Department of Mathematics\ Mathematical Analysis

Chapter 4

THE INFINITE REAL SERIES

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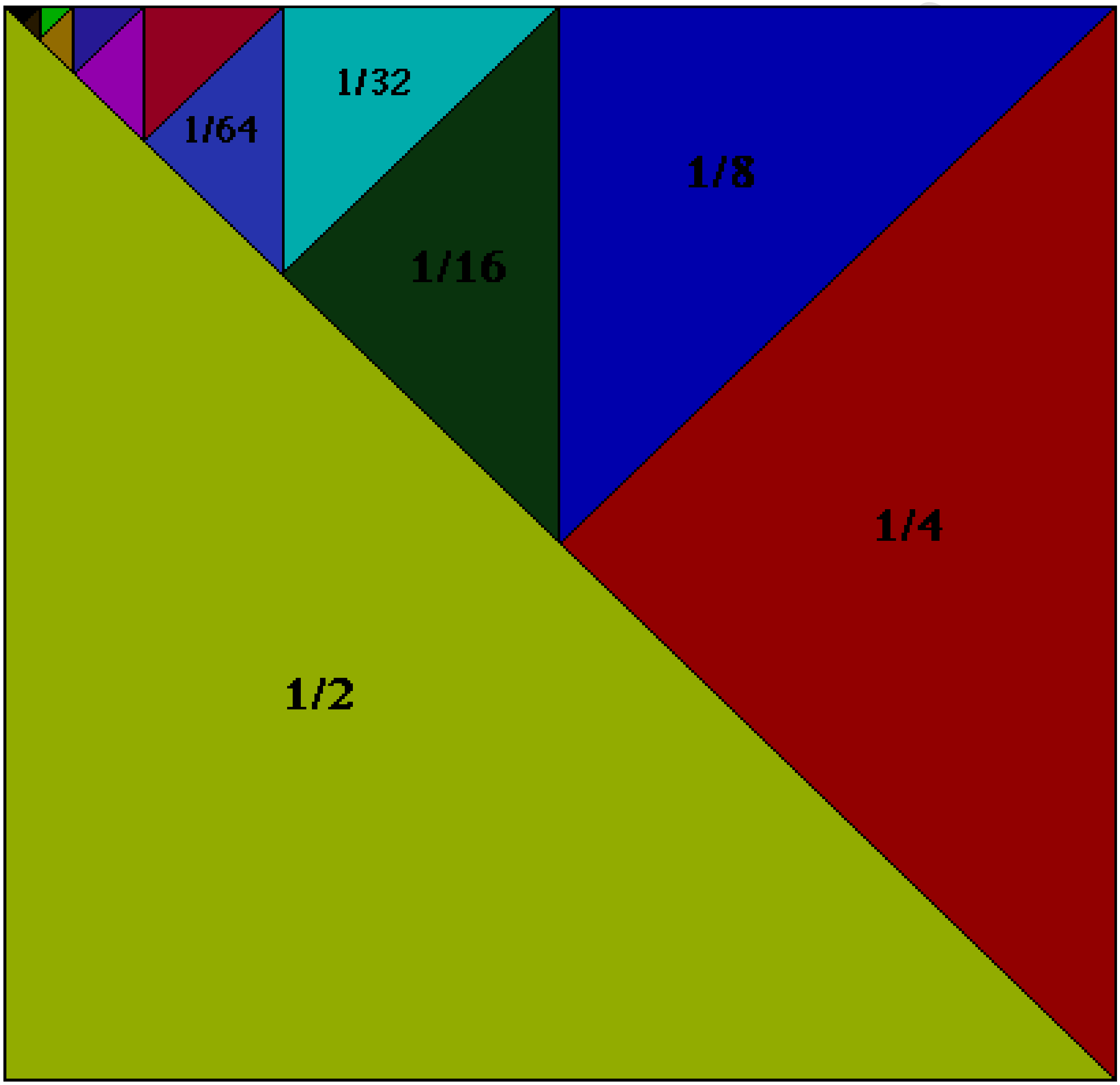

Definition

If $\langle x_n \rangle$ be a real sequence, then $x_1 + x_2 + x_3 + \dots$ is called infinite, and it is written as $\sum_{n=1}^{\infty} x_n$.

If the series of the form $x_1 + x_2 + x_3 + \dots + x_n$, then it is called a finite series and written as $\sum_{k=1}^{n} x_k$.

Definition

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series, the sequence $\langle s_n \rangle$ is called the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$, where:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$.$$
$$.$$
$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

Definition

1. An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be: -

i. Convergent if $\langle s_n \rangle$ is convergent

ii. Divergent if $\langle s_n \rangle$ is divergent

2. If $\langle s_n \rangle$ converges to s, then $\sum_{n=1}^{\infty} a_n$ =s.

Examples

Check the convergence of the following series: -

1. Let $a_n = 1, \forall n \Rightarrow \sum_{n=1}^{\infty} a_n = 1 + 1 + 1 + \dots$

$$s_1 = a_1 = 1$$
$$s_2 = a_1 + a_2 = 1 + 1 = 2$$
$$s_3 = a_1 + a_2 + a_3 = 1 + 1 + 1 = 3$$

So, $s_n = a_1 + a_2 + a_3 + \dots + a_n = 1 + 1 + 1 + \dots + 1 = n$

$\Rightarrow$ The sequence of partial sums is $\langle s_n \rangle = \langle n \rangle$ iinfiniteinfinites divergent, since it is unbounded

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

2. Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$a_1 = \frac{1}{1(1+1)} = \frac{1}{1(2)} = 1 - \frac{1}{2}$

$a_2 = \frac{1}{2(2+1)} = \frac{1}{2(3)} = \frac{1}{2} - \frac{1}{3}$

$a_3 = \frac{1}{3(3+1)} = \frac{1}{3(4)} = \frac{1}{3} - \frac{1}{4}$

$a_4 = \frac{1}{4(4+1)} = \frac{1}{4(5)} = \frac{1}{4} - \frac{1}{5}$

.

.

$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$

$s_1 = a_1 = 1 - \frac{1}{2}$

$s_2 = a_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3}$

$s_3 = a_1 + a_2 + a_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4}$

$s_4 = a_1 + a_2 + a_3 + a_4 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) = 1 - \frac{1}{5}$

Therefore, $s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 - \frac{1}{n+1}$.

$\Rightarrow$ the sequence of partial sums of $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is $\langle s_n \rangle = \left\langle 1 - \frac{1}{n+1} \right\rangle$ convergent, since

$\lim_{x\to\infty} s_n = \lim_{x\to\infty} (1 - \frac{1}{n+1}) = 1 \Rightarrow \langle s_n \rangle$ is a convergent sequence $\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

3. Harmonic series

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ divergent series

Proof:

$s_1 = a_1 = 1$

$s_2 = a_1 + a_2 = 1 + \frac{1}{2}$

$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3}$

.

.

$s_n = a_1 + a_2 + a_3 + \dots + a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

$s_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1}$

$s_{n+2} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$

.

.

$s_{n+n} = s_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$

Let m=2n

$$|s_m - s_n| = \left|(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}) - (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n})\right|$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n}$$

$$= n.\frac{1}{2n} = \frac{1}{2}$$

If $\varepsilon = \frac{1}{2} \Rightarrow |s_m - s_n| > \in$

So $\langle s_n \rangle$ is not a Cauchy sequence $\Rightarrow$ $\langle s_n \rangle$ is not a convergent sequence

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

4. Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$, where $a > 0$, r is called the base of the series. The sequence of partial sums is

$s_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

. $if\ |r| = 1$then $s_n = \pm a \pm a \ldots \pm a = \pm na \quad \Rightarrow \langle s_n \rangle = \langle \pm na \rangle$ is a divergent sequence.

$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent.

. $if\ |r| < 1 \Rightarrow s_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$\Rightarrow rs_n = ar + ar^2 + ar^3 + \ldots + ar^n$

$\Rightarrow s_n - rs_n = a - ar^n$

$\Rightarrow s_n(1 - r) = a(1 - r^n)$

$\Rightarrow s_n = \frac{a(1-r^n)}{(1-r)}$

Then when $n \to \infty \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \frac{a(1-0)}{(1-r)} = \frac{a}{1-r}$.

$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ is convergent

$if\ |r| > 1 \Rightarrow s_n = \frac{a(1-r^n)}{(1-r)}$. When $n \to \infty \Rightarrow r^n = \pm\infty \Rightarrow s_n \to \infty$.

$\Rightarrow s_n$ divergent $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent.

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} diverget, & if\ |r| \geq 1 \\ converget,\ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} & if |r| < 1 \end{cases}$$

Examples

1. $\sum_{n=1}^{\infty} a_n = 1 + \frac{7}{3} + \left(\frac{7}{3}\right)^2 + \left(\frac{7}{3}\right)^3 + \ldots$ *Geometric series*

$a = 1,\ r = \frac{7}{3},\ |r| = \left|\frac{7}{3}\right| = \frac{7}{3} > 1$

$\therefore \sum_{n=1}^{\infty} a_n\ is\ diverge$

$2.\sum_{n=1}^{\infty} a_n = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \ldots$ *Geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = 1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^4 + \ldots$$

$$a = 1,\ r = -\frac{2}{3},\ |r| = \left|-\frac{2}{3}\right| = \frac{2}{3} < 1$$

$$\therefore \sum_{n=1}^{\infty} a_n \text{ is converge and } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{1}{1+\frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$$

The relation between the convergence of the series $\sum a_n$ and the sequence $\langle a_n \rangle$:

Theorem (1)

If $\sum_{n=1}^{\infty} a_n$ convergent, then $\lim_{n\to\infty} a_n = 0$ $(i.e., \forall \varepsilon > 0, \exists k \in N \ni |a_n - 0| < \varepsilon \ \forall n > k)$

Proof

Suppose the $s_n = a_1 + a_2 + a_3 + \ldots + a_n$

$$\sum_{n=1}^{\infty} a_n \text{ convergent} \Rightarrow \langle s_n \rangle \text{ convergent} \Rightarrow \langle s_n \rangle \text{ cauchy sequence}$$

$$\Rightarrow \forall \varepsilon > 0,\ \exists k \in N \ni |s_m - s_n| < \varepsilon \ \ \forall n, m > k$$

$$let\ m = n + 1$$

$$\Rightarrow |s_m - s_n| < \varepsilon \Rightarrow |s_{n+1} - s_n| = |a_{n+1}| < \varepsilon \ \ \forall n > k$$

$$\Rightarrow |a_n| < \varepsilon \ \ \forall n > k \Rightarrow |a_n - 0| < \varepsilon \ \forall n > k$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Remark (1)

The converse of Theorem (1) is <u>not</u> necessarily true. For example:

$$\langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle \to 0$$

$$but \ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \ \ diverge.$$

Corollary (1)

$$If \ \lim_{n\to\infty} a_n \neq 0, \ then \ \sum_{n=1}^{\infty} a_n \ is \ diverge$$

Proof: Suppose that $\sum_{n=1}^{\infty} a_n$ converge

$$\Rightarrow \lim_{n\to\infty} a_n = 0 \ \ by\ Th.(1)$$

$$\Rightarrow \text{C!}$$

Remark (2)

The converse of Corollary (1) is not necessarily true. For example, let

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n-1})$

We see that $a_1 = \sqrt{1} - \sqrt{1-1} = 1$

$a_2 = \sqrt{2} - \sqrt{2-1} = \sqrt{2} - 1$

$a_3 = \sqrt{3} - \sqrt{3-1} = \sqrt{3} - \sqrt{2}$

…..

Therefore, $s_1 = a_1 = 1$

$s_2 = a_1 + a_2 = 1 + \sqrt{2} - 1 = \sqrt{2}$

$s_3 = a_1 + a_2 + a_3 = 1 + \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} = \sqrt{3}$

…….

So, $s_n = \sqrt{n}, \forall n \Rightarrow \langle s_n \rangle = \langle \sqrt{n} \rangle$ unbounded $\Rightarrow \langle s_n \rangle$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n-1})$ is divergent, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\sqrt{n} - \sqrt{n-1})$ is a convergent sequence since $\lim_{n\to\infty} (\sqrt{n} - \sqrt{n-1}) . \frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n}+\sqrt{n-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}} = 0$

Theorem (2)

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series and* $k \in R$, *then*:

1. $\sum_{n=1}^{\infty} (a_n + b_n)$ *convergent and* $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
2. $\sum_{n=1}^{\infty} k a_n$ *convergent and* $\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n$

Proof (1)

Let $\langle s_n \rangle$ be a sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\langle t_n \rangle$ be a sequence of partial sums of $\sum_{n=1}^{\infty} b_n$

Since $\sum_{n=1}^{\infty} a_n$ convergent, then $\exists s \in R \ni \sum_{n=1}^{\infty} a_n = s$ *and* $\langle s_n \rangle \to s \Rightarrow \lim_{n\to\infty} s_n = s.$

Since $\sum_{n=1}^{\infty} b_n$ convergent, then $\exists t \in R \ni \sum_{n=1}^{\infty} b_n = t$ *and* $\langle t_n \rangle \to t \Rightarrow \lim_{n\to\infty} t_n = t.$

$\lim_{n\to\infty}(s_n+t_n)\to s+t$, but $\langle s_n+t_n\rangle$ is the sequence of partial sums of $\sum_{n=1}^{\infty}(a_n+b_n)$

$$\Rightarrow \sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n=s+t$$

Proof (2)

Let $\langle s_n\rangle$ be a sequence of partial sums of $\sum_{n=1}^{\infty}a_n$. Since $\sum_{n=1}^{\infty}a_n$ convergent, then

$$\exists s\in R \ni \sum_{n=1}^{\infty}a_n=s \text{ and } \langle s_n\rangle\to s\Rightarrow \lim_{n\to\infty}s_n=s.$$

$$\lim_{n\to\infty}ks_n=k\lim_{n\to\infty}s_n=ks\Rightarrow\langle ks_n\rangle\to ks$$

$$\Rightarrow \sum_{n=1}^{\infty}ka_n=ks=k\sum_{n=1}^{\infty}a_n$$

$$\Rightarrow \sum_{n=1}^{\infty}ka_n=k\sum_{n=1}^{\infty}a_n$$

Exercise

Give an example of two divergent series, but their sum is a convergent series.

Suppose that

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{n} \text{ and } \sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{-1}{n}$$

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}0=0 \text{ convergent}\to 0$$

Tests for series with positive terms

Theorem (3) **Comparison test**

If $0\le a_n\le b_n$ for all $n\in N$, then

a. $\sum_{n=1}^{\infty}b_n$ *convergent* $\Rightarrow \sum_{n=1}^{\infty}a_n$ convergent.

b. $\sum_{n=1}^{\infty}a_n$ *divergent* $\Rightarrow \sum_{n=1}^{\infty}b_n$ divergent.

Proof (a)

Let $\langle s_n\rangle$ be a sequence of partial sums of $\sum_{n=1}^{\infty}a_n$ and $\langle t_n\rangle$ be a sequence of partial sums of $\sum_{n=1}^{\infty}b_n$.

$\because 0 \le a_n \le b_n \Rightarrow s_n = a_1 + a_2 + a_3 + ... + a_n$

$\le b_1 + b_2 + b_3 + ... + b_n$

$= t_n$

$\Rightarrow s_n \le t_n, \ \forall n$

$\because \sum_{n=1}^{\infty} b_n$ *convergent* $\Rightarrow \langle t_n \rangle \to t$ *as* $n \to \infty$.

$\because b_n \ge 0 \Rightarrow \langle t_n \rangle$ increasing seq. and $t_n \le t, \forall n$

$\because s_n \le t_n, \forall n \Rightarrow s_n \le t, \forall n \Rightarrow \langle s_n \rangle$ *is bounded*

$\Rightarrow \langle s_n \rangle$ *is bounded and increasing (monoton)*

$\Rightarrow \langle s_n \rangle$ *is convergent seq.*

$\Rightarrow \sum_{n=1}^{\infty} a_n$ *convergent series.*

Proof (b)

Suppose that $\sum_{n=1}^{\infty} b_n$ convergent series $\Rightarrow$ By (a) $\sum_{n=1}^{\infty} a_n$ convergent series $\Rightarrow C!$

Theorem (4) P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \ p > 0 \ , \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ..$$

There are three cases :-

i. $p = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n}$ *Harmonic* $\Rightarrow$ *diverge*

ii. *if* $0 < p < 1$ *we comparsion with* $\sum_{n=1}^{\infty} \frac{1}{n}$

$\because n^p < n, \forall n \Rightarrow \frac{1}{n^p} > \frac{1}{n}$ (since $p < 1$)

$\Rightarrow a_n = \frac{1}{n} < \frac{1}{n^p} = b_n, \forall n$ *and* $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ *divergent.*

$\therefore$ *by Thm* (3) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ *divergent if* $0 < p < 1$.

$iii.if\ \ p>1$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}+\ldots$$

$$<1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\ldots$$

$$=1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\frac{16}{16^p}+\ldots$$

$$=1+\frac{2}{2(2^{p-1})}+\frac{4}{4(4^{p-1})}+\frac{8}{8(8^{p-1})}+\frac{16}{16(16^{p-1})}+\ldots$$

$$=1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{4}\right)^{p-1}+\left(\frac{1}{8}\right)^{p-1}+\left(\frac{1}{16}\right)^{p-1}+\ldots$$

$$=1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2^2}\right)^{p-1}+\left(\frac{1}{2^3}\right)^{p-1}+\left(\frac{1}{2^4}\right)^{p-1}+\ldots$$

$$=1+\left(\frac{1}{2}\right)^{p-1}+\left(\left(\frac{1}{2}\right)^{p-1}\right)^2+\left(\left(\frac{1}{2}\right)^{p-1}\right)^3+\left(\left(\frac{1}{2}\right)^{p-1}\right)^4+\ldots$$

This is a geometric series with $a=1, r=\left(\frac{1}{2}\right)^{p-1}$

$\because p>1\Rightarrow p-1>0\Rightarrow r=\frac{1}{2^{p-1}}<1$ $\Longrightarrow$ It is convergent.

Since each term in it is larger than the corresponding term of $\sum_{n=1}^{\infty}\frac{1}{n^p}$.

By Theorem (3), $\sum_{n=1}^{\infty}\frac{1}{n^p}$ convergent, if p>1.

$$\Rightarrow p-series\begin{cases} converge & if\ \ p>1 \\ diverge & if\ 0<p\le 1\end{cases}$$

Examples

$1.\sum_{n=1}^{\infty}\frac{1}{2n^2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2},\ p=2>1$

$\Rightarrow p-series, p=2>1\Rightarrow convergent.$

$2.\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}, p<1$

$\Rightarrow p-series, p=\frac{1}{2}<1\Rightarrow divergent.$

Theorem (5) **Comparison Limit Test**

$$Let \sum_{n=1}^{\infty} a_n \ and \sum_{n=1}^{\infty} b_n \ be\ positive-term\ series\ such\ that:-$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

$$then \sum_{n=1}^{\infty} a_n \ convergent \Leftrightarrow \sum_{n=1}^{\infty} b_n \ convergent.$$

Without proof

Theorem (6) **Ratio Test**

$$Let \sum_{n=1}^{\infty} a_n \ be\ a\ series, a_n > 0, \forall n \ and \ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = b,$$

$1. if\ b < 1 \Rightarrow \sum a_n \ convergent,$

$2. if\ b > 1 \Rightarrow \sum a_n \ divergent,\ and$

$3. if\ b = 1 \Rightarrow no\ \inf ormations$

Without proof

Theorem (7) **Root Test**

$$Let \sum_{n=1}^{\infty} a_n \ be\ a\ series, a_n > 0, \forall n\ if\ \exists b \in R \ni \lim_{n\to\infty} \sqrt[n]{a_n} = b,$$

$1. if\ b < 1 \Rightarrow \sum a_n \ convergent,$

$2. if\ b > 1 \Rightarrow \sum a_n \ divergent,\ and$

$3. if\ b = 1 \Rightarrow no\ \inf ormations$

Without proof

Remark: Is there a convergent series in one test but not in another test?

The Number e

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

Remark: The series $\sum_{n=0}^{\infty}\frac{1}{n!}$ is a convergent series.

Since

$$s_n=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\frac{1}{n!}$$

$$=1+1+\frac{1}{2\times1}+\frac{1}{3\times2\times1}+\frac{1}{4\times3\times2\times1}+\ldots\frac{1}{n!}$$

$$=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\ldots\frac{1}{n!}$$

$$<1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\frac{1}{2^{(n-1)}}$$

$$=1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\frac{1}{2^{(n-1)}}$$

This is a geometric series with $a=\frac{1}{2}, r=\frac{1}{2}$ and its sum is $1=\frac{1/2}{1/2}$

$$\therefore s_n<1+1+1=3$$

$$\Rightarrow s_n<3\Rightarrow\langle s_n\rangle \textit{ bounded and increasing}\Rightarrow\langle s_n\rangle \textit{ converge}\Rightarrow\sum\frac{1}{n!}\textit{ converge.}$$

Theorem (8) $e=\lim_{n\to\infty}(1+\frac{1}{n})^n$

Proof:

$$Let\ s_n=\sum_{k=0}^{n}\frac{1}{k!}=1+1+\frac{1}{2}+\frac{1}{3!}+\ldots+\frac{1}{n!}$$

$$and\ t_n=(1+\frac{1}{n})^n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\ldots+\left(\frac{1}{n}\right)^n$$

$$\therefore t_n=1+1+\frac{1}{2!}\left(\frac{n(n-1)}{n^2}\right)+\frac{1}{3!}\left(\frac{n(n-1)(n-2)}{n^3}\right)+\ldots+\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(\frac{(n-1)}{n}\right)+\frac{1}{3!}\left(\frac{(n-1)(n-2)}{n^2}\right)+\ldots+\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n^n}$$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$ [since each term from the previous step t_n is multiplied by a number less than 1, therefore

$$\therefore t_n \le s_n$$

$$when\ n \to \infty \Rightarrow \lim_{n\to\infty} t_n \le \lim_{n\to\infty} s_n = e \Rightarrow \lim_{n\to\infty} t_n \le e \quad ....(1)$$

$$if\ n \ge m$$

$$t_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{2!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+..+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{m-1}{n}\right)+...+\frac{1}{n^n}$$

$$\ge 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{2!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+..+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{m-1}{n}\right)$$

Since we cut the terms at the term m.

Now, fix m and let $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} t_n \ge 1+1+\frac{1}{2!}+\frac{1}{3!}+..+\frac{1}{m!} = s_m$$

$$\Rightarrow \lim_{n\to\infty} t_n \ge e \ \ when\ \ m \to \infty \qquad ....(2)$$

$$by\ (1)\ and\ (2) \Rightarrow \lim_{n\to\infty} t_n = e$$

$$\Rightarrow e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

Example: Prove that e is an irrational number.

Solution:

Suppose that e is a rational number

$$\Rightarrow \exists m, n > 0 \ \ni e = \frac{m}{n}$$

$$\because e = \sum_{n=0}^{\infty}\frac{1}{n!} \Rightarrow s_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$$

$$e - s_n = \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+...$$

$$= \frac{1}{(n+1)!}+\frac{1}{(n+2)(n+1)!}+\frac{1}{(n+3)(n+2)(n+1)!}+...$$

$$= \frac{1}{(n+1)!}[1+\frac{1}{(n+2)}+\frac{1}{(n+3)(n+2)}+...\ ]$$

$$< \frac{1}{(n+1)!}[1+\frac{1}{(n+1)}+\frac{1}{(n+1)^2}+...\ ]$$

$$= \frac{1}{(n+1)!}\cdot\frac{n+1}{n} = \frac{1}{(n+1)n!}\cdot\frac{n+1}{n} = \frac{1}{n\cdot n!}$$

$$\Rightarrow 0 < e - s_n < \frac{1}{n\cdot n!}$$

$$\Rightarrow (e - s_n)n! < \frac{1}{n} \quad .....(1)$$

We can see that $(n!)e \in N$ since:

$$n!e = n!\frac{m}{n} = n(n-1)!\frac{m}{n} = (n-1)!m \in N$$

$$\text{and } \; n!s_n = n!(1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!})$$

$$= n!+n!+\frac{n!}{2!}+\frac{n!}{3!}+\ldots+1$$

Since $n \geq 1 \Rightarrow \exists$ natural number $(e-s_n)n!$ *such that* $0 < e-s_n < \frac{1}{n} < 1 \;\; (by\,(1)) \Rightarrow C!$

$\Rightarrow e$ is irrational number.

Example Is the series convergent or divergent?

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

Solution:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$$

$$\Rightarrow a_n = \left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}} = \lim_{n\to\infty}\left[\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\left(\frac{n+1}{n}\right)^{n}-1\right)\right]^{-1}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-1}\left(\left(\frac{n+1}{n}\right)^{n}-1\right)^{-1}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)^{-1}\left(\left(\frac{n}{n}+\frac{1}{n}\right)^{n}-1\right)^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} - \lim_{n\to\infty}(1)\right)^{-1}$$

$$= (1+0)^{-1}\left(e-1\right)^{-1} = 1(e-1)^{-1}$$

$$= \frac{1}{e-1} < 1$$

$$\Rightarrow \sum a_n \; convergent.$$

Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

Theorem (9) **(Alternating Series Test)**

The Series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent if:

1. $a_n > 0, \forall n$
2. $a_{n+1} \le a_n, \forall n$
3. $\lim_{n\to\infty} a_n = 0$

Proof

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

$$let \;\; n = 2m \; (even)$$

$$s_n = s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2m-1} - a_{2m})$$

$$\because a_{n+1} \le a_n \Rightarrow a_n - a_{n+1} \ge 0 \Rightarrow (a_i - a_{i+1}) \ge 0$$

$\Rightarrow \langle s_n \rangle$ non-increasing series ………(1)

Since

$$s_n = a_1 - (a_2 + a_3) - (a_4 - a_5) - \dots - a_{2m}$$

$$= a_1 - \left((a_2 + a_3) + (a_4 - a_5) + \dots + a_{2m}\right)$$

$$< a_1$$

$\Rightarrow \langle s_n \rangle$ *is bounded* ……..(2)

By (1) and (2) $\Rightarrow \langle s_n \rangle$ is a monotone and bounded sequence [by Theorem 3 in Chapter 3]

$\Rightarrow \langle s_n \rangle = \langle s_{2m} \rangle$ convergent sequence and $\lim_{n\to\infty} s_n = s$

Similarly, if n=2m+1 (odd)

$\Rightarrow \langle s_{2m+1} \rangle$ convergent to t.

$\because s - t = \lim_{m\to\infty} s_{2m} - \lim_{m\to\infty} s_{2m+1}$

$= \lim_{m\to\infty} (s_{2m} - s_{2m+1})$

$= \lim_{m\to\infty} a_{2m+1} = 0\,(by\,hyp\,(3))$

$\Rightarrow s = t \Rightarrow \langle s_n \rangle \to s\ and \sum_{n=1}^{\infty} (-1)^{n-1} a_n = s.$

Examples

Is the following series convergent?

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

Solution :−

$a_n = \frac{1}{n} > 0,\ a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n,\ \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\ convergent.$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$?

3. $\sum_{n=2}^{\infty} \frac{(-1)^n}{Ln(n)}$?

Absolute and Conditional Convergence

Definition (Absolutely Convergent)

A series $\sum a_n$ is called absolutely convergent if the associated series $\sum |a_n|$ convergent.

Definition (Conditionally Convergent)

A series $\sum a_n$ is called conditionally if the associated series $\sum a_n$ convergent but $\sum |a_n|$ divergent.

Examples

$$1.Let \sum a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

$$\Rightarrow \sum |a_n| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n} \quad Geometric\ series\ r = \frac{1}{2}, a = 1 \Rightarrow con.$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \quad absolutely\ convergent.$$

$$2.Let \sum a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$\Rightarrow \sum |a_n| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad Harmonic\ series \Rightarrow div.$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad not\ absolutely\ convergent.$$

$$Now, \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$a_n = \frac{1}{n+1},\ a_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad condtitionally\ convergent.$$

Theorem (10) If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

Proof:

Since $\sum a_n$ is absolutely convergent $\Rightarrow \sum |a_n|$ is convergent.

$\Rightarrow$ the seq. of partial sums $\langle s_n \rangle$ of $\sum |a_n|$ is convergent.

$\Rightarrow \langle s_n \rangle$ is Cauchy seq.

$\Rightarrow s_n = |a_1| + |a_2| + |a_3| + \ldots + |a_n|,\ s_m = |a_1| + |a_2| + |a_3| + \ldots + |a_m|$

$\because \langle s_n \rangle$ *caushy sequence* $\Rightarrow \forall \varepsilon > 0, \exists k \in N \ni |s_n - s_m| < \varepsilon,\ \forall n, m > k$

let $m = n + t,\ t \geq 1 \Rightarrow \left\| a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \ldots + |a_{n+t} \right\| < \varepsilon$

$\Rightarrow |a_{n+1} + a_{n+2} + \ldots + a_{n+t}| \leq |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \ldots + |a_{n+t}| < \varepsilon$ ....(*)

If $\langle t_n \rangle$ *is a seq.of partial sums of* $\sum a_n \Rightarrow t_n = a_1 + a_2 + \ldots + a_n$ *and*

$|t_n - t_m| = |a_{n+1} + a_{n+2} + \ldots + a_{n+t}| < \varepsilon,\ \forall n > k$ (*by* (*))

$\Rightarrow \langle t_n \rangle$ *cauchy seq.*

$\Rightarrow \langle t_n \rangle$ *convergent*

$\Rightarrow \sum a_n$ *convergent.*

General Test of Series

Since R is complete, this test states that the series is convergent if and only if the partial sums s_n is a Cauchy sequence. Also, it is used for complex series ₵ is complete

Theorem (11) (General Principle of Convergence)

The necessary and sufficient condition for the convergence of an infinite series

$\sum a_n \Leftrightarrow \forall \varepsilon > 0, \exists k \in N \ni |a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon, \forall n > k, p \geq 0.$

Proof: without proof

Product of series

If $\sum a_n, \sum b_n$ convergent series. Is $\sum a_n \cdot \sum b_n$ convergent?

We see that:

$$\sum a_n \cdot \sum b_n = (a_1 + a_2 + ...) \cdot (b_1 + b_2 + ...) =$$

$$= a_1(b_1 + b_2 + ...) + a_2(b_1 + b_2 + ...) + ...$$

In this form, the product series is difficult.

<u>Definition</u> (Cauchy product of series)

Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be two series and $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0$. We say that $\sum_{n=0}^{\infty} c_n$ is the product of $\sum a_n$ *and* $\sum b_n$.

Here, we compute some terms of the product:

$n = 0$

$$c_0 = \sum_{k=0}^{0} a_k \cdot b_{n-k} = a_0 b_0$$

$n = 1$

$$c_1 = \sum_{k=0}^{1} a_k \cdot b_{n-k} = a_0 b_1 + a_1 b_0$$

$n = 2$

$$c_2 = \sum_{k=0}^{2} a_k \cdot b_{n-k} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

Then

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = c_0 + c_1 + c_2 + ...$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + ...$$

We will give an example of two series that are conditionally convergent but whose product is a divergent series.

<u>Example</u>

Let $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. Is $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n$ convergent?

Solution

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + ...$$

is conditionally convergent and not absolutely convergent (check).

Suppose that $\sum c_n$ (Cauchy product) convergent

$$\Rightarrow \sum c_n = a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \ldots$$

$$= 1 - (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}) - (\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}) + \ldots$$

$$\Rightarrow c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

$$\because \sum c_n \ \textit{convergent} \Rightarrow \lim_{n\to\infty} c_n = 0$$

$$\Rightarrow |c_n - 0| = |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)}} < \varepsilon, \ \forall \varepsilon$$

$$(n-k+1)(k+1) = nk - k^2 + k + n - k + 1 = nk - k^2 + n + 1$$

$$= nk - k^2 + (n + 1 + \frac{n^2}{4} - \frac{n^2}{4}) = (\frac{n^2}{4} + n + 1) - (\frac{n^2}{4} - nk + k^2)$$

$$= \left(\frac{n+2}{2}\right)^2 - \left(\frac{n}{2} - k\right)^2 \le \left(\frac{n}{2} + 1\right)^2 = \left(\frac{n+2}{2}\right)^2$$

$$\Rightarrow \sqrt{(n-k+1)(k+1)} \le \frac{n+2}{2}$$

$$\Rightarrow \frac{1}{\sqrt{(n-k+1)(k+1)}} \ge \frac{2}{n+2} \Rightarrow |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}} \ge \sum_{k=0}^{n} \frac{2}{n+2} = (n+1)\frac{2}{n+2}$$

$$\textit{if } \ \varepsilon = \frac{2(n+1)}{n+2} \Rightarrow |c_n - 0| > \varepsilon \Rightarrow \text{C!}$$

$$\Rightarrow \sum \text{c}_\text{n} \ \textit{not convergent}.$$

Theorem (12)

Suppose that $\sum a_n$ converges absolutely and $\sum a_n = A, \sum b_n = B$ ($\sum b_n$ *convergent*). Then the Cauchy product $\sum c_n$ =AB.

Proof without

Exercise

let $\sum a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$, $\sum b_n = \sum_{n=0}^{\infty} \frac{1}{n(n+1)}$. *Find* $\sum c_n$ *and prove that* $\sum c_n$ *converges*

and $\sum c_n = \sum a_n \sum b_n$.

Power series

A power series (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \qquad \text{where } x \in R \text{ is called a power series in } x$$

or

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \dots \qquad \text{where } x \in R\text{, c constant.}$$

Note: You studied the Maclaurin series and the Taylor series in the previous stage, and their convergence.

Examples

Geometric series, exponential function and sine

The geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$

which is valid for $|x| < 1$, is one of the most important examples of a power series, as are the exponential function formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and the sine formula

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

<u>Definition</u>: Interval of convergence

1. The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the interval of x-values that can make a power series, a convergent series.
2. The center of the interval of convergence is always the center point c of the power series.
3. The radius of convergence is half of the length of the interval of convergence. If the radius of convergence is L , then the interval of convergence:$(c-L, c+L)$.

Examples

Find the convergence intervals:

$$1.\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-...$$

Solution

$$|a_n|=\left|(-1)^{n-1}\frac{x^n}{n}\right|=\frac{x^n}{n},\ |a_{n+1}|=\left|(-1)^n\frac{x^{n+1}}{n+1}\right|=\frac{x^{n+1}}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}/n+1}{x^n/n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|=\lim_{n\to\infty}\left|x\cdot\frac{n}{n+1}\right|=\lim_{n\to\infty}\frac{n}{n+1}|x|$$

$$=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x|=|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|<1\Rightarrow -1<x<1$$

[In the ratio test, for a series to be convergent, the limit must be < 1.]

Now, check when $x=\pm 1$

$$if\ x=1\Rightarrow\sum(-1)^{n-1}\frac{x^n}{n}=\sum(-1)^{n-1}\frac{1}{n}$$

$$but\ \sum(-1)^{n-1}\frac{1}{n}\ convergent\ (since\sum(-1)^{n-1}\frac{1}{n}\ Alternating\ series)$$

$$if\ x=-1\Rightarrow\sum(-1)^{n-1}\frac{x^n}{n}=\sum(-1)^{n-1}\frac{(-1)^n}{n}=\sum\frac{(-1)^{2n-1}}{n}$$

not absolute and not conditional convergent $\Rightarrow$ *divergent*

$\Rightarrow$ *the convergent interval is* $x\in(-1,1]$

$$2.\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-...$$

Solution

$$|a_n|=\left|(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right|=\frac{x^{2n-1}}{2n-1}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^{2n-1}}{2n-1}\right|}=\lim_{n\to\infty}\frac{(|x|^{2n-1})^{\frac{1}{n}}}{\sqrt[n]{|2n-1|}}$$

$$=\lim_{n\to\infty}\frac{|x|^{\frac{2n-1}{n}}}{\sqrt[n]{|2n-1|}}=\lim_{n\to\infty}\frac{|x|^{\frac{2n}{n}-\frac{1}{n}}}{\sqrt[n]{|2n-1|}}=\frac{|x|^2}{1}=x^2$$

In the root test, for a series to be convergent, the limit must be <1.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^{2n-1}}{2n-1}\right|}<1\Rightarrow x^2<1$$

$$\Rightarrow -1<x<1$$

Now, check when $x=\pm1$

$$if\ x=1\Rightarrow\sum(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=\sum(-1)^{n-1}\frac{1}{2n-1}$$

$$but\ \sum(-1)^{n-1}\frac{1}{2n-1}\ convergent\ (since\sum(-1)^{n-1}\frac{1}{2n-1}\ Alternating\ series)$$

$$if\ x=-1\Rightarrow\sum(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=\sum(-1)^{n-1}\frac{(-1)^{2n-1}}{2n-1}=\sum\frac{(-1)^{3n-2}}{2n-1}$$

$$it\ is\ convergent\ (since\sum(-1)^{3n-2}\frac{1}{2n-1}\ Alternating\ series)$$

$$\Rightarrow the\ convergent\ interval\ is\ x\in[-1,1]$$

Exercise

1. $Prove\ that\sum_{n=0}^{\infty}\frac{x^n}{n!}\ is\ cnvergent\ \forall x?$

2. $Prove\ that\sum_{n=0}^{\infty}n!\cdot x^n\ is\ cnvergent\ only\ x=0?$

3. $Find\ the\ convergece\ interval\ for\sum_{n=1}^{\infty}\frac{(2x-5)^n}{n^2}?\ (by\ y=2x-5)$

Properties of Power series

- If the two power series
  $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$
  and
  $g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$
  have same interval of convergence (-L, L) then
  $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$
- If the two power series
  $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$
  and
  $g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$
  have same interval of convergence (-L, L) then
  $f(x)g(x) = f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$
  where
  $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$
- A power series is a continuous function of x within its interval of convergence.
- A power series can be integrated term by term within the limits of (-L, L).
- Uniqueness of power series: If two power series have same radius of convergence, and converges to the same function then the power series are identical.