3.1 PLANE CURVES AND PARAMETRIC EQUATIONS

■ Plane Curves and Parametric Equations ■ Eliminating the Parameter
■ Finding Parametric Equations for a Curve

Plane Curves and Parametric Equations

We can think of a curve as the path of a point moving in the plane; the x- and y-coordinates of the point are then functions of time. This idea leads to the following definition.

PLANE CURVES AND PARAMETRIC EQUATIONS

If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations

$$x = f(t) \qquad y = g(t)$$

where $t \in I$, are **parametric equations** for the curve, with **parameter** t.

EXAMPLE 1 ■ Sketching a Plane Curve

Sketch the curve defined by the parametric equations

$$x = t^2 - 3t \qquad y = t - 1$$

SOLUTION For every value of t we get a point on the curve. For example, if $t = 0$, then $x = 0$ and $y = -1$, so the corresponding point is $(0, -1)$. In Figure 1 we plot the points (x, y) determined by the values of t shown in the following table.

t	x	y
-2	10	-3
-1	4	-2
0	0	-1
1	-2	0
2	-2	1
3	0	2
4	4	3
5	10	4

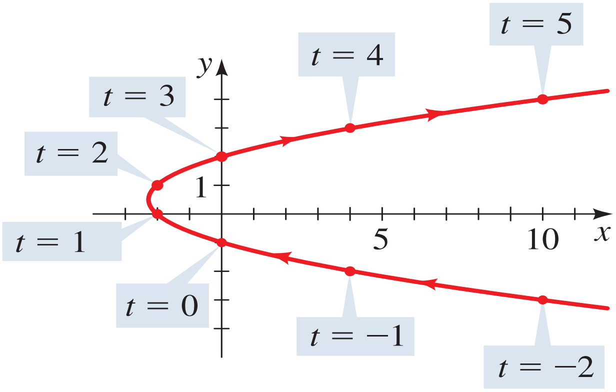


FIGURE 1

As t increases, a particle whose position is given by the parametric equations moves along the curve in the direction of the arrows.

If we replace t by $-t$ in Example 1, we obtain the parametric equations

$$x = t^2 + 3t \qquad y = -t - 1$$

The graph of these parametric equations (see Figure 2) is the same as the curve in Figure 1 but traced out in the opposite direction. On the other hand, if we replace t by $2t$ in Example 1, we obtain the parametric equations

$$x = 4t^2 - 6t \qquad y = 2t - 1$$

The graph of these parametric equations (see Figure 3) is again the same but is traced out "twice as fast." *Thus a parametrization contains more information than just the shape of the curve; it also indicates* how *the curve is being traced out.*

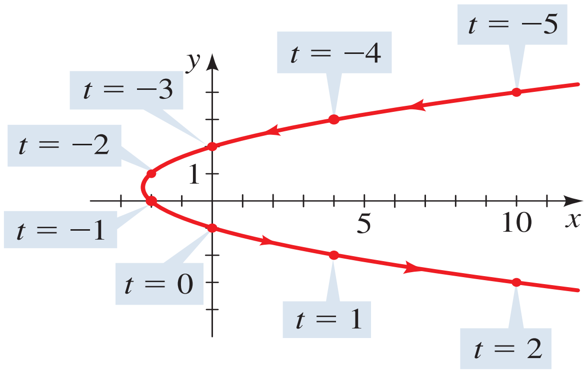


FIGURE 2 $x = t^2 + 3t,\ y = -t - 1$

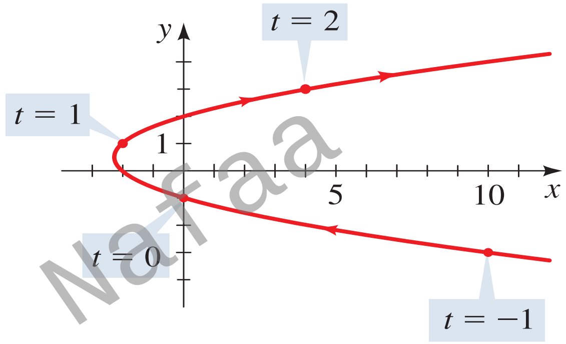


FIGURE 3 $x = 4t^2 - 6t,\ y = 2t - 1$

Eliminating the Parameter

Often a curve given by parametric equations can also be represented by a single rectangular equation in x and y. The process of finding this equation is called *eliminating the parameter*. One way to do this is to solve for t in one equation, then substitute into the other.

EXAMPLE 2 ■ Eliminating the Parameter

Eliminate the parameter in the parametric equations of Example 1.

SOLUTION First we solve for t in the simpler equation, then we substitute into the other equation. From the equation $y = t - 1$ we get $t = y + 1$. Substituting into the equation for x, we get

$$x = t^2 - 3t = (y + 1)^2 - 3(y + 1) = y^2 - y - 2$$

Thus the curve in Example 1 has the rectangular equation $x = y^2 - y - 2$, so it is a parabola.

EXAMPLE 3 ■ Modeling Circular Motion

The following parametric equations model the position of a moving object at time t (in seconds):

$$x = \cos t \qquad y = \sin t \qquad t \geq 0$$

Describe and graph the path of the object.

SOLUTION To identify the curve, we eliminate the parameter. Since $\cos^2 t + \sin^2 t = 1$ and since $x = \cos t$ and $y = \sin t$ for every point (x, y) on the curve, we have

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

This means that all points on the curve satisfy the equation $x^2 + y^2 = 1$, so the graph is a circle of radius 1 centered at the origin. As t increases from 0 to 2π, the point given by the parametric equations starts at $(1, 0)$ and moves counterclockwise once around the circle, as shown in Figure 4. So the object completes one revolution around the circle in 2π seconds. Notice that the parameter t can be interpreted as the angle shown in the figure.

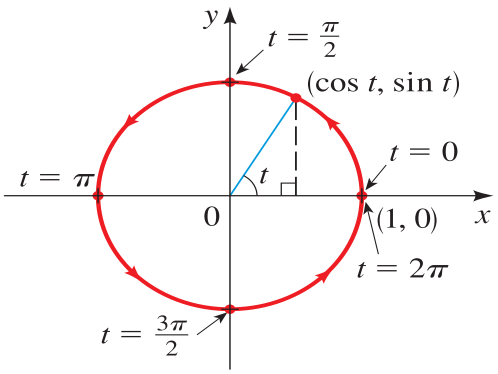


FIGURE 4

EXAMPLE 4 ■ Sketching a Parametric Curve

Eliminate the parameter, and sketch the graph of the parametric equations

$$x = \sin t \qquad y = 2 - \cos^2 t$$

SOLUTION To eliminate the parameter, we first use the trigonometric identity $\cos^2 t = 1 - \sin^2 t$ to change the second equation:

$$y = 2 - \cos^2 t = 2 - (1 - \sin^2 t) = 1 + \sin^2 t$$

Now we can substitute $\sin t = x$ from the first equation to get

$$y = 1 + x^2$$

so the point (x, y) moves along the parabola $y = 1 + x^2$. However, since $-1 \leq \sin t \leq 1$, we have $-1 \leq x \leq 1$, so the parametric equations represent only the part of the parabola between $x = -1$ and $x = 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, 2 - \cos^2 t)$ moves back and forth infinitely often along the parabola between the points $(-1, 2)$ and $(1, 2)$, as shown in Figure 5.

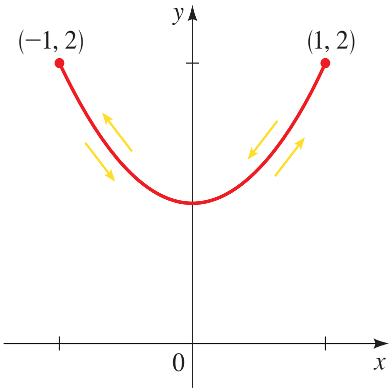


FIGURE 5

Finding Parametric Equations for a Curve

It is often possible to find parametric equations for a curve by using some geometric properties that define the curve, as in the next two examples.

EXAMPLE 5 ■ Finding Parametric Equations for a Graph

Find parametric equations for the line of slope 3 that passes through the point $(2, 6)$.

SOLUTION Let's start at the point $(2, 6)$ and move up and to the right along this line. Because the line has slope 3, for every 1 unit we move to the right, we must move up 3 units. In other words, if we increase the x-coordinate by t units, we must correspondingly increase the y-coordinate by $3t$ units. This leads to the parametric equations

$$x = 2 + t \qquad y = 6 + 3t$$

To confirm that these equations give the desired line, we eliminate the parameter. We solve for t in the first equation and substitute into the second to get

$$y = 6 + 3(x - 2) = 3x$$

Thus the slope-intercept form of the equation of this line is $y = 3x$, which is a line of slope 3 that does pass through $(2, 6)$ as required. The graph is shown in Figure 6.

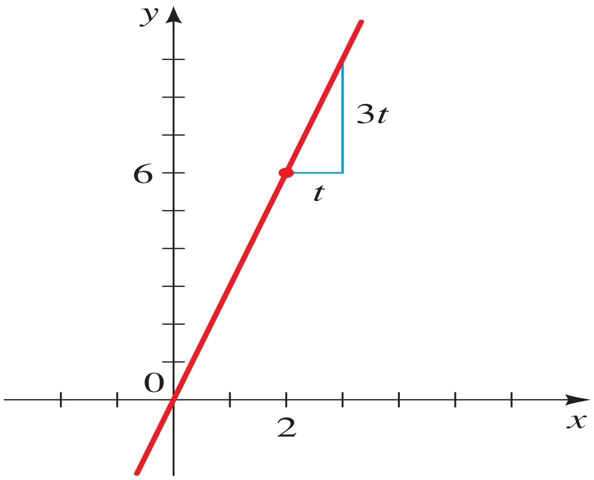


FIGURE 6

3.2 VECTORS IN TWO DIMENSIONS

Geometric Description of Vectors ■ Vectors in the Coordinate Plane

Geometric Description of Vectors

A **vector** in the plane is a line segment with an assigned direction. We sketch a vector as shown in Figure 1 with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point A is the **initial point**, and B is the **terminal point** of the vector $\overrightarrow{AB}$. The length of the line segment AB is called the **magnitude** or **length** of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus we write $\mathbf{u} = \overrightarrow{AB}$.

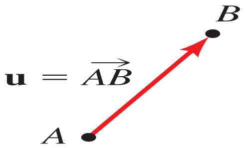


FIGURE 1

Two vectors are considered **equal** if they have equal magnitude and the same direction. Thus all the vectors in Figure 2 are equal. This definition of equality makes sense if we think of a vector as representing a displacement. Two such displacements are the same if they have equal magnitudes and the same direction. So the vectors in Figure 2 can be thought of as the *same* displacement applied to objects in different locations in the plane.

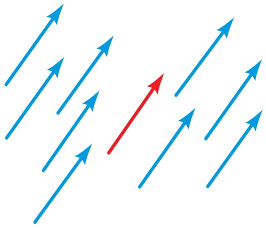


FIGURE 2

If the displacement $\mathbf{u} = \overrightarrow{AB}$ is followed by the displacement $\mathbf{v} = \overrightarrow{BC}$, then the resulting displacement is $\overrightarrow{AC}$ as shown in Figure 3. In other words, the single displacement represented by the vector $\overrightarrow{AC}$ has the same effect as the other two displacements together. We call the vector $\overrightarrow{AC}$ the **sum** of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and we write $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. (The **zero vector**, denoted by $\mathbf{0}$, represents no displacement.) Thus to find the sum of any two vectors $\mathbf{u}$ and $\mathbf{v}$, we sketch vectors equal to $\mathbf{u}$ and $\mathbf{v}$ with the initial point of one at the terminal point of the other (see Figure 4(a)). If we draw $\mathbf{u}$ and $\mathbf{v}$ starting at the same point, then $\mathbf{u} + \mathbf{v}$ is the vector that is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 4(b).

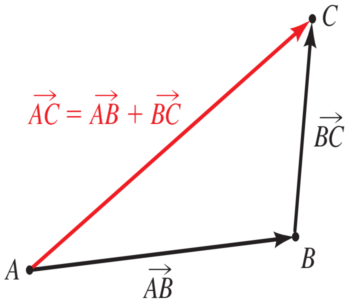
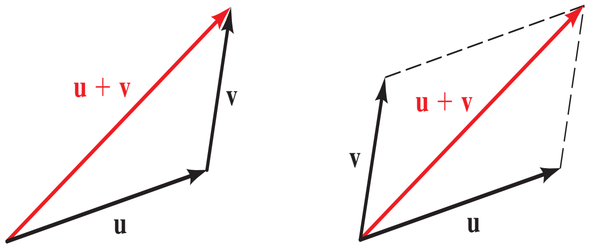


FIGURE 3



(a) (b)

FIGURE 4 Addition of vectors

If c is a real number and $\mathbf{v}$ is a vector, we define a new vector $c\mathbf{v}$ as follows: The vector $c\mathbf{v}$ has magnitude $|c||\mathbf{v}|$ and has the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$. If $c = 0$, then $c\mathbf{v} = \mathbf{0}$, the zero vector. This process is called **multiplication of a vector by a scalar**. Multiplying a vector by a scalar has the effect of stretching or shrinking the vector. Figure 5 shows graphs of the vector $c\mathbf{v}$ for different values of c. We write the vector $(-1)\mathbf{v}$ as $-\mathbf{v}$. Thus $-\mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but with the opposite direction.

The **difference** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Figure 6 shows that the vector $\mathbf{u} - \mathbf{v}$ is the other diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

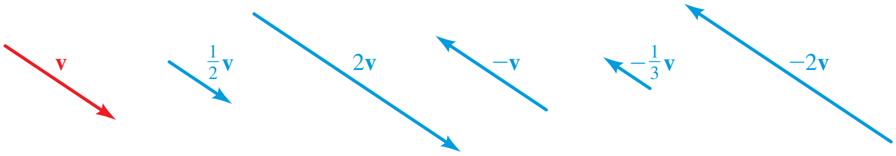


FIGURE 5 Multiplication of a vector by a scalar

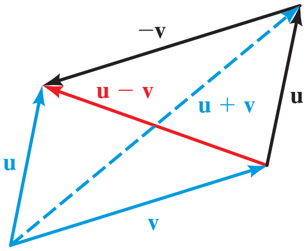


FIGURE 6 Subtraction of vectors

Vectors in the Coordinate Plane

So far, we've discussed vectors geometrically. By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). In Figure 7(a), to go from the initial point of the vector $\mathbf{v}$ to the terminal point, we move a_1 units to the right and a_2 units upward. We represent $\mathbf{v}$ as an ordered pair of real numbers.

Note the distinction between the *vector* $\langle a_1, a_2 \rangle$ and the *point* (a_1, a_2).

$$\mathbf{v} = \langle a_1, a_2 \rangle$$

where a_1 is the **horizontal component** of $\mathbf{v}$ and a_2 is the **vertical component** of $\mathbf{v}$. Remember that a vector represents a magnitude and a direction, not a particular arrow in the plane. Thus the vector $\langle a_1, a_2 \rangle$ has many different representations, depending on its initial point (see Figure 7(b)).

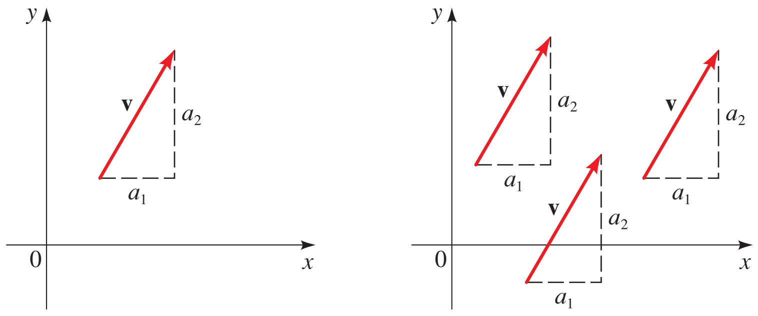


FIGURE 7

(a)

(b)

Using Figure 8, we can state the relationship between a geometric representation of a vector and the analytic one as follows.

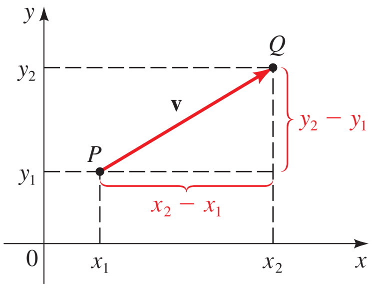


FIGURE 8

COMPONENT FORM OF A VECTOR

If a vector $\mathbf{v}$ is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

EXAMPLE 1 ■ Describing Vectors in Component Form

(a) Find the component form of the vector $\mathbf{u}$ with initial point $(-2, 5)$ and terminal point $(3, 7)$.

(b) If the vector $\mathbf{v} = \langle 3, 7 \rangle$ is sketched with initial point $(2, 4)$, what is its terminal point?

(c) Sketch representations of the vector $\mathbf{w} = \langle 2, 3 \rangle$ with initial points at $(0, 0)$, $(2, 2)$, $(-2, -1)$, and $(1, 4)$.

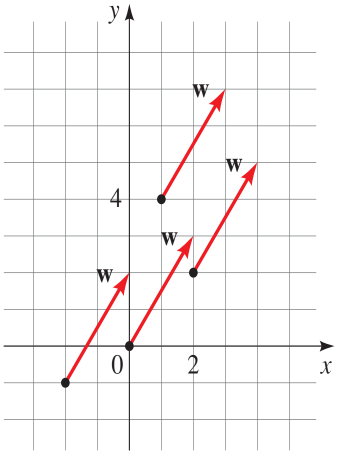


FIGURE 9

SOLUTION

(a) The desired vector is

$$\mathbf{u} = \langle 3 - (-2), 7 - 5 \rangle = \langle 5, 2 \rangle$$

(b) Let the terminal point of $\mathbf{v}$ be (x, y). Then

$$\langle x - 2, y - 4 \rangle = \langle 3, 7 \rangle$$

So $x - 2 = 3$ and $y - 4 = 7$, or $x = 5$ and $y = 11$. The terminal point is $(5, 11)$.

(c) Representations of the vector $\mathbf{w}$ are sketched in Figure 9.

MAGNITUDE OF A VECTOR

The **magnitude** or **length** of a vector $\mathbf{v} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$$

EXAMPLE 2 ■ Magnitudes of Vectors

Find the magnitude of each vector.

(a) $\mathbf{u} = \langle 2, -3 \rangle$ **(b)** $\mathbf{v} = \langle 5, 0 \rangle$ **(c)** $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

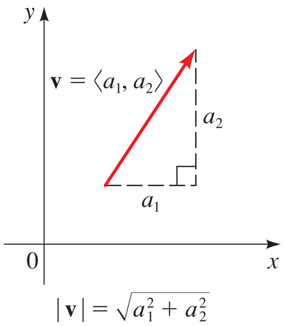


FIGURE 10

SOLUTION

(a) $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

(c) $|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

ALGEBRAIC OPERATIONS ON VECTORS

If $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{u} = \langle ca_1, ca_2 \rangle \qquad c \in \mathbb{R}$$

EXAMPLE 3 ■ Operations with Vectors

If $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

SOLUTION By the definitions of the vector operations we have

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle 2, -3 \rangle + \langle -1, 2 \rangle = \langle 1, -1 \rangle \\ \mathbf{u} - \mathbf{v} &= \langle 2, -3 \rangle - \langle -1, 2 \rangle = \langle 3, -5 \rangle \\ 2\mathbf{u} &= 2\langle 2, -3 \rangle = \langle 4, -6 \rangle \\ -3\mathbf{v} &= -3\langle -1, 2 \rangle = \langle 3, -6 \rangle \\ 2\mathbf{u} + 3\mathbf{v} &= 2\langle 2, -3 \rangle + 3\langle -1, 2 \rangle = \langle 4, -6 \rangle + \langle -3, 6 \rangle = \langle 1, 0 \rangle \end{aligned}$$

The following properties for vector operations can be easily proved from the definitions. The **zero vector** is the vector $\mathbf{0} = \langle 0, 0 \rangle$. It plays the same role for addition of vectors as the number 0 does for addition of real numbers.

PROPERTIES OF VECTORS

Vector addition	**Multiplication by a scalar**
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$1\mathbf{u} = \mathbf{u}$
Length of a vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u} \rvert = \lvert c \rvert \lvert \mathbf{u} \rvert$	$c\mathbf{0} = \mathbf{0}$

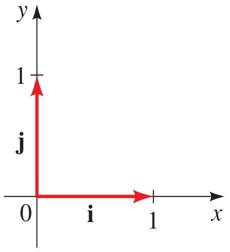


FIGURE 12

A vector of length 1 is called a **unit vector**. For instance, in Example 2(c) the vector $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ is a unit vector. Two useful unit vectors are **i** and **j**, defined by

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \mathbf{j} = \langle 0, 1 \rangle$$

(See Figure 12.) These vectors are special because any vector can be expressed in terms of them. (See Figure 13.)

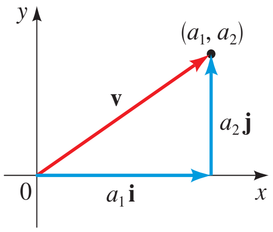


FIGURE 13

VECTORS IN TERMS OF i AND j

The vector $\mathbf{v} = \langle a_1, a_2 \rangle$ can be expressed in terms of **i** and **j** by

$$\mathbf{v} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

EXAMPLE 4 ■ Vectors in Terms of i and j

(a) Write the vector $\mathbf{u} = \langle 5, -8 \rangle$ in terms of **i** and **j**.

(b) If $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$, write $2\mathbf{u} + 5\mathbf{v}$ in terms of **i** and **j**.

SOLUTION

(a) $\mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

$$\begin{aligned} 2\mathbf{u} + 5\mathbf{v} &= 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j}) \\ &= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j}) \\ &= \mathbf{i} + 34\mathbf{j} \end{aligned}$$

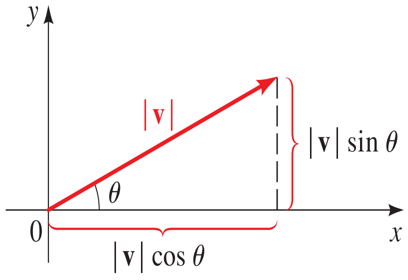


FIGURE 14

HORIZONTAL AND VERTICAL COMPONENTS OF A VECTOR

Let $\mathbf{v}$ be a vector with magnitude $|\mathbf{v}|$ and direction θ. Then $\mathbf{v} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$, where

$$a_1 = |\mathbf{v}| \cos\theta \qquad \text{and} \qquad a_2 = |\mathbf{v}| \sin\theta$$

Thus we can express $\mathbf{v}$ as

$$\mathbf{v} = |\mathbf{v}| \cos\theta\,\mathbf{i} + |\mathbf{v}| \sin\theta\,\mathbf{j}$$

EXAMPLE 5 ■ Components and Direction of a Vector

(a) A vector $\mathbf{v}$ has length 8 and direction $\pi/3$. Find the horizontal and vertical components, and write $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Find the direction of the vector $\mathbf{u} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$.

SOLUTION

(a) We have $\mathbf{v} = \langle a, b \rangle$, where the components are given by

$$a = 8\cos\frac{\pi}{3} = 4 \qquad \text{and} \qquad b = 8\sin\frac{\pi}{3} = 4\sqrt{3}$$

Thus $\mathbf{v} = \langle 4, 4\sqrt{3} \rangle = 4\mathbf{i} + 4\sqrt{3}\,\mathbf{j}$.

(b) From Figure 15 we see that the direction θ has the property that

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Thus the reference angle for θ is $\pi/6$. Since the terminal point of the vector $\mathbf{u}$ is in Quadrant II, it follows that $\theta = 5\pi/6$.

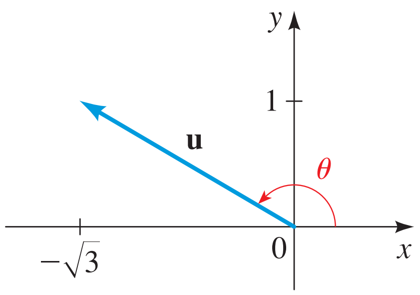


FIGURE 15

3.3 THREE-DIMENSIONAL COORDINATE GEOMETRY

■ The Three-Dimensional Rectangular Coordinate System ■ Distance Formula in Three Dimensions ■ The Equation of a Sphere

■ The Three-Dimensional Rectangular Coordinate System

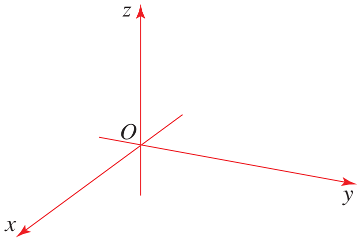


FIGURE 1 Coordinate axes

To represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 2(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane is the plane that contains the y- and z-axes; the xz-plane is the plane that contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**.

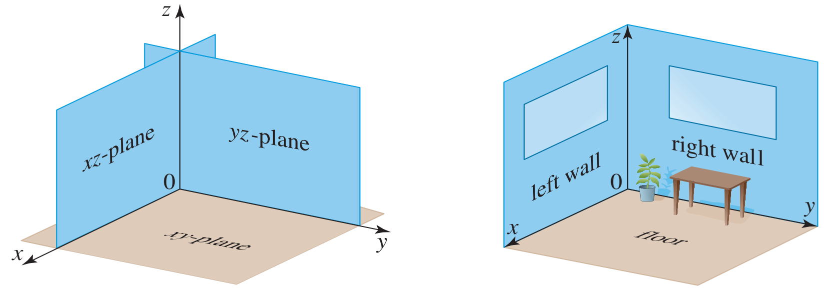


FIGURE 2 (a) Coordinate planes (b) Coordinate "walls"

Because people often have difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following (see Figure 2(b)). Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall; the y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls.

Now any point P in space can be located by a unique **ordered triple** of real numbers (a, b, c), as shown in Figure 3. The first number a is the x-coordinate of P, the second number b is the y-coordinate of P, and the third number c is the z-coordinate of P. The set of all ordered triples $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ forms the **three-dimensional rectangular coordinate system**.

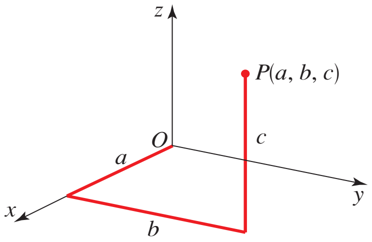


FIGURE 3 Point $P(a, b, c)$

EXAMPLE 1 ■ Plotting Points in Three Dimensions

Plot the points $(2, 4, 7)$ and $(-4, 3, -5)$.

SOLUTION The points are plotted in Figure 4.

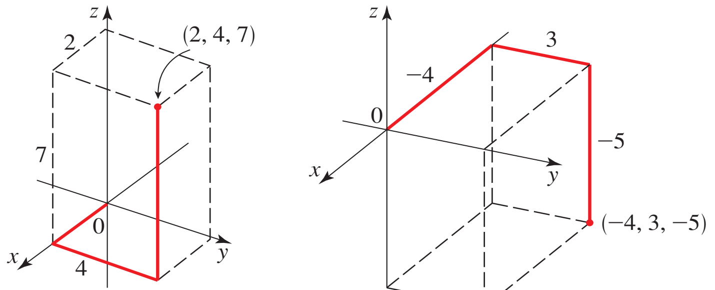


FIGURE 4

In two-dimensional geometry the graph of an equation involving x and y is a *curve* in the plane. In three-dimensional geometry an equation in x, y, and z represents a *surface* in space.

EXAMPLE 2 ■ Surfaces in Three-Dimensional Space

Describe and sketch the surfaces represented by the following equations:

(a) $z = 3$ **(b)** $y = 5$

SOLUTION

(a) The surface consists of the points $P(x, y, z)$ where the z-coordinate is 3. This is the horizontal plane that is parallel to the xy-plane and three units above it, as in Figure 5.

(b) The surface consists of the points $P(x, y, z)$ where the y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it, as in Figure 6.

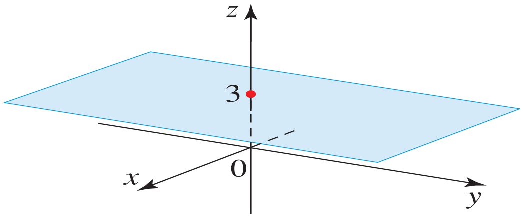


FIGURE 5 The plane $z = 3$

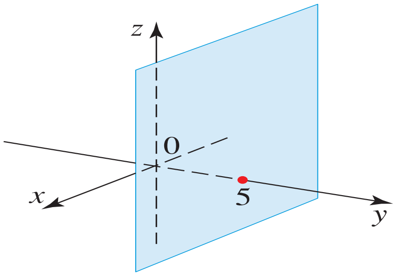


FIGURE 6 The plane $y = 5$

Distance Formula in Three Dimensions

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

DISTANCE FORMULA IN THREE DIMENSIONS

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

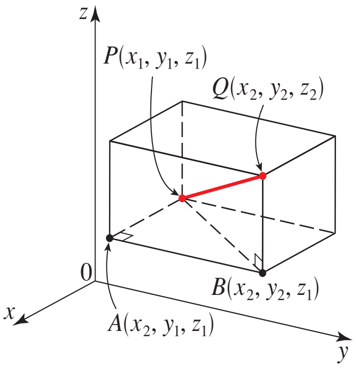


FIGURE 7

Proof To prove this formula, we construct a rectangular box as in Figure 7, where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are diagonally opposite vertices and the faces of the box are parallel to the coordinate planes. If A and B are the vertices of the box that are indicated in the figure, then

$$d(P, A) = |x_2 - x_1| \qquad d(A, B) = |y_2 - y_1| \qquad d(Q, B) = |z_2 - z_1|$$

Triangles PAB and PBQ are right triangles, so by the Pythagorean Theorem we have

$$(d(P, Q))^2 = (d(P, B))^2 + (d(Q, B))^2$$

$$(d(P, B))^2 = (d(P, A))^2 + (d(A, B))^2$$

Combining these equations, we get

$$\begin{aligned}(d(P, Q))^2 &= (d(P, A))^2 + (d(A, B))^2 + (d(Q, B))^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2\end{aligned}$$

Therefore

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 3 ■ Using the Distance Formula

Find the distance between the points $P(2, -1, 7)$ and $Q(1, -3, 5)$.

SOLUTION We use the Distance Formula:

$$d(P, Q) = \sqrt{(1-2)^2 + (-3-(-1))^2 + (5-7)^2} = \sqrt{1+4+4} = 3$$

■ The Equation of a Sphere

We can use the Distance Formula to find an equation for a sphere in a three-dimensional coordinate space.

EQUATION OF A SPHERE

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

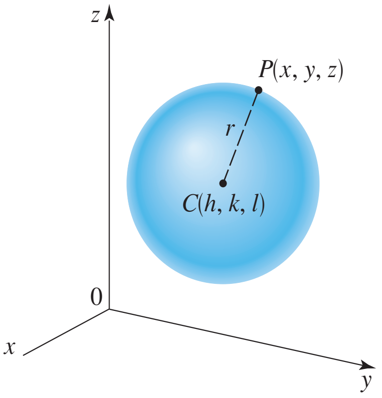


FIGURE 8 Sphere with radius r and center $C(h, k, l)$

EXAMPLE 4 ■ Finding the Equation of a Sphere

Find an equation of a sphere with radius 5 and center $C(-2, 1, 3)$.

SOLUTION We use the general equation of a sphere, with $r = 5$, $h = -2$, $k = 1$, and $l = 3$:

$$(x+2)^2 + (y-1)^2 + (z-3)^2 = 25$$

EXAMPLE 5 ■ Finding the Center and Radius of a Sphere

Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We complete the squares in the x-, y-, and z-terms to rewrite the given equation in the form of an equation of a sphere.

$$\begin{aligned} x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 &= 0 && \text{Given equation} \\ (x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) &= -6 + 4 + 9 + 1 && \text{Complete squares} \\ (x+2)^2 + (y-3)^2 + (z+1)^2 &= 8 && \text{Factor into squares} \end{aligned}$$

Comparing this with the standard equation of a sphere, we can see that the center is $(-2, 3, -1)$ and the radius is $\sqrt{8} = 2\sqrt{2}$.

The intersection of a sphere with a plane is called the **trace** of the sphere in the plane.

EXAMPLE 6 ■ Finding the Trace of a Sphere

Describe the trace of the sphere $(x-2)^2+(y-4)^2+(z-5)^2=36$ in **(a)** the xy-plane and **(b)** the plane $z=9$.

SOLUTION

(a) In the xy-plane the z-coordinate is 0. So the trace of the sphere in the xy-plane consists of all the points on the sphere whose z-coordinate is 0. We replace z by 0 in the equation of the sphere and get

$$\begin{aligned} (x-2)^2+(y-4)^2+(0-5)^2 &= 36 && \text{Replace } z \text{ by } 0\\ (x-2)^2+(y-4)^2+25 &= 36 && \text{Calculate}\\ (x-2)^2+(y-4)^2 &= 11 && \text{Subtract } 25 \end{aligned}$$

Thus the trace of the sphere is the circle

$$(x-2)^2+(y-4)^2=11 \qquad z=0$$

which is a circle of radius $\sqrt{11}$ that is in the xy-plane, centered at $(2, 4, 0)$ (see Figure 9(a)).

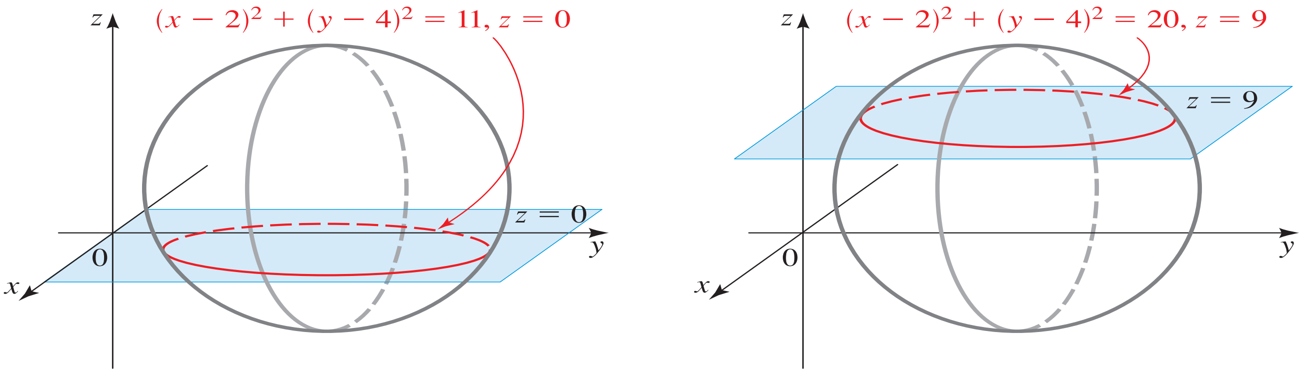
(b) The trace of the sphere in the plane $z=9$ consists of all the points on the sphere whose z-coordinate is 9. So we replace z by 9 in the equation of the sphere and get

$$\begin{aligned} (x-2)^2+(y-4)^2+(9-5)^2 &= 36 && \text{Replace } z \text{ by } 0\\ (x-2)^2+(y-4)^2+16 &= 36 && \text{Calculate}\\ (x-2)^2+(y-4)^2 &= 20 && \text{Subtract } 16 \end{aligned}$$

Thus the trace of the sphere is the circle

$$(x-2)^2+(y-4)^2=20 \qquad z=9$$

which is a circle of radius $\sqrt{20}$ that is 9 units above the xy-plane, centered at $(2, 4, 9)$ (see Figure 9(b)).



(a) (b)

FIGURE 9 The trace of a sphere in the planes $z=0$ and $z=9$

3.4 VECTORS IN THREE DIMENSIONS

■ Vectors in Space ■ Combining Vectors in Space ■ The Dot Product for Vectors in Space ■ Direction Angles of a Vector

Vectors in Space

Recall from Section 3.1 that a vector can be described geometrically by its initial point and terminal point. When we place a vector **v** in space with its initial point at the origin, we can describe it algebraically as an ordered triple:

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle$$

where a_1, a_2, and a_3 are the **components** of **v** (see Figure 1). Recall also that a vector has many different representations, depending on its initial point. The following definition gives the relationship between the algebraic and geometric representations of a vector.

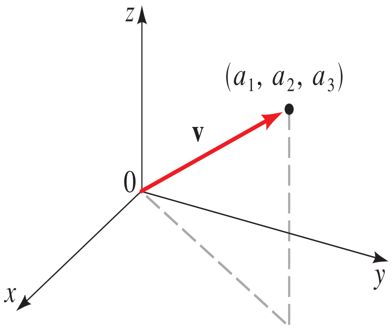


FIGURE 1 $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$

COMPONENT FORM OF A VECTOR IN SPACE

If a vector **v** is represented in space with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

EXAMPLE 1 ■ Describing Vectors in Component Form

(a) Find the components of the vector **v** with initial point $P(1, -4, 5)$ and terminal point $Q(3, 1, -1)$.

(b) If the vector $\mathbf{w} = \langle -2, 1, 3 \rangle$ has initial point $(2, 1, -1)$, what is its terminal point?

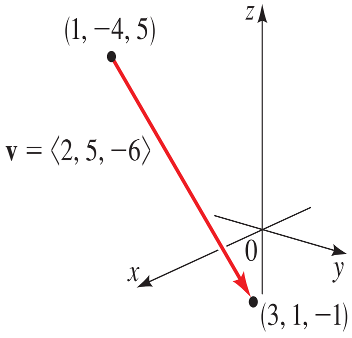


FIGURE 2 $\mathbf{v} = \langle 2, 5, -6 \rangle$

SOLUTION

(a) The desired vector is

$$\mathbf{v} = \langle 3 - 1, 1 - (-4), -1 - 5 \rangle = \langle 2, 5, -6 \rangle$$

See Figure 2.

(b) Let the terminal point of **w** be (x, y, z). Then

$$\mathbf{w} = \langle x - 2, y - 1, z - (-1) \rangle$$

Since $\mathbf{w} = \langle -2, 1, 3 \rangle$, we have $x - 2 = -2$, $y - 1 = 1$, and $z + 1 = 3$. So $x = 0$, $y = 2$, and $z = 2$, and the terminal point is $(0, 2, 2)$.

The following formula is a consequence of the Distance Formula, since the vector $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$ in standard position has initial point $(0, 0, 0)$ and terminal point (a_1, a_2, a_3)

.

MAGNITUDE OF A VECTOR IN THREE DIMENSIONS

The magnitude of the vector $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

EXAMPLE 2 ■ Magnitude of Vectors in Three Dimensions

Find the magnitude of the given vector.

(a) $\mathbf{u} = \langle 3, 2, 5 \rangle$ **(b)** $\mathbf{v} = \langle 0, 3, -1 \rangle$ **(c)** $\mathbf{w} = \langle 0, 0, -1 \rangle$

SOLUTION

(a) $|\mathbf{u}| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$

(b) $|\mathbf{v}| = \sqrt{0^2 + 3^2 + (-1)^2} = \sqrt{10}$

(c) $|\mathbf{w}| = \sqrt{0^2 + 0^2 + (-1)^2} = 1$

Combining Vectors in Space

We now give definitions of the algebraic operations involving vectors in three dimensions.

ALGEBRAIC OPERATIONS ON VECTORS IN THREE DIMENSIONS

If $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$, and c is a scalar, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\mathbf{u} = \langle ca_1, ca_2, ca_3 \rangle$$

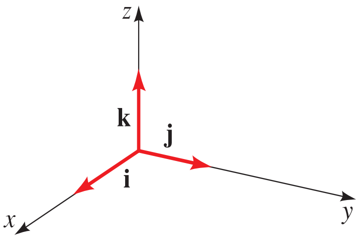


FIGURE 3

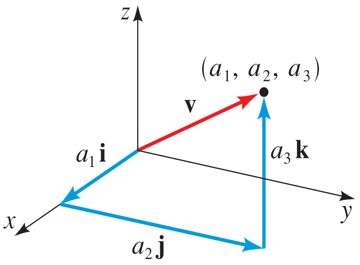


FIGURE 4

EXAMPLE 3 ■ Operations with Three-Dimensional Vectors

If $\mathbf{u} = \langle 1, -2, 4 \rangle$ and $\mathbf{v} = \langle 6, -1, 1 \rangle$ find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $5\mathbf{u} - 3\mathbf{v}$.

SOLUTION Using the definitions of algebraic operations, we have

$$\mathbf{u} + \mathbf{v} = \langle 1 + 6, -2 - 1, 4 + 1 \rangle = \langle 7, -3, 5 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle 1 - 6, -2 - (-1), 4 - 1 \rangle = \langle -5, -1, 3 \rangle$$

$$5\mathbf{u} - 3\mathbf{v} = 5\langle 1, -2, 4 \rangle - 3\langle 6, -1, 1 \rangle = \langle 5, -10, 20 \rangle - \langle 18, -3, 3 \rangle = \langle -13, -7, 17 \rangle$$

Recall that a unit vector is a vector of length 1. The vector $\mathbf{w}$ in Example 2(c) is an example of a unit vector. Some other unit vectors in three dimensions are

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

as shown in Figure 3. Any vector in three dimensions can be written in terms of these three vectors (see Figure 4).

EXPRESSING VECTORS IN TERMS OF i, j, AND k

The vector $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$ can be expressed in terms of **i**, **j**, and **k** by

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

All the properties of vectors in section 3.1 hold for vectors in three dimensions as well. We use these properties in the next example.

EXAMPLE 4 ■ Vectors in Terms of i, j, and k

(a) Write the vector $\mathbf{u} = \langle 5, -3, 6 \rangle$ in terms of **i**, **j**, and **k**.

(b) If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{u} + 3\mathbf{v}$ in terms of **i**, **j**, and **k**.

SOLUTION

(a) $\mathbf{u} = 5\mathbf{i} + (-3)\mathbf{j} + 6\mathbf{k} = 5\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

(b) We use the properties of vectors to get the following:

$$\begin{aligned} 2\mathbf{u} + 3\mathbf{v} &= 2(2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 4\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} \\ &= 16\mathbf{i} + 4\mathbf{j} + 15\mathbf{k} \end{aligned}$$

■ The Dot Product for Vectors in Space

DEFINITION OF THE DOT PRODUCT FOR VECTORS IN THREE DIMENSIONS

If $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$ are vectors in three dimensions, then their **dot product** is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3$$

EXAMPLE 5 ■ Calculating Dot Products for Vectors in Three Dimensions

Find the given dot product.

(a) $\langle -1, 2, 3 \rangle \cdot \langle 6, 5, -1 \rangle$

(b) $(2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 8\mathbf{k})$

SOLUTION

(a) $\langle -1, 2, 3 \rangle \cdot \langle 6, 5, -1 \rangle = (-1)(6) + (2)(5) + (3)(-1) = 1$

(b) $$\begin{aligned} (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}) &= \langle 2, -3, -1 \rangle \cdot \langle -1, 2, 8 \rangle \\ &= (2)(-1) + (-3)(2) + (-1)(8) = -16 \end{aligned}$$

ANGLE BETWEEN TWO VECTORS

Let **u** and **v** be vectors in space, and let θ be the angle between them. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

In particular, **u** and **v** are **perpendicular** (or **orthogonal**) if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

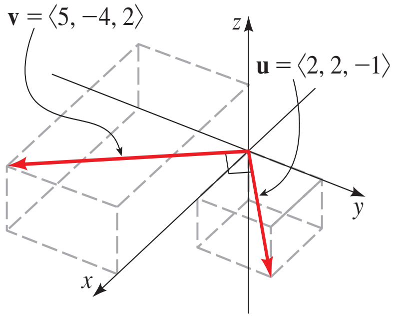


FIGURE 5 The vectors **u** and **v** are perpendicular.

EXAMPLE 6 ■ Checking Whether Two Vectors Are Perpendicular

Show that the vector $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION We find the dot product.

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = (2)(5) + (2)(-4) + (-1)(2) = 0$$

Since the dot product is 0, the vectors are perpendicular. See Figure 5.

■ Direction Angles of a Vector

The **direction angles** of a nonzero vector $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ are the angles α, β, and γ in the interval $[0, \pi]$ that the vector **v** makes with the positive x-, y-, and z-axes (see Figure 6). The cosines of these angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector **v**. By using the formula for the angle between two vectors, we can find the direction cosines of **v**:

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{v}|} \qquad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|\,|\mathbf{j}|} = \frac{a_2}{|\mathbf{v}|} \qquad \cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}|\,|\mathbf{k}|} = \frac{a_3}{|\mathbf{v}|}$$

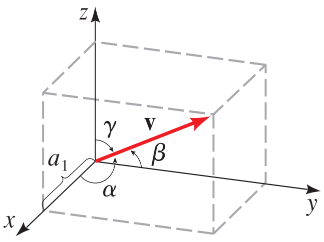


FIGURE 6 Direction angles of the vector **v**

DIRECTION ANGLES OF A VECTOR

If $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a nonzero vector in space, the direction angles α, β, and γ satisfy

$$\cos\alpha = \frac{a_1}{|\mathbf{v}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{v}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{v}|}$$

In particular, if $|\mathbf{v}| = 1$, then the direction cosines of **v** are simply the components of **v**.

EXAMPLE 7 ■ Finding the Direction Angles of a Vector

Find the direction angles of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

SOLUTION The length of the vector **v** is $|\mathbf{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. From the above box we get

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

Since the direction angles are in the interval $[0, \pi]$ and since $\cos^{-1}$ gives angles in that same interval, we get α, β, and γ by simply taking $\cos^{-1}$ of the above equations.

$$\alpha = \cos^{-1}\frac{1}{\sqrt{14}} \approx 74° \qquad \beta = \cos^{-1}\frac{2}{\sqrt{14}} \approx 58° \qquad \gamma = \cos^{-1}\frac{3}{\sqrt{14}} \approx 37°$$

The direction angles of a vector uniquely determine its direction but not its length. If we also know the length of the vector **v**, the expressions for the direction cosines of **v** allow us to express the vector as

$$\boxed{\mathbf{v} = \langle |\mathbf{v}|\cos\alpha, |\mathbf{v}|\cos\beta, |\mathbf{v}|\cos\gamma \rangle}$$

From this we get

$$\mathbf{v} = |\mathbf{v}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Since $\mathbf{v}/|\mathbf{v}|$ is a unit vector, we get the following.

PROPERTY OF DIRECTION COSINES

The direction angles α, β, and γ of a nonzero vector **v** in space satisfy the following equation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

This property indicates that if we know two of the direction cosines of a vector, we can find the third up to its sign.

An angle θ is **acute** if $0 \le \theta < \pi/2$ and is **obtuse** if $\pi/2 < \theta \le \pi$.

EXAMPLE 8 ■ Finding the Direction Angles of a Vector

A vector makes an angle $\alpha = \pi/3$ with the positive x-axis and an angle $\beta = 3\pi/4$ with the positive y-axis. Find the angle γ that the vector makes with the positive z-axis, given that γ is an obtuse angle.

SOLUTION By the property of the direction angles we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\frac{\pi}{3} + \cos^2\frac{3\pi}{4} + \cos^2\gamma = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$$

$$\cos^2\gamma = \frac{1}{4}$$

$$\cos\gamma = \frac{1}{2} \qquad \text{or} \qquad \cos\gamma = -\frac{1}{2}$$

$$\gamma = \frac{\pi}{3} \qquad \text{or} \qquad \gamma = \frac{2\pi}{3}$$

Since we require γ to be an obtuse angle, we conclude that $\gamma = 2\pi/3$.

3.5 THE CROSS PRODUCT

- The Cross Product
- Properties of the Cross Product
- Area of a Parallelogram
- Volume of a Parallelepiped

In this section we define an operation on vectors that allows us to find a vector which is perpendicular to two given vectors.

The Cross Product

Given two vectors $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$, we often need to find a vector $\mathbf{w}$ perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. If we write $\mathbf{w} = \langle c_1, c_2, c_3 \rangle$, then $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0$$

You can check that one of the solutions of this system of equations is the vector $\mathbf{w} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$. This vector is called the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ and is denoted by $\mathbf{u} \times \mathbf{v}$.

THE CROSS PRODUCT

If $\mathbf{u} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{v} = \langle b_1, b_2, b_3 \rangle$ are three-dimensional vectors, then the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

We can write the definition of the cross product using determinants as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

EXAMPLE 1 ■ Finding a Cross Product

If $\mathbf{u} = \langle 0, -1, 3 \rangle$ and $\mathbf{v} = \langle 2, 0, -1 \rangle$, find $\mathbf{u} \times \mathbf{v}$.

SOLUTION We use the formula above to find the cross product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$

$$= (1 - 0)\mathbf{i} - (0 - 6)\mathbf{j} + (0 - (-2))\mathbf{k}$$

$$= \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$$

So the desired vector is $\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

Properties of the Cross Product

One of the most important properties of the cross product is the following theorem.

CROSS PRODUCT THEOREM

The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal (perpendicular) to both $\mathbf{u}$ and $\mathbf{v}$.

Proof To show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$, we compute their dot product and show that it is 0.

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0\end{aligned}$$

A similar computation shows that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$. Therefore $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and to $\mathbf{v}$. ■

EXAMPLE 2 ■ Finding an Orthogonal Vector

If $\mathbf{u} = -\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{k}$, find a unit vector that is orthogonal to the plane containing the vectors $\mathbf{u}$ and $\mathbf{v}$.

SOLUTION By the Cross Product Theorem the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to the plane containing the vectors $\mathbf{u}$ and $\mathbf{v}$. (See Figure 1.) In Example 1 we found $\mathbf{u} \times \mathbf{v} = \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$. To obtain an orthogonal unit vector, we multiply $\mathbf{u} \times \mathbf{v}$ by the scalar $1/|\mathbf{u} \times \mathbf{v}|$:

$$\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{1^2 + 6^2 + 2^2}} = \frac{\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{41}}$$

So the desired vector is $\dfrac{1}{\sqrt{41}}(\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$.

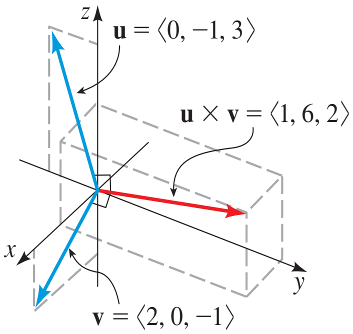


FIGURE 1 The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 3 ■ Finding a Vector Perpendicular to a Plane

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION By the Cross Product Theorem the vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know that

$$\overrightarrow{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$
$$\overrightarrow{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$
$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Notice that any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane.

If $\mathbf{u}$ and $\mathbf{v}$ are represented by directed line segments with the same initial point (as in Figure 2), then the Cross Product Theorem says that the cross product $\mathbf{u} \times \mathbf{v}$ points in a direction perpendicular to the plane through $\mathbf{u}$ and $\mathbf{v}$. It turns out that the direction of $\mathbf{u} \times \mathbf{v}$ is given by the *right-hand rule:* If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{u}$ to $\mathbf{v}$, then your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$ (as in Figure 2). You can check that the vector $\mathbf{u} \times \mathbf{v}$ in Figure 1 satisfies the right-hand rule.

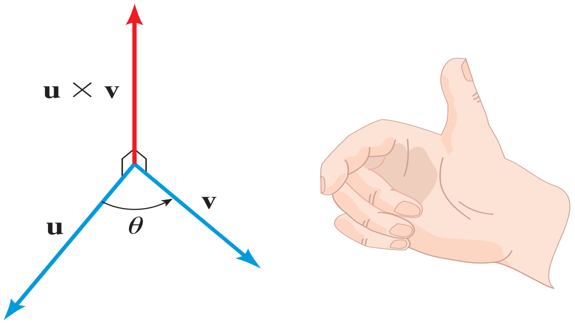


FIGURE 2 Right-hand rule

Now that we know the direction of the vector $\mathbf{u} \times \mathbf{v}$, the remaining thing we need is the length $|\mathbf{u} \times \mathbf{v}|$.

LENGTH OF THE CROSS PRODUCT

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

In particular, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if

$$\mathbf{u} \times \mathbf{v} = \mathbf{0}$$

Proof We apply the definitions of the cross product and length of a vector. You can verify the algebra in the first step by expanding the right-hand sides of the first and second lines and then comparing the results.

$$\begin{aligned}
|\mathbf{u}\times\mathbf{v}|^2 &= (a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2 && \text{Definitions}\\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2 && \text{Verify algebra}\\
&= |\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2 && \text{Definitions}\\
&= |\mathbf{u}|^2|\mathbf{v}|^2-|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta && \text{Property of Dot Product}\\
&= |\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta) && \text{Factor}\\
&= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta && \text{Pythagorean Identity}
\end{aligned}$$

The result follows by taking square roots and observing that $\sqrt{\sin^2\theta}=\sin\theta$ because $\sin\theta\geq 0$ when $0\leq\theta\leq\pi$. ■

We have now completely determined the vector $\mathbf{u}\times\mathbf{v}$ geometrically. The vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and its orientation is determined by the right-hand rule. The length of $\mathbf{u}\times\mathbf{v}$ is $|\mathbf{u}||\mathbf{v}|\sin\theta$.

Area of a Parallelogram

We can use the cross product to find the area of a parallelogram. If $\mathbf{u}$ and $\mathbf{v}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{u}|$, altitude $|\mathbf{v}|\sin\theta$, and area

$$A=|\mathbf{u}|(|\mathbf{v}|\sin\theta)=|\mathbf{u}\times\mathbf{v}|$$

(See Figure 3.) Thus we have the following way of interpreting the magnitude of a cross product.

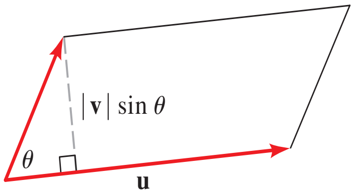


FIGURE 3 Parallelogram determined by **u** and **v**.

AREA OF A PARALLELOGRAM

The length of the cross product $\mathbf{u}\times\mathbf{v}$ is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 4 ■ Finding the Area of a Triangle

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle -40, -15, 15\rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$|\overrightarrow{PQ}\times\overrightarrow{PR}|=\sqrt{(-40)^2+(-15)^2+15^2}=5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

3.6 EQUATIONS OF LINES AND PLANES

■ Equations of Lines ■ Equations of Planes

In this section we find equations for lines and planes in a three-dimensional coordinate space. We use vectors to help us find such equations.

Equations of Lines

A line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is described by a vector $\mathbf{v}$ parallel to L. If we let $\mathbf{r}_0$ be the position vector of P_0 (that is, the vector $\overrightarrow{OP_0}$), then for all real numbers t the terminal points P of the position vectors $\mathbf{r}_0 + t\mathbf{v}$ trace out a line parallel to $\mathbf{v}$ and passing through P_0 (see Figure 1). Each value of the parameter t gives a point P on L. So the line L is given by the position vector $\mathbf{r}$, where

The **position vector** of a point (a_1, a_2, a_3) is the vector $\langle a_1, a_2, a_3\rangle$; that is, it is the vector from the origin to the point.

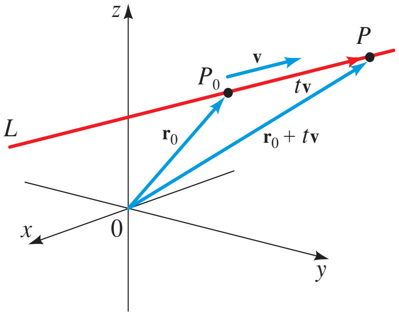


FIGURE 1

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

for $t \in \mathbb{R}$. This is the **vector equation of a line**.

Let's write the vector $\mathbf{v}$ in component form $\mathbf{v} = \langle a, b, c\rangle$ and let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Then the vector equation of the line becomes

$$\begin{aligned}\langle x, y, z\rangle &= \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle\\ &= \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle\end{aligned}$$

Since two vectors are equal if and only if their corresponding components are equal, we have the following result.

PARAMETRIC EQUATIONS FOR A LINE

A line passing through the point $P(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ is described by the parametric equations

$$\begin{aligned}x &= x_0 + at\\ y &= y_0 + bt\\ z &= z_0 + ct\end{aligned}$$

where t is any real number.

EXAMPLE 1 ■ Equations of a Line

Find parametric equations for the line that passes through the point $(5, -2, 3)$ and is parallel to the vector $\mathbf{v} = \langle 3, -4, 2\rangle$.

SOLUTION We use the above formula to find the parametric equations:

$$\begin{aligned}x &= 5 + 3t\\ y &= -2 - 4t\\ z &= 3 + 2t\end{aligned}$$

where t is any real number. (See Figure 2.)

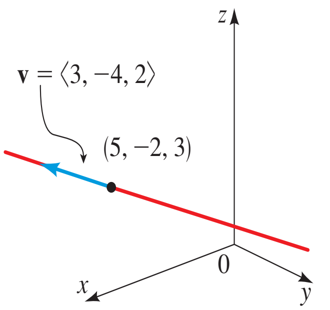


FIGURE 2 Line through $(5, -2, 3)$ with direction $\mathbf{v} = \langle 3, -4, 2\rangle$

EXAMPLE 2 ■ Equations of a Line

Find parametric equations for the line that passes through the points $(-1, 2, 6)$ and $(2, -3, -7)$.

SOLUTION We first find a vector determined by the two points:

$$\mathbf{v} = \langle 2 - (-1), -3 - 2, -7 - 6\rangle = \langle 3, -5, -13\rangle$$

Now we use $\mathbf{v}$ and the point $(-1, 2, 6)$ to find the parametric equations:

$$x = -1 + 3t$$
$$y = 2 - 5t$$
$$z = 6 - 13t$$

where t is any real number. A graph of the line is shown in Figure 3.

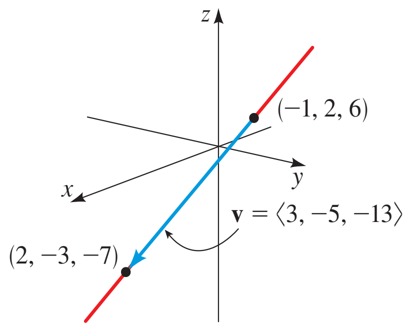


FIGURE 3 Line through $(-1, 2, 6)$ and $(2, -3, -7)$

Equations of Planes

Although a line in space is determined by a point and a direction, the "direction" of a plane cannot be described by a vector in the plane. In fact, different vectors in a plane can have different directions. But a vector perpendicular to a plane *does* completely specify the direction of the plane. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. To determine whether a point $P(x, y, z)$ is in the plane, we check whether the vector $\overrightarrow{P_0P}$ with initial point P_0 and terminal point P is orthogonal to the normal vector. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P, respectively. Then the vector $\overrightarrow{P_0P}$ is represented by $\mathbf{r} - \mathbf{r}_0$ (see Figure 4). So the plane is described by the tips of the vectors $\mathbf{r}$ satisfying

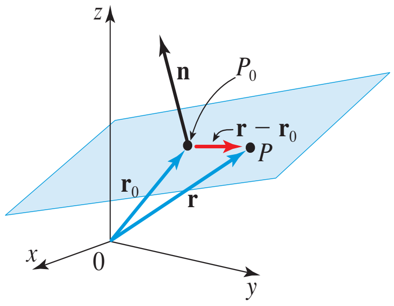


FIGURE 4

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$$

This is the **vector equation of the plane**.

Let's write the normal vector $\mathbf{n}$ in component form $\mathbf{n} = \langle a, b, c\rangle$ and let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Then the vector equation of the plane becomes

$$\langle a, b, c\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Performing the dot product, we arrive at the following equation of the plane in the variables x, y, and z.

EXAMPLE 3 ■ Finding an Equation for a Plane

A plane has normal vector $\mathbf{n} = \langle 4, -6, 3\rangle$ and passes through the point $P(3, -1, -2)$.

(a) Find an equation of the plane.

(b) Find the intercepts, and sketch a graph of the plane.

SOLUTION

(a) By the above formula for the equation of a plane we have

$$
\begin{aligned}
4(x-3)-6(y-(-1))+3(z-(-2)) &= 0 && \text{Formula}\\
4x-12-6y-6+3z+6 &= 0 && \text{Expand}\\
4x-6y+3z &= 12 && \text{Simplify}
\end{aligned}
$$

Thus an equation of the plane is $4x - 6y + 3z = 12$.

(b) To find the x-intercept, we set $y = 0$ and $z = 0$ in the equation of the plane and solve for x. Similarly, we find the y- and z-intercepts.

x-intercept: Setting $y = 0$, $z = 0$, we get $x = 3$.

y-intercept: Setting $x = 0$, $z = 0$, we get $y = -2$.

z-intercept: Setting $x = 0$, $y = 0$, we get $z = 4$.

So the graph of the plane intersects the coordinate axes at the points $(3, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$. This enables us to sketch the portion of the plane shown in Figure 5.

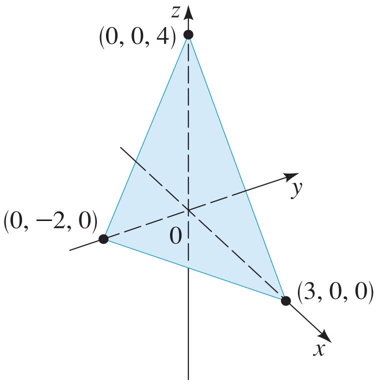


FIGURE 5 The plane $4x - 6y + 3z = 32$

EXAMPLE 4 ■ Finding an Equation for a Plane

Find an equation of the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. In Example 3 of Section 3.5 we found $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15\rangle$. Using the formula for an equation of a plane, we have

$$
\begin{aligned}
-40(x-1)-15(y-4)+15(z-6) &= 0 && \text{Formula}\\
-40x+40-15y+60+15z-90 &= 0 && \text{Expand}\\
-40x-15y+15z &= -10 && \text{Simplify}\\
8x+3y-3z &= 2 && \text{Divide by } -5
\end{aligned}
$$

So an equation of the plane is $8x + 3y - 3z = 2$. A graph of this plane is shown in Figure 6.

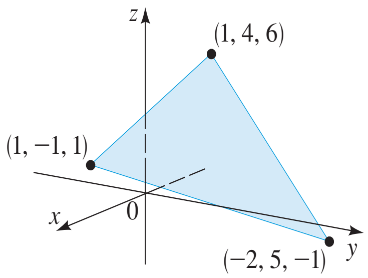


FIGURE 6 A plane through three points