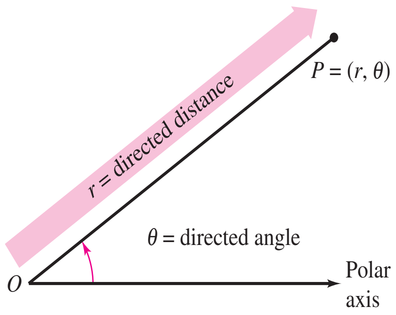
## 2.1 POLAR COORDINATES

- Polar Coordinates
- Relationship Between Polar and Rectangular Coordinates
- Polar Equations




Polar coordinates
**Figure 2.1**

### Polar Coordinates

So far, you have been representing graphs as collections of points $(x, y)$ on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section, you will study a coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point $O$, called the **pole** (or **origin**), and construct from $O$ an initial ray called the **polar axis,** as shown in Figure 2.1. Then each point $P$ in the plane can be assigned **polar coordinates** $(r, \theta)$, as follows.

$r$ = *directed distance* from $O$ to $P$

$\theta$ = *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

Figure 2.2 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.



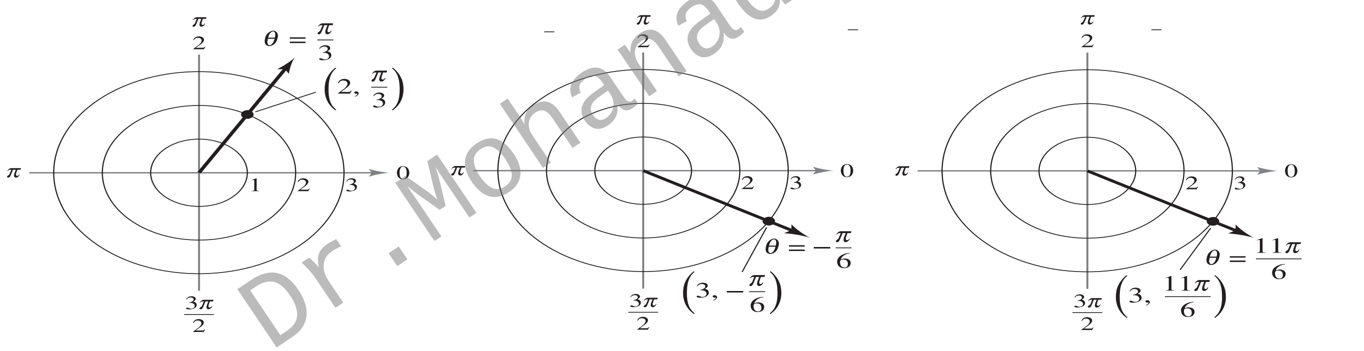



**Figure 2.2**

With rectangular coordinates, each point $(x, y)$ has a unique representation. This is not true with polar coordinates. For instance, the coordinates

$(r, \theta)$ and $(r, 2\pi + \theta)$

represent the same point [see parts (b) and (c) in Figure 2.2]. Also, because $r$ is a *directed distance,* the coordinates

$(r, \theta)$ and $(-r, \theta + \pi)$

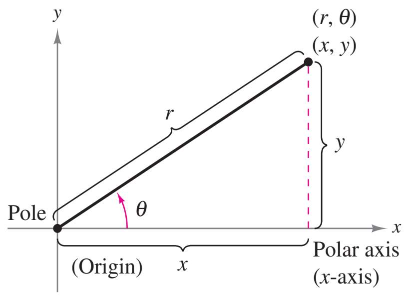
represent the same point. In general, the point $(r, \theta)$ can be written as

$$(r, \theta) = (r, \theta + 2n\pi)$$

or

$$(r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where $n$ is any integer. Moreover, the pole is represented by $(0, \theta)$, where $\theta$ is any angle.




Relating polar and rectangular coordinates
**Figure 2.3**

## Relationship Between Polar and Rectangular Coordinates

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive $x$-axis and the pole with the origin, as shown in Figure 2.3. Because $(x, y)$ lies on a circle of radius $r$, it follows that

$$r^2 = x^2 + y^2.$$

Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan\theta = \frac{y}{x}, \quad \cos\theta = \frac{x}{r}, \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

You can show that the same relationships hold for $r < 0$.

**THEOREM 2.1 Coordinate Conversion**

The polar coordinates $(r, \theta)$ of a point are related to the rectangular coordinates $(x, y)$ of the point as follows.

| **Polar-to-Rectangular** | **Rectangular-to-Polar** |
|---|---|
| $x = r\cos\theta$ | $\tan\theta = \dfrac{y}{x}$ |
| $y = r\sin\theta$ | $r^2 = x^2 + y^2$ |



**EXAMPLE 1 Polar-to-Rectangular Conversion**

**a.** For the point $(r, \theta) = (2, \pi)$,

$$x = r\cos\theta = 2\cos\pi = -2 \quad \text{and} \quad y = r\sin\theta = 2\sin\pi = 0.$$

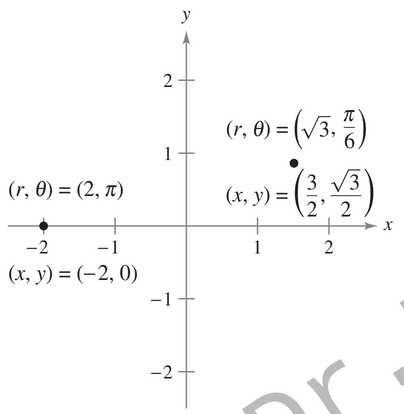
So, the rectangular coordinates are $(x, y) = (-2, 0)$.

**b.** For the point $(r, \theta) = \left(\sqrt{3}, \pi/6\right)$,

$$x = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2} \quad \text{and} \quad y = \sqrt{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are $(x, y) = \left(3/2, \sqrt{3}/2\right)$.

See Figure 2.4 .




To convert from polar to rectangular coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$.
**Figure 2.4**

**EXAMPLE 2 Rectangular-to-Polar Conversion**

**a.** For the second-quadrant point $(x, y) = (-1, 1)$,

$$\tan\theta = \frac{y}{x} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}.$$

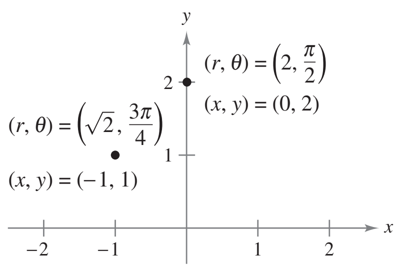
Because $\theta$ was chosen to be in the same quadrant as $(x, y)$, you should use a positive value of $r$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

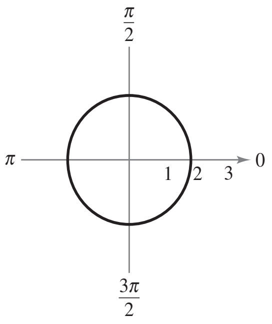
This implies that *one* set of polar coordinates is $(r, \theta) = \left(\sqrt{2}, 3\pi/4\right)$.

**b.** Because the point $(x, y) = (0, 2)$ lies on the positive $y$-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$.

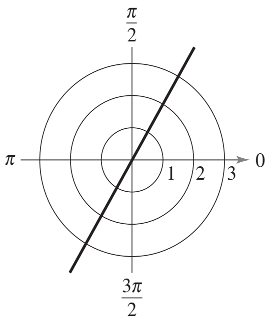
See Figure 2.5.



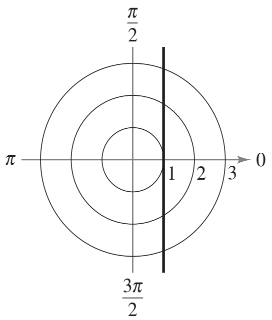

To convert from rectangular to polar coordinates, let $\tan\theta = y/x$ and $r = \sqrt{x^2 + y^2}$.
**Figure 2.5**




**(a)** Circle: $r = 2$




**(b)** Radial line: $\theta = \frac{\pi}{3}$




**(c)** Vertical line: $r = \sec\theta$

**Figure 2.6**

## Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

### EXAMPLE 3 Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

**a.** $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = \sec\theta$

**Solution**

**a.** The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. So, this graph is a circle centered at the origin with a radius of 2. [See Figure 2.6(a).] You can confirm this by using the relationship $r^2 = x^2 + y^2$ to obtain the rectangular equation

$$x^2 + y^2 = 2^2. \qquad \text{Rectangular equation}$$

**b.** The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive $x$- axis. [See Figure 2.6(b).] You can confirm this by using the relationship $\tan\theta = y/x$ to obtain the rectangular equation

$$y = \sqrt{3}x. \qquad \text{Rectangular equation}$$

**c.** The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship $r\cos\theta = x$.

$$r = \sec\theta \qquad \text{Polar equation}$$
$$r\cos\theta = 1$$
$$x = 1 \qquad \text{Rectangular equation}$$

From the rectangular equation, you can see that the graph is a vertical line. [See Figure 2.6(c).]



## Polar Equations

In Examples 1 and 2 we converted points from one coordinate system to the other. Now we consider the same problem for equations.

### EXAMPLE 4 Converting an Equation from Rectangular to Polar Coordinates

Express the equation $x^2 = 4y$ in polar coordinates.

**SOLUTION** We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = 4y \qquad \text{Rectangular equation}$$
$$(r\cos\theta)^2 = 4(r\sin\theta) \qquad \text{Substitute } x = r\cos\theta,\ y = r\sin\theta$$
$$r^2\cos^2\theta = 4r\sin\theta \qquad \text{Expand}$$
$$r = 4\frac{\sin\theta}{\cos^2\theta} \qquad \text{Divide by } r\cos^2\theta$$

**EXAMPLE 5** Converting Equations from Polar to Rectangular Coordinates

Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

**(a)** $r = 5 \sec \theta$ **(b)** $r = 2 \sin \theta$ **(c)** $r = 2 + 2 \cos \theta$

**SOLUTION**

**(a)** Since $\sec \theta = 1/\cos \theta$, we multiply both sides by $\cos \theta$.

$$r = 5 \sec \theta \qquad \text{Polar equation}$$

$$r \cos \theta = 5 \qquad \text{Multiply by } \cos \theta$$

$$x = 5 \qquad \text{Substitute } x = r \cos \theta$$

The graph of $x = 5$ is the vertical line in Figure 2.7

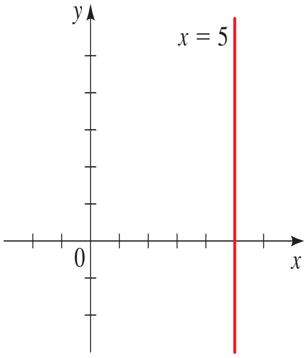



FIGURE 2.7

**(b)** We multiply both sides of the equation by $r$, because then we can use the formulas $r^2 = x^2 + y^2$ and $r \sin \theta = y$.

$$r = 2 \sin \theta \qquad \text{Polar equation}$$

$$r^2 = 2r \sin \theta \qquad \text{Multiply by } r$$

$$x^2 + y^2 = 2y \qquad r^2 = x^2 + y^2 \text{ and } r \sin \theta = y$$

$$x^2 + y^2 - 2y = 0 \qquad \text{Subtract } 2y$$

$$x^2 + (y - 1)^2 = 1 \qquad \text{Complete the square in } y$$

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 8.

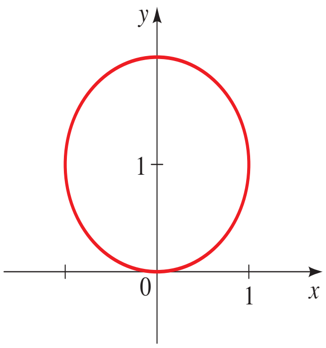



FIGURE 8



**(c)** We first multiply both sides of the equation by $r$:

$$r^2 = 2r + 2r \cos \theta$$

Using $r^2 = x^2 + y^2$ and $x = r \cos \theta$, we can convert two terms in the equation into rectangular coordinates, but eliminating the remaining $r$ requires more work.

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r \cos \theta = x$$

$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$

$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

# 2.2 GRAPHS OF POLAR EQUATIONS

■ Graphing Polar Equations ■ Symmetry

The **graph of a polar equation** $r = f(\theta)$ consists of all points $P$ that have at least one polar representation $(r, \theta)$ whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations than by rectangular equations.

## Graphing Polar Equations

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a)). To plot points in polar coordinates, it is convenient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as in Figure 1(b). We will use such grids to help us sketch polar graphs.

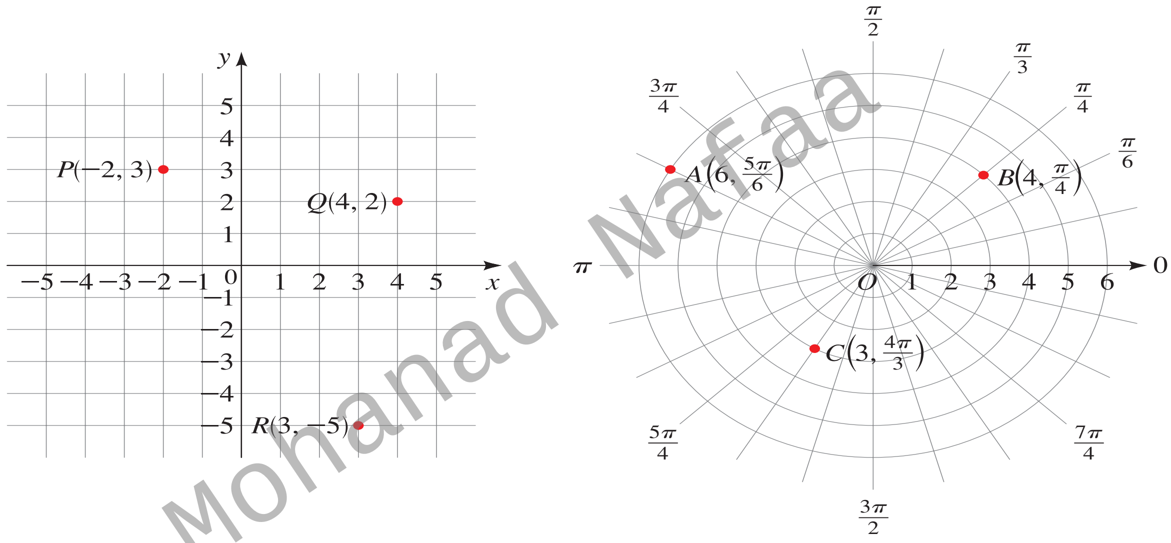



**FIGURE 1** (a) Grid for rectangular coordinates (b) Grid for polar coordinates



### EXAMPLE 1 Sketching the Graph of a Polar Equation

Sketch a graph of the polar equation $r = 2 \sin \theta$.

**SOLUTION** We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

| $\theta$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $r = 2 \sin \theta$ | 0 | 1 | $\sqrt{2}$ | $\sqrt{3}$ | 2 | $\sqrt{3}$ | $\sqrt{2}$ | 1 | 0 |

We plot these points in Figure 2 and then join them to sketch the curve. The graph appears to be a circle. We have used values of $\theta$ only between 0 and $\pi$, since the same points (this time expressed with negative $r$-coordinates) would be obtained if we allowed $\theta$ to range from $\pi$ to $2\pi$.

The polar equation $r = 2\sin\theta$ in rectangular coordinates is

$$x^2 + (y-1)^2 = 1$$

From the rectangular form of the equation we see that the graph is a circle of radius 1 centered at $(0, 1)$.

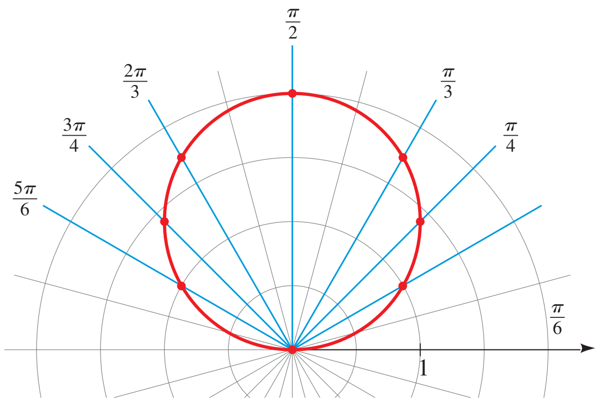



**FIGURE 2** $r = 2\sin\theta$

In general, the graphs of equations of the form

$$r = 2a\sin\theta \qquad \text{and} \qquad r = 2a\cos\theta$$

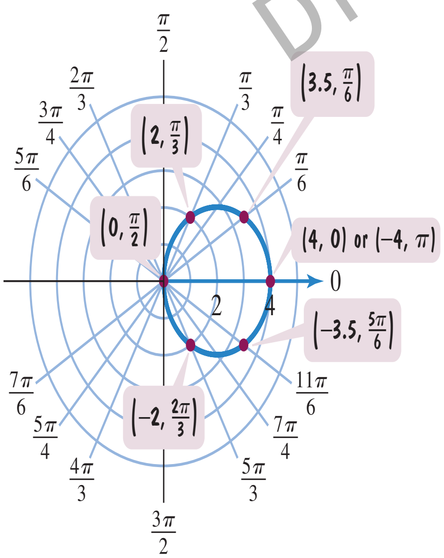
are **circles** with radius $|a|$ centered at the points with polar coordinates $(a, \pi/2)$ and $(a, 0)$, respectively.



## EXAMPLE 2 Graphing an Equation Using the Point-Plotting Method

Graph the polar equation $r = 4\cos\theta$ with $\theta$ in radians.

**Solution** We construct a partial table of coordinates for $r = 4\cos\theta$ using multiples of $\frac{\pi}{6}$. Then we plot the points and join them with a smooth curve.




The graph of $r = 4\cos\theta$

| $\theta$ | $r = 4\cos\theta$ | $(r, \theta)$ |
|---|---|---|
| $0$ | $4\cos 0 = 4\cdot 1 = 4$ | $(4, 0)$ |
| $\frac{\pi}{6}$ | $4\cos\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3} \approx 3.5$ | $\left(3.5, \frac{\pi}{6}\right)$ |
| $\frac{\pi}{3}$ | $4\cos\frac{\pi}{3} = 4\cdot\frac{1}{2} = 2$ | $\left(2, \frac{\pi}{3}\right)$ |
| $\frac{\pi}{2}$ | $4\cos\frac{\pi}{2} = 4\cdot 0 = 0$ | $\left(0, \frac{\pi}{2}\right)$ |
| $\frac{2\pi}{3}$ | $4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ | $\left(-2, \frac{2\pi}{3}\right)$ |
| $\frac{5\pi}{6}$ | $4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \approx -3.5$ | $\left(-3.5, \frac{5\pi}{6}\right)$ |
| $\pi$ | $4\cos\pi = 4(-1) = -4$ | $(-4, \pi)$ |
| Values of $r$ repeat. | | |

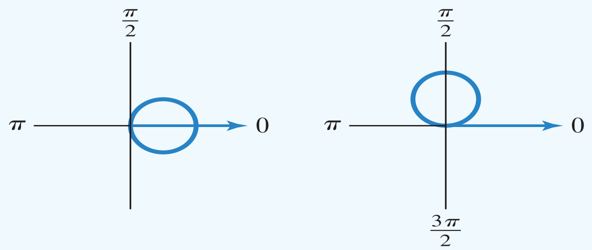
## Circles in Polar Coordinates

The graphs of

$$r = 2a\cos\theta \qquad \text{and} \qquad r = 2a\sin\theta$$

are circles.

$$r = 2a\cos\theta \qquad \text{and} \qquad r = 2a\sin\theta$$




### EXAMPLE 3 Sketching the Graph of a Cardioid

Sketch a graph of $r = 2 + 2\cos\theta$.

**SOLUTION** Instead of plotting points as in Example 1, we first sketch the graph of $r = 2 + 2\cos\theta$ in *rectangular* coordinates in Figure 3. We can think of this graph as a table of values that enables us to read at a glance the values of $r$ that correspond to increasing values of $\theta$. For instance, we see that as $\theta$ increases from 0 to $\pi/2$, $r$ (the distance from $O$) decreases from 4 to 2, so we sketch the corresponding part of the polar graph in Figure (a). As $\theta$ increases from $\pi/2$ to $\pi$, Figure 3 shows that $r$ decreases from 2 to 0, so we sketch the next part of the graph as in Figure (b). As $\theta$ increases from $\pi$ to $3\pi/2$, $r$ increases from 0 to 2, as shown in part (c). Finally, as $\theta$ increases from $3\pi/2$ to $2\pi$, $r$ increases from 2 to 4, as shown in part (d). If we let $\theta$ increase beyond $2\pi$ or decrease beyond 0, we would simply retrace our path. Combining the portions of the graph from parts (a) through (d) of Figure 4, we sketch the complete graph in part (e).

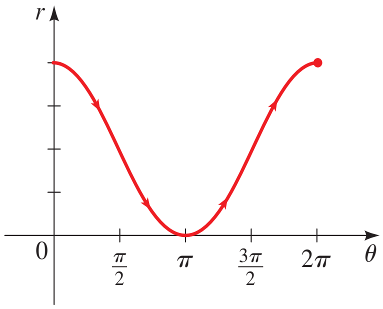



**FIGURE 3** $r = 2 + 2\cos\theta$

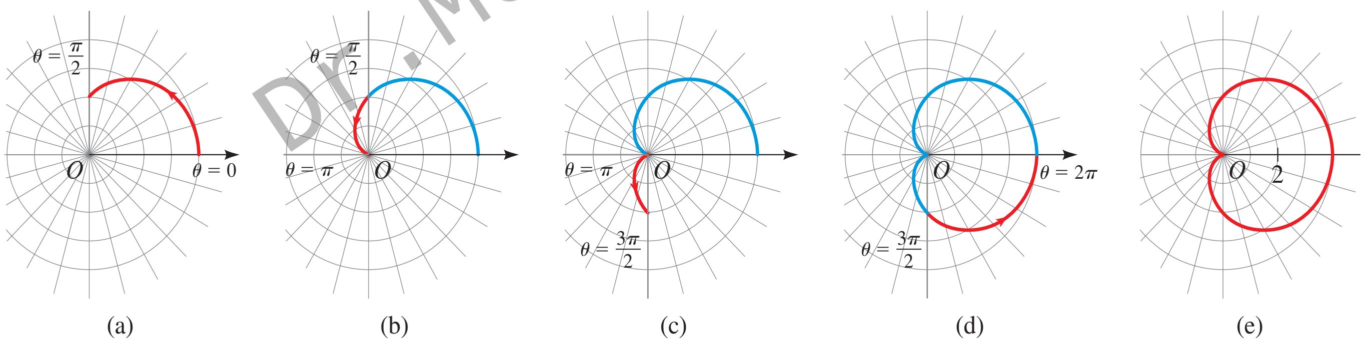



**FIGURE 4** Steps in sketching $r = 2 + 2\cos\theta$

The polar equation $r = 2 + 2\cos\theta$ in rectangular coordinates is

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

The curve in Figure 4 is called a **cardioid** because it is heart-shaped. In general, the graph of any equation of the form

$$r = a(1 \pm \cos\theta) \qquad \text{or} \qquad r = a(1 \pm \sin\theta)$$

is a cardioid.

## EXAMPLE 4 Sketching the Graph of a Four-Leaved Rose

Sketch the curve $r = \cos 2\theta$.

**SOLUTION** we first sketch the graph of $r = \cos 2\theta$ in *rectangular* coordinates, as shown in Figure 5. As $\theta$ increases from 0 to $\pi/4$, Figure 5 shows that $r$ decreases from 1 to 0, so we draw the corresponding portion of the polar curve in Figure 6. As $\theta$ increases from $\pi/4$ to $\pi/2$, the value of $r$ goes from 0 to $-1$. This means that the distance from the origin increases from 0 to 1, but instead of being in Quadrant I, this portion of the polar curve lies on the opposite side of the origin in Quadrant III. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in

which the portions are traced out. The resulting curve has four petals and is called a **four-leaved rose**.

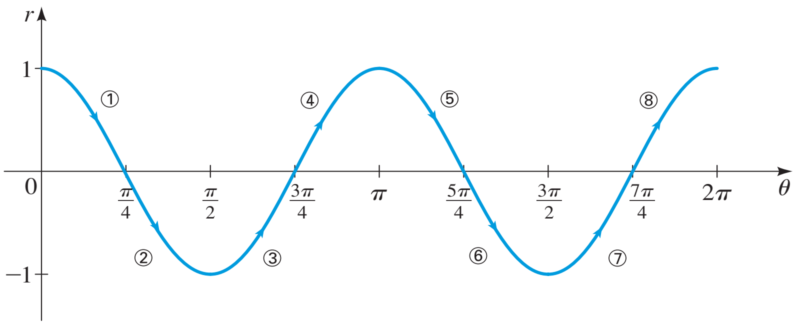



**FIGURE 5** Graph of $r = \cos 2\theta$ sketched in rectangular coordinates

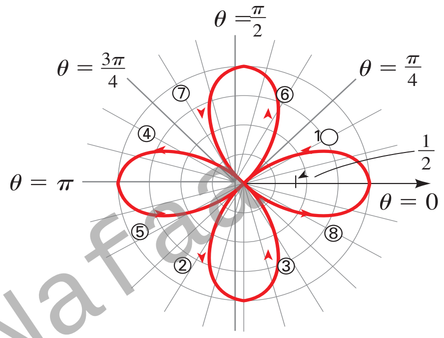



**FIGURE 6** Four-leaved rose $r = \cos 2\theta$ sketched in polar coordinates

In general, the graph of an equation of the form

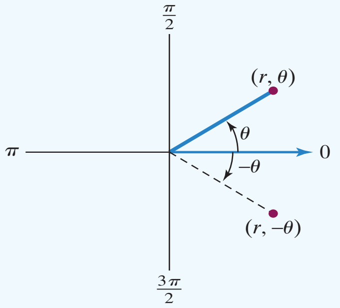
$$r = a \cos n\theta \qquad \text{or} \qquad r = a \sin n\theta$$

is an ***n*-leaved rose** if $n$ is odd or a $2n$-leaved rose if $n$ is even.

## Symmetry

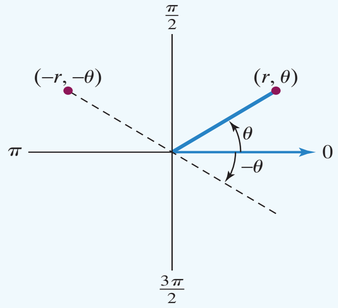
### Tests for Symmetry in Polar Coordinates

**Symmetry with Respect to the Polar Axis ($x$-Axis)**



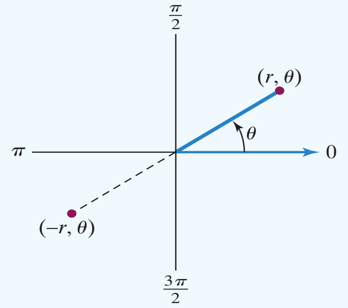

Replace $\theta$ with $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis.

**Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$ ($y$-Axis)**




Replace $(r, \theta)$ with $(-r, -\theta)$. If an equivalent equation results, the graph is symmetric with respect to $\theta = \frac{\pi}{2}$.

**Symmetry with Respect to the Pole (Origin)**




Replace $r$ with $-r$. If an equivalent equation results, the graph is symmetric with respect to the pole.

## EXAMPLE 5 Graphing a Polar Equation Using Symmetry

Check for symmetry and then graph the polar equation:

$$r = 1 - \cos\theta.$$

**Solution** We apply each of the tests for symmetry.

**Polar Axis:** Replace $\theta$ with $-\theta$ in $r = 1 - \cos\theta$:

$r = 1 - \cos(-\theta)$ Replace $\theta$ with $-\theta$ in $r = 1 - \cos\theta$.

$r = 1 - \cos\theta$ The cosine function is even: $\cos(-\theta) = \cos\theta$.

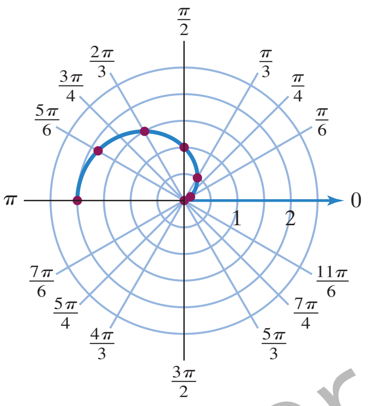
Because the polar equation does not change when $\theta$ is replaced with $-\theta$, the graph is symmetric with respect to the polar axis.

**The Line $\theta = \frac{\pi}{2}$:** Replace $(r, \theta)$ with $(-r, -\theta)$ in $r = 1 - \cos\theta$:

$-r = 1 - \cos(-\theta)$ Replace $r$ with $-r$ and $\theta$ with $-\theta$ in $r = 1 - \cos\theta$.

$-r = 1 - \cos\theta$ $\cos(-\theta) = \cos\theta$.

$r = \cos\theta - 1$ Multiply both sides by $-1$.



Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when $(r, \theta)$ is replaced with $(-r, -\theta)$, the equation fails this symmetry test. The graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

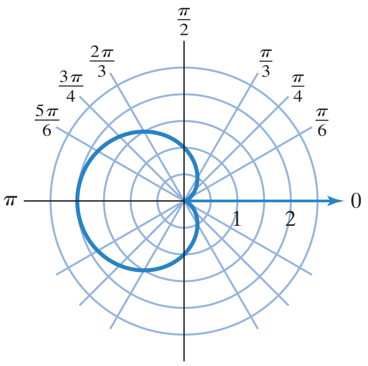
**The Pole:** Replace $r$ with $-r$ in $r = 1 - \cos\theta$:

$-r = 1 - \cos\theta$ Replace $r$ with $-r$ in $r = 1 - \cos\theta$.

$r = \cos\theta - 1$ Multiply both sides by $-1$.

Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when $r$ is replaced with $-r$, the equation fails this symmetry test. The graph may or may not be symmetric with respect to the pole.

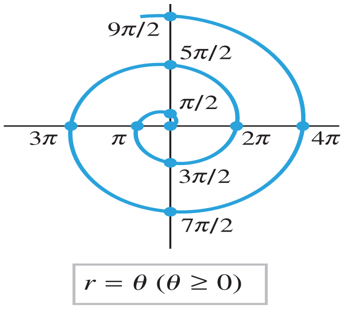
Now we are ready to graph $r = 1 - \cos\theta$. Because the period of the cosine function is $2\pi$, we need not consider values of $\theta$ beyond $2\pi$. Recall that we discovered the graph of the equation $r = 1 - \cos\theta$ has symmetry with respect to the polar axis. Because the graph has this symmetry, we can obtain a complete graph by plotting fewer points. Let's start by finding the values of $r$ for values of $\theta$ from 0 to $\pi$.



| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | 0.13 | 0.5 | 1 | 1.5 | 1.87 | 2 |

**EXAMPLE 7** **Sketch the graph of $r = \theta$ ($\theta \geq 0$) in polar coordinates by plotting points.**

***Solution.*** Observe that as $\theta$ increases, so does $r$; thus, the graph is a curve that spirals out from the pole as $\theta$ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the points that correspond to values of $\theta$ that are integer multiples of $\pi/2$, keeping in mind that the value of $r$ is always equal to the value of $\theta$ (Figure 12). ◀




▲ **Figure 12**

## POLAR EQUATIONS OF CIRCLES

We have already had considerable experience in transforming the rectangular equation of a given curve into an equivalent polar equation for the same curve.

Consider, for example, the circle (Fig. 13 , left) with center $(a, 0)$ and ra-dius $a$:

$$(x - a)^2 + y^2 = a^2 \quad \text{or} \quad x^2 + y^2 = 2ax. \tag{1}$$

Since $x^2 + y^2 = r^2$ and $x = r\cos\theta$, this equation becomes

$$r^2 = 2ar\cos\theta,$$

which is equivalent to

$$r = 2a\cos\theta \tag{2}$$

because the origin $r = 0$ lies on the graph of (2).

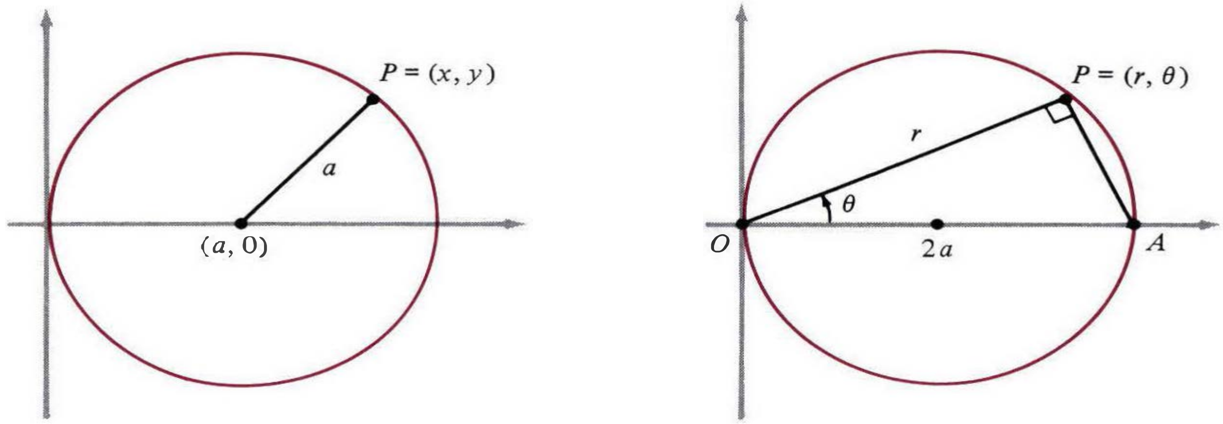
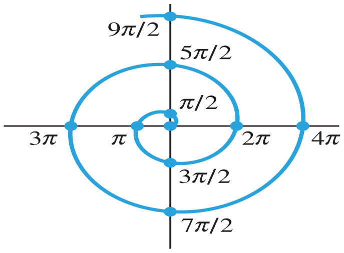



Figure 13

**EXAMPLE 7** Sketch the graph of $r = \theta$ ($\theta \geq 0$) in polar coordinates by plotting points.

***Solution.*** Observe that as $\theta$ increases, so does $r$; thus, the graph is a curve that spirals out from the pole as $\theta$ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the points that correspond to values of $\theta$ that are integer multiples of $\pi/2$, keeping in mind that the value of $r$ is always equal to the value of $\theta$ (Figure 1□). ◀




$r = \theta$ ($\theta \geq 0$)

▲ **Figure 10**

## POLAR EQUATIONS OF CIRCLES

We have already had considerable experience in transforming the rectangular equation of a given curve into an equivalent polar equation for the same curve.

Consid er, forexample, the circle (Fig. 1□, left) with center $(a, 0)$ and ra-dius $a$:

$$(x - a)^2 + y^2 = a^2 \quad \text{or} \quad x^2 + y^2 = 2ax. \tag{1}$$

Since $x^2 + y^2 = r^2$ and $x = r\cos\theta$, this equation becomes

$$r^2 = 2ar\cos\theta,$$

which is equivalent to

$$r = 2a\cos\theta \tag{2}$$

because the origin $r = 0$ lies on the graph of (2).

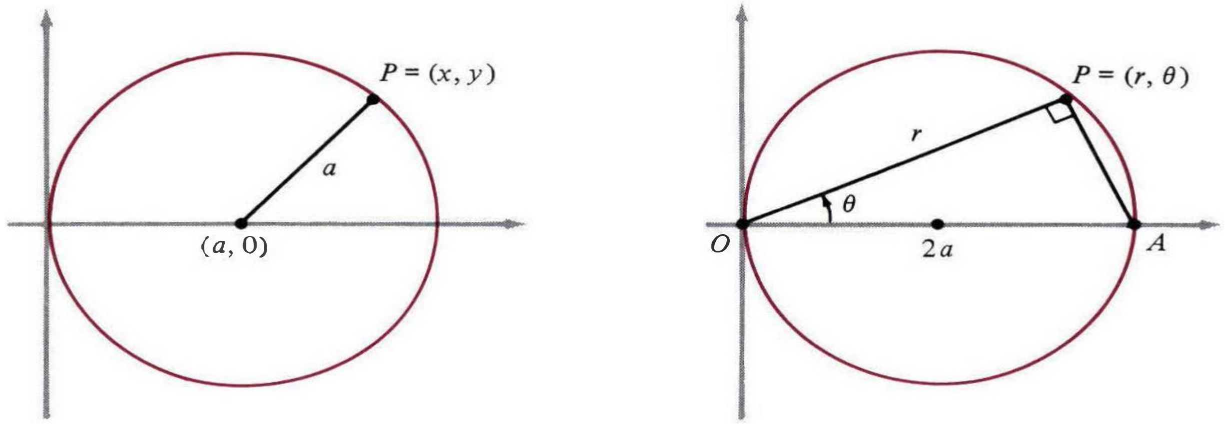



**Figure 1□**