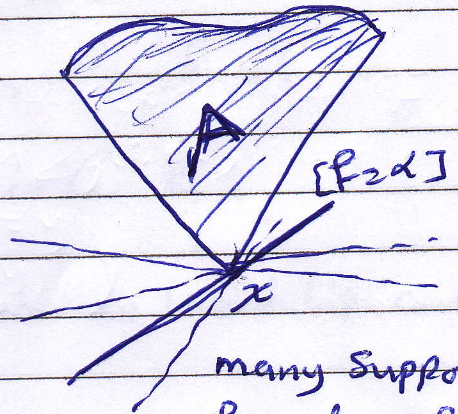
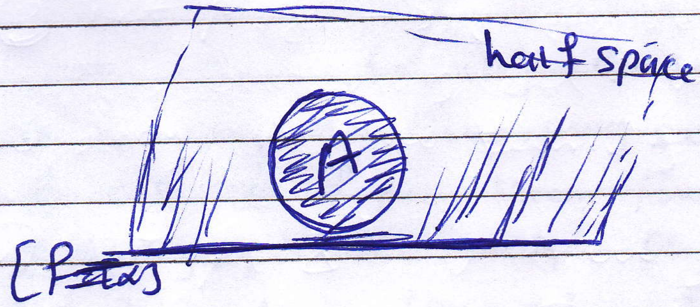
A support ......

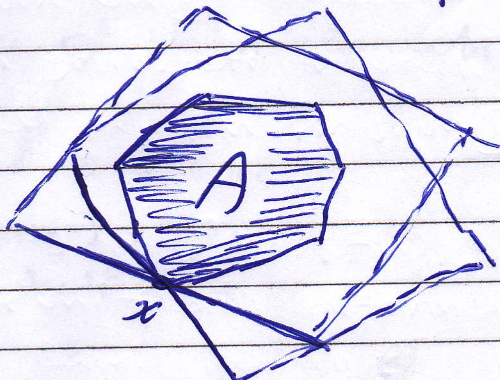
i.e. $[f = \alpha]$ supporting a set A at max(or min) x if $[f = \alpha]$ through at x and all points of the set A are in the same half closed plane of $[f = \alpha]$





many support function f





more than one half space contain A

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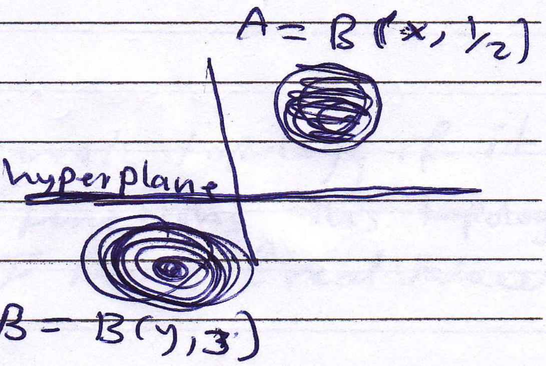
Separates ~~two~~ sets A & B pag 198

properly separates A and B

$x = (1,1) \in \mathbb{R}^2$

$y = (-20,-10)$



lemma 5.55 without proof

$[f = \alpha]$ in T.V.S is closed $\Leftrightarrow$ f cont.

dense $\Leftrightarrow$ f discont.

Example 5.56 pag 199

lemma (5.57) without proof

f separates A & B $\Rightarrow$ f bounded above (or below)

$\Rightarrow$ A linear subspace $\Rightarrow f|_A = 0$.

Th 5.61 if A, B convex then A, B can be properly separated by nonzero linear functional. without proof ←

Th. 5.67 without proof.

Weak Topology:- let $\mathcal{X} \neq \emptyset$ and $\{(Y_i, \tau_i)\}_{i \in I}$ be a family of Top spaces $\forall i$

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and $\forall i$, $f_i : \mathcal{X} \to Y_i$ be a function. The weak topology (initial top.) on $\mathcal{X}$ generated by the family $\{f_i\}_{i \in I}$ is the weakest top. on $\mathcal{X}$ that makes f_i continuous functions. This top. generated by the family of sets $\{f_i^{-1}(V) : i \in I, V \in \tau_i\}$

Another subbase of this topology is

$\{f_i^{-1}(V) : i \in I,\ V \in S_i\}$, S_i is subbase of τ_i.

Remark

① As special case of weak topology if it is generated by real valued functions this topology denoted by $\sigma(X, F)$, F family of real valued functions. Its subbase is

$$\{U(f,x,\epsilon) : f \in F,\ x \in X,\ \epsilon > 0\}$$ where
$$U(f,x,\epsilon) = \{y \in X : |f(y) - f(x)| < \epsilon\}$$

② The weak Top. $\sigma(X, F)$ is Hausdorff iff F is total (or separates points in X, i.e., if $f(x) = f(y)\ \forall f \in F$ then $x = y$ or if $\forall x \neq y$, $\exists f \in F$ $f(x) \neq f(y)$)

③ Let $p : X \to \mathbb{R}$ is seminorm on X then p is uniformly cont. We can generate a weak topology by the family of all seminorms on X.

④ Let $f \in X'$, then $\rho : X \to \mathbb{R}$, $\rho(x) = |f(x)|$ is seminorm on X. And the unit ball determined by ρ is $\{x \in X : \rho(x) < 1\}$. The family of all f linear functional on $X = X'$ generates a weak Topology on X denoted by $\sigma(X, X')$ this topology will be Hausdorff. Also any T.v.s. is L.C. $\Longleftrightarrow$ its topology generated by seminorms.

Note that

1. this topology also called Seminorm top.
2. weak* topology $\sigma(X', X)$ is the

Def if $U \subseteq \mathcal{X}$ ($\mathcal{X}$ is T.V.S) & f convex function on U a functional $f' \in \mathcal{X}'$ is called subgradient at x_0 in U if $f(x) - f(x_0) \geq f^{-}(x - x_0)$

The set of all subgradient at x_0 = subdifferential at x_0

$= \partial f(x_0)$.

$\Rightarrow \partial f(x_0)$ has many properties - - - - -

Example for f which has no subgradient i.e. $\partial f = \phi$??

$$f : \mathbb{R} \to [-\infty, \infty] \ni f(x) = \begin{cases} -\sqrt{1-|x|^2}, & |x| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

Then f convex not subdiff everywhere (i.e. differen——).

Remark

① $\partial f(x) = \nabla f(x)$ if f differentiable at x

② If f is real function then

$\partial f(x) = [f'_{+}(x), f^{*}_{-}(x)]$, $\forall x \in (\mathrm{dom}\, f)^{\circ}$

③ if α satisfies $f(x) \geq f(x_0) + \alpha(x - x_0)$ $\forall x \in \mathcal{X}$, then we say that f has a supporting hyperplane at x_0 with gradient α.

Albundi 5.5